Lattices From Graph Associahedra

Emily Barnard

Joint with Thomas McConville

DePaul University

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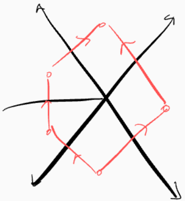
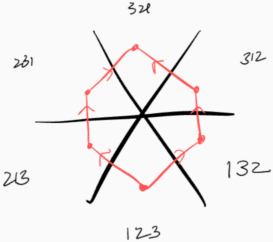
My favorite posets

Tamari Lattice T_3



weak order on S_3



Posets From Polytopes

Definition

Let P a polytope with vertex set V, and fix a linear function λ.

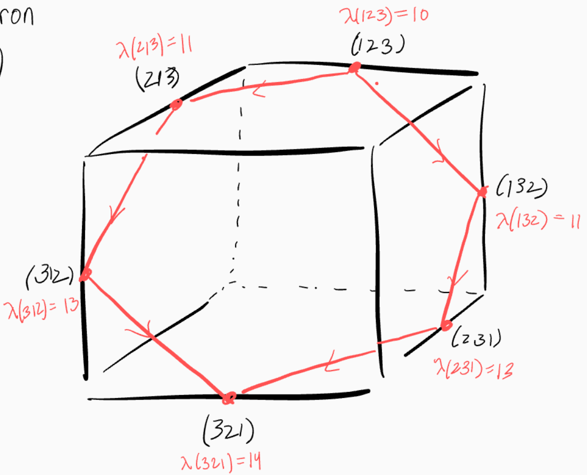
Let $L(P, \lambda)$ denote the partial order on V obtained by taking the transitive and reflexive closure of $\mathbf{x} \leqslant \mathbf{y}$ when

- $[\mathbf{x}, \mathbf{y}]$ is an edge of P and
- $\lambda(\mathbf{x}) \leqslant \lambda(\mathbf{y})$.

Posets from Polytopes

Permutahedron

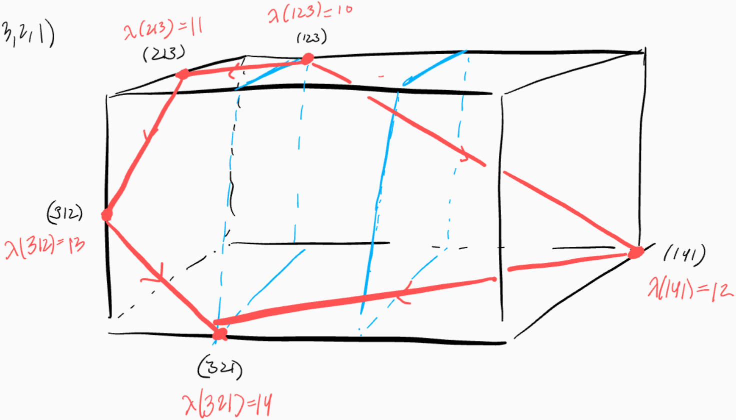
$\lambda = (3,2,1)$



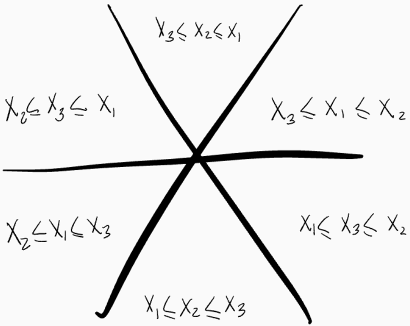
Posets from Polytopes

Associahedron

$\lambda = (3,2,1)$

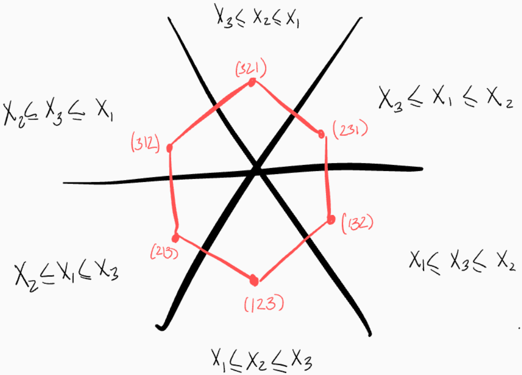


Posets from (normal) fans



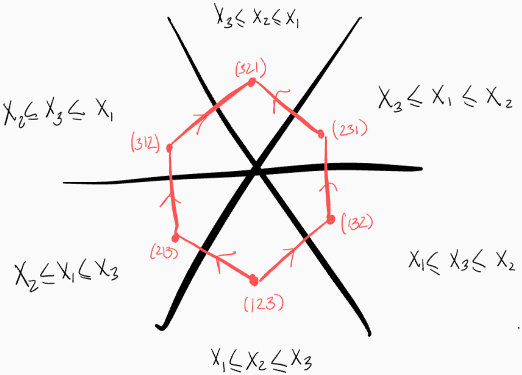
Normal fan of the permutahedron

Posets from (normal) fans



Draw in the dual graph.

Posets from (normal) fans



Orient the dual graph.

Motivation

Properties of the weak order on $\mathfrak{S}_n$ and the Tamari lattice

- The Hasse diagram is (an orientation of) the one-skeleton of a polytope.
- Both posets are lattices.

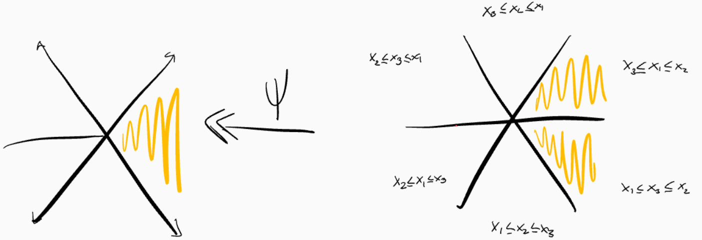
Fact

- The normal fan of the associahedron coarsens the normal fan of the permutahedron.
- Thus, there is a canonical surjection from $\mathfrak{S}_n$ onto the Tamari lattice T_n which we denote by Ψ.

The Canonical Surjection

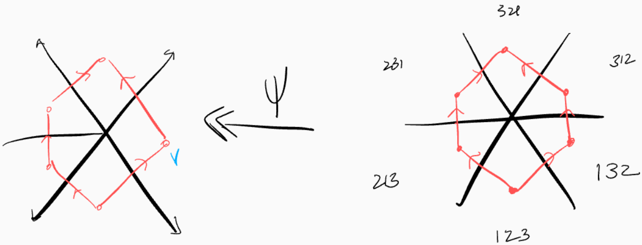
Normal Fan of the associahedron

Normal Fan of the permutaheron



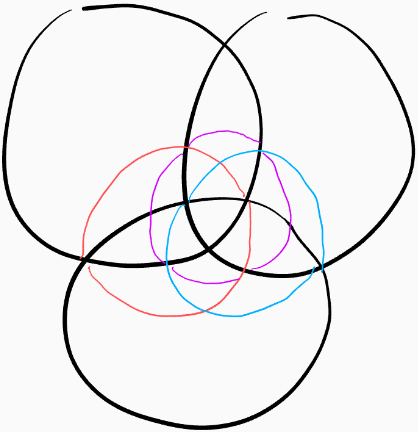
The Canonical Surjection

Tamari Lattice T_3 weak order on S_3

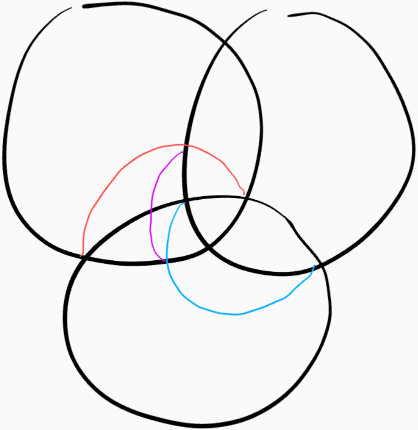


$$\psi(132) = \psi(312) = V$$

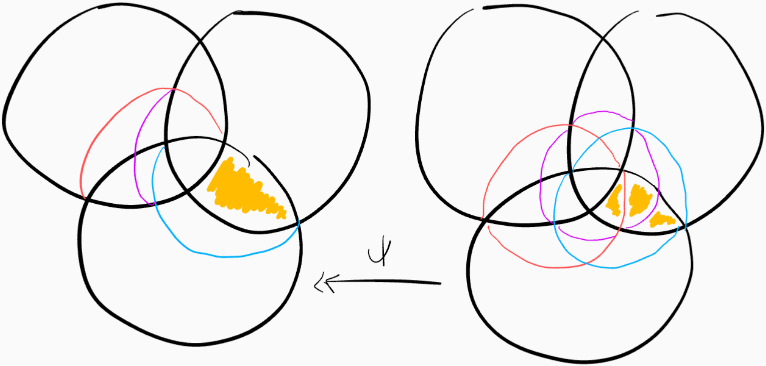
The Canonical Surjection



The Canonical Surjection



The Canonical Surjection



Normal Fan of the associahedron

Braid Arrangmt

Goal of the talk

Theorem [Reading]

The canonical surjection $\Psi : \mathfrak{S}_n \to T_n$ is a lattice quotient.
That is:

- $\Psi(w \vee w') = \Psi(w) \vee \Psi(w')$
- $\Psi(w \wedge w') = \Psi(w) \wedge \Psi(w')$

Set Up

Given a graph G, we will construct a **graph associahedron** P_G, a polytope whose normal fan coarsens the normal fan of the permutahedron. Then we will construct an analogous poset L_G.

Question

For which G is the canonical surjection $\Psi_G : \mathfrak{S}_n \to L_G$ a lattice quotient?

Notation

- Write $[n]$ for the set $\{1, 2, \ldots, n\}$.
- G is a graph with vertex set $[n]$.
- Let Δ_I denote the simplex with vertex set $\{\mathbf{e_i} : i \in I \subseteq [n]\}$.

Definition/Recall

Let P and Q be polytopes. The **Minkowski Sum** is the polytope

$$P + Q = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in P \text{ and } \mathbf{y} \in Q\}.$$

The normal fan of P is a coarsening of the normal fan of $P + Q$.

Graph Associahedra

Definition

A **tube** is a nonempty subset I of vertices such that the induced subgraph $G|_I$ is connected.

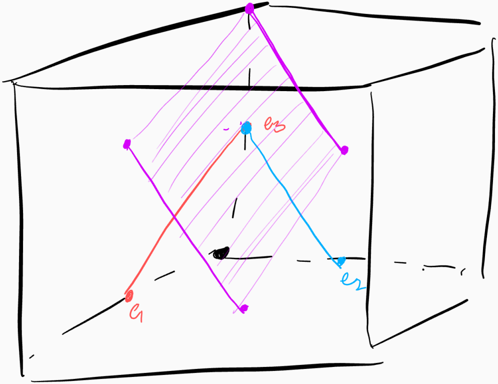
The Graph Assocciahedron

The **Graph Associahedron** P_G is the Minkowski sum

$$P_G = \sum_{I \text{ is a tube of } G} \Delta_I .$$

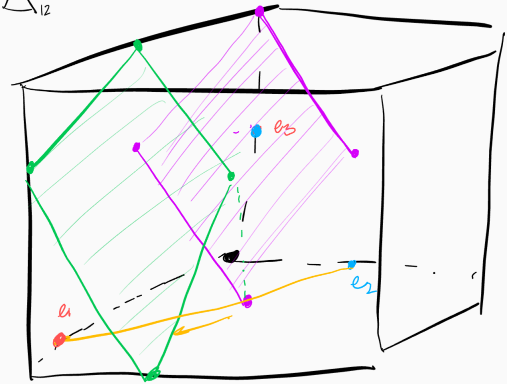
Examples: The Complete Graph

$\Delta_{13} + \Delta_{23}$



Examples: The Complete Graph

$(\Delta_{13} + \Delta_{23}) + \Delta_{12}$



Examples: The Complete Graph

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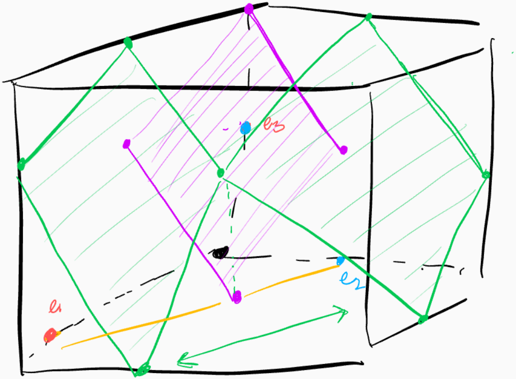

Examples: The Complete Graph

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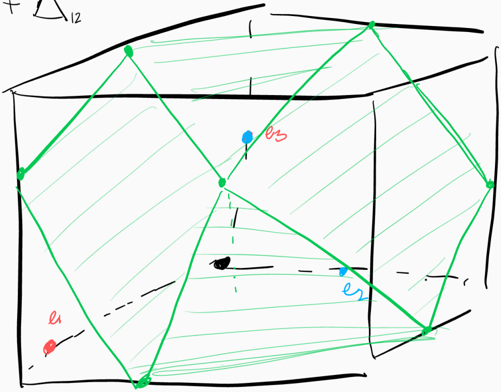
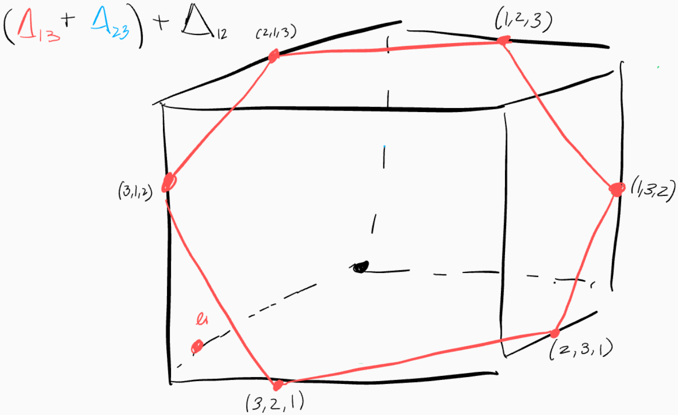


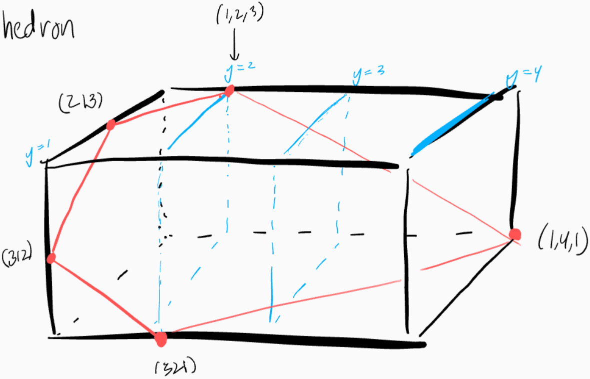
Examples: The Complete Graph



Examples

$G = 1 — 2 — 3$

$P_G =$ Associahedron



Context: Topology and Geometry

The Bergman Complex

Let M be an oriented matroid. The Bergman complex $\mathcal{B}(M)$ and the positive Bergman complex $\mathcal{B}^+(M)$ generalize the notions of a tropical variety and positive tropical variety to matroids.

Theorem[Ardila, Reiner, Williams]

Let Φ be a the root system associated to a (possibly infinite) Coxeter system (W, S) and let Γ be the associated Coxeter diagram. The positive Bergman complex $\mathcal{B}^+(M_\Phi)$ is dual to the graph associahedron P_Γ.

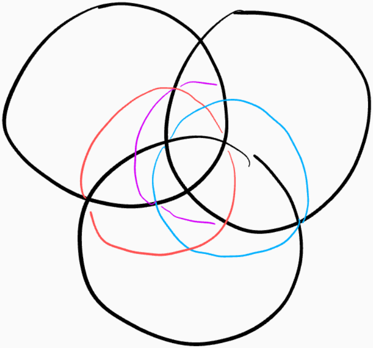
The Poset L_G

Definition

Fix $\lambda = (n, n-1, \ldots, 2, 1)$. The poset L_G is the partial order on the vertex set of P_G obtained by taking the transitive and reflexive closure of $\mathbf{x} \leqslant \mathbf{y}$ when

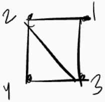
- $[\mathbf{x}, \mathbf{y}]$ is an edge of P_G and
- $\lambda(\mathbf{x}) \leqslant \lambda(\mathbf{y})$.

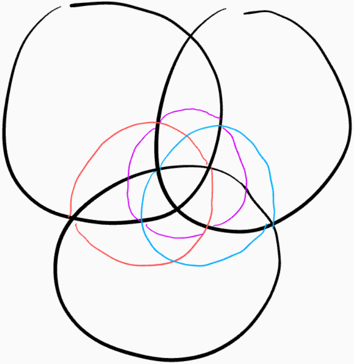
The poset L_G



Normal Fan of the P_G

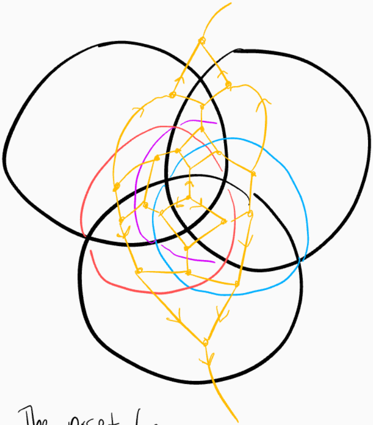
$G =$



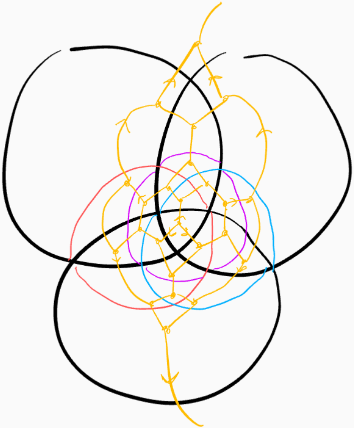


Normal Fan of the permutahedron

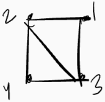
The poset L_G



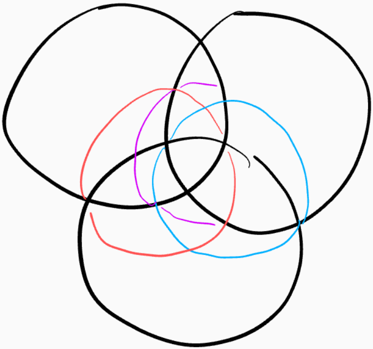
The poset L_G



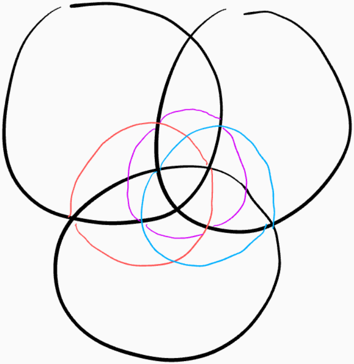
The weak order on S_4

$G=$ 

The canonical surjection

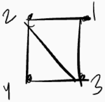


Normal Fan of the P_G

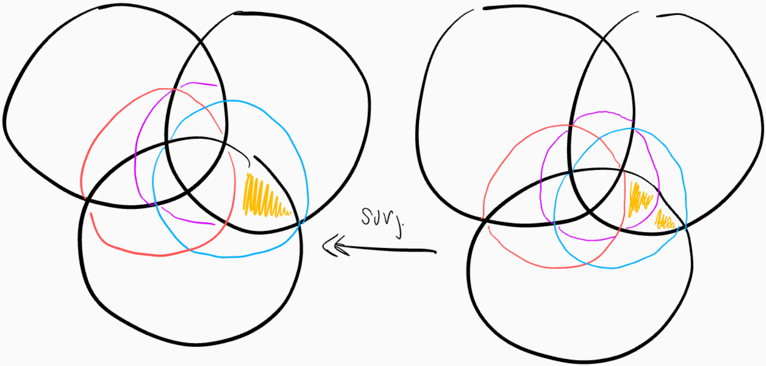


Normal Fan of the permutahedron

$G=$



The canonical surjection



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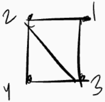
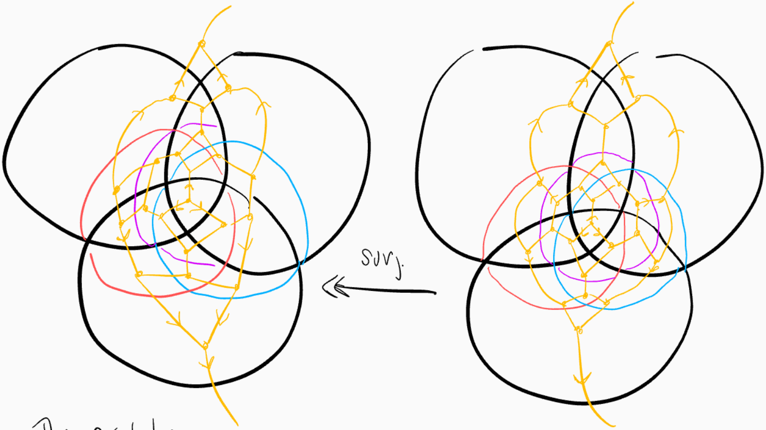
$G =$ 

The weak order on S_4

The canonical surjection



The poset L_G

The weak order on S_4

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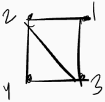
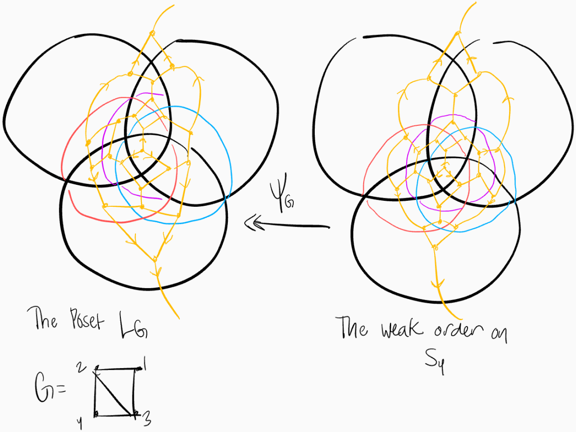


The Canonical Surjection

Let Ψ_G denote the surjection from $\mathfrak{S}_n$ to the poset L_G.



Recap: Main Question

Theorem [Reading]

Let G be the path graph, let L_G be the associated poset. Then canonical surjection $\Psi_G : \mathfrak{S}_n \to L_G$ is a lattice quotient.
That is:

- $\Psi_G(w \vee w') = \Psi_G(w) \vee \Psi_G(w')$
- $\Psi_G(w \wedge w') = \Psi_G(w) \wedge \Psi_G(w')$

Question

For which G is the canonical surjection Ψ_G a lattice quotient?

Main Results

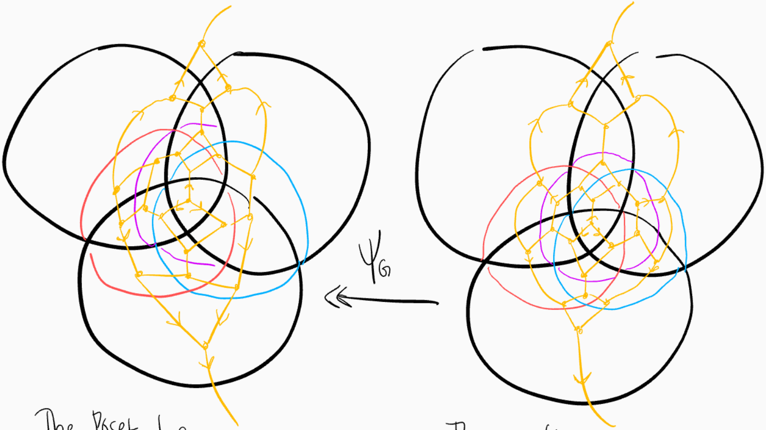
Definition

We say a graph G is **filled** if for each edge $\{i, k\}$ in G, the edges $\{i, j\}$ and $\{j, k\}$ are also in G for all $i < j < k$.

Theorem [B., McConville]

The map Ψ_G is a lattice quotient if and only if G is filled.

A filled graph



The Poset L_G

The weak order on S_4

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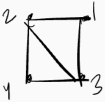

Proof Sketch

Each permutation maps to the bottom element of its ψ_G-fiber

$$\begin{tikzcd}
S_n \arrow[rr, two heads, "\psi_G"] \arrow[dr, dashed, two heads] & & L_G \\
& \text{Subposet of } S_n \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Hvala! Thank you!

When is L_G a lattice?

Definition

Two tubes I, J are said to be **compatible** if either

- they are *nested*: $I \subseteq J$ or $J \subseteq I$, or
- they are *separated*: $I \cup J$ is not a tube.

A (maximal) **tubing** $\mathcal{X}$ of G is a (maximal) collection of pairwise compatible tubes.

Definition/Theorem

Each cover relation in L_G is encoded by a **flip** $\mathcal{X} \to \mathcal{Y}$ defined by:

- $\mathcal{Y} = \mathcal{X} \backslash \{I\} \cup \{I'\}$
- $\text{top}_{\mathcal{X}}(I) < \text{top}_{\mathcal{Y}}(I')$

When is L_G a lattice?