Reverse plane partitions via representations of quivers

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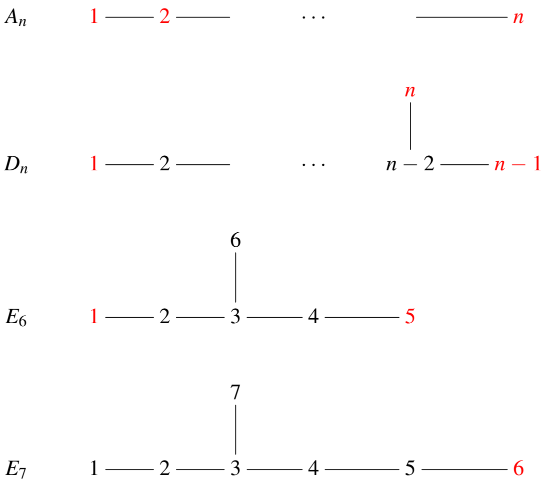
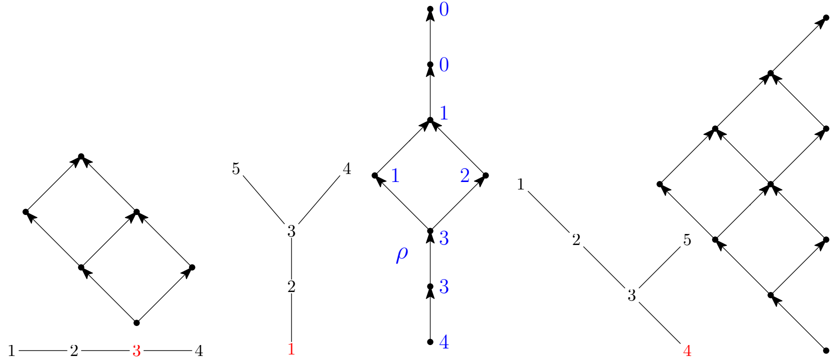
Outline

- minuscule posets
- Auslander–Reiten quivers
- nilpotent endomorphisms of quiver representations
- promotion on reverse plane partitions

A minuscule poset is defined by choosing a simply-laced Dynkin diagram and a **minuscule vertex** m.





$\mathsf{P}_{A_4,3}$ $\mathsf{P}_{D_5,1}$ $\mathsf{P}_{D_5,4}$

A **reverse plane partition** is an order-reversing map $\rho : \mathsf{P} \to \mathbb{Z}_{\geqslant 0}$.

Theorem (Proctor '84)

For any minuscule poset P, the generating function for reverse plane partitions on P is

$$\sum_{\rho: P \to \mathbb{Z}_{\geqslant 0} \in RPP(P)} q^{|\rho|} = \prod_{x \in P} \frac{1}{1 - q^{rk(x)}}$$

where $|\rho| := \sum_{x \in P} \rho(x)$ and $rk : P \to \mathbb{Z}_{\geqslant 1}$ is the rank function on P.

- Analogous identities for order filters of certain minuscule posets (Stanley '71, Hillman–Grassl '76, Gansner '81, Pak '01, Sulzgruber '17)
- Analogous identities for "skew shapes" (Morales–Pak–Panova '15, Naruse–Okada '18)

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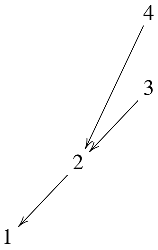
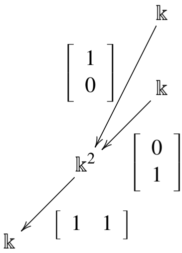
We will interpret this identity in terms of quiver representations.



Q

a **quiver**



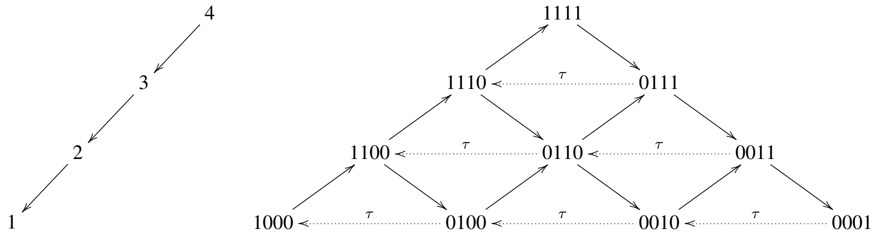
V

a **representation** of Q

$\mathbf{dim}(V) = 1211$

dimension vector of V

Any quiver Q has an **Auslander–Reiten quiver** $\Gamma(Q)$ whose vertices are the isomorphism classes of indecomposable representations of Q.



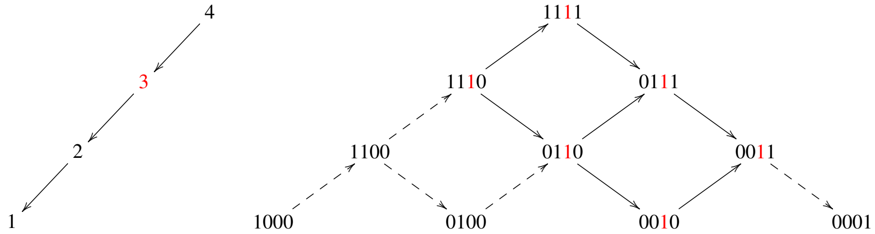
Q $\Gamma(Q)$ - the Auslander–Reiten quiver of Q

- There is a map τ called the **Auslander–Reiten translation**.
- The Auslander–Reiten translation partitions the indecomposables into τ**-orbits**.

$$\{\text{vertices of } Q\} \longleftrightarrow \{\tau\text{-orbits}\}$$

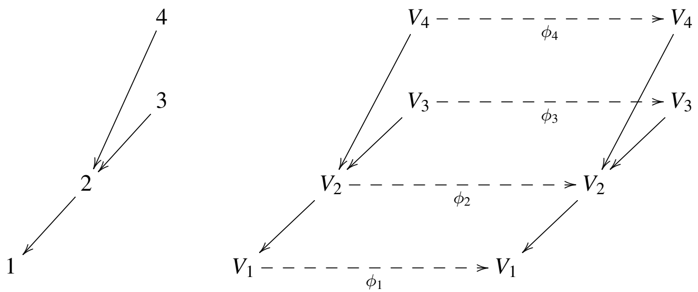
Lemma

Given a Dynkin quiver Q and a minuscule vertex m, the Hasse quiver of the minscule poset $P_{\overline{Q},m}$ is isomorphic to the full subquiver of $\Gamma(Q)$ on the representations supported at m.



Q $\Gamma(Q)$ - the Auslander–Reiten quiver of Q

Let $\mathcal{C}_{Q,m}$ denote the category of all representations of Q, each of whose indecomposable summands is supported at m.



- Let $\phi = (\phi_i)_i \in \mathrm{NEnd}(V) := \{\text{nilpotent endomorphisms of } V\}$.
- Each $\phi_i \rightsquigarrow \lambda^i = (\lambda^i_1 \geqslant \cdots \geqslant \lambda^i_r)$ where partition λ^i records the sizes of the Jordan blocks of ϕ_i.

$$\mathrm{JF}(\phi) := (\lambda^1, \ldots, \lambda^n) \quad \text{the } \textbf{Jordan form data} \text{ of } \phi$$

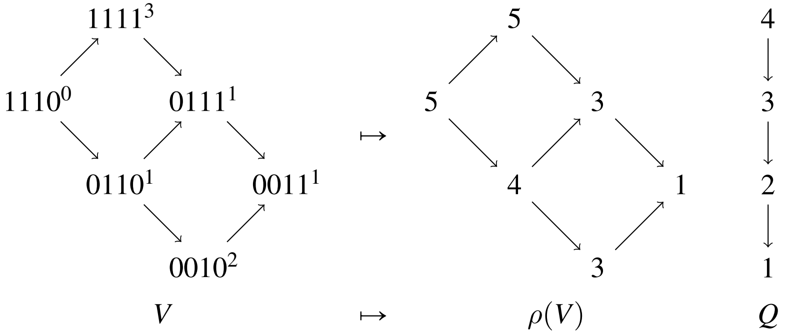
Theorem (G.–Patrias–Thomas, '18)

There is a unique maximum value of $JF(\cdot)$ on $NEnd(V)$ with respect to componentwise dominance order, denoted by $GenJF(V)$. Moreover, it is attained on a dense open subset of $NEnd(V)$.

Theorem (G.–Patrias–Thomas, '18)

The objects of $\mathcal{C}_{Q,m}$ *are in bijection with* $RPP(P_{\overline{Q},m})$ *via*

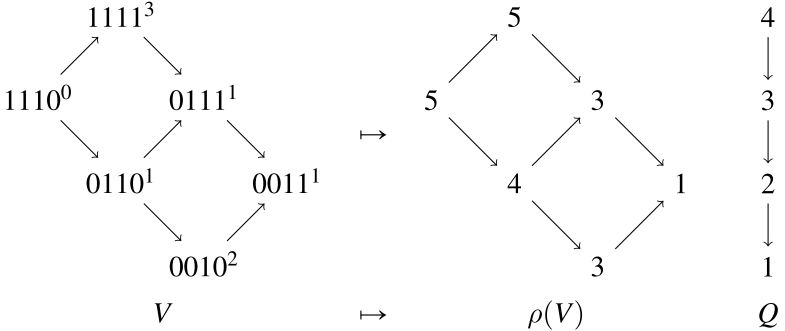
$$V \mapsto \rho(V) \text{ – reverse plane partition from filling the } \tau\text{-orbits of } P_{\overline{Q},m} \text{ with the Jordan block sizes in } GenJF(V)$$



$$\mathbf{dim}(V) = 3585$$
$$\mathrm{GenJF}(V) = ((3), (4,1), (5,3), (5))$$

Theorem (G.–Patrias–Thomas, '18)

The objects of $\mathcal{C}_{Q,m}$ *are in bijection with* $RPP(P_{\overline{Q},m})$.



$$\mathrm{GenJF}(V) = ((3), (4,1), (5,3), (5))$$

Corollary

$$\sum_{\rho \in RPP(P)} q^{|\rho|} = \sum_{V \in \mathcal{C}_{Q,m}} q^{dim(V)} = \prod_{V^i \in ind(\mathcal{C}_{Q,m})} \frac{1}{1 - q^{dim(V^i)}} = \prod_{x \in P} \frac{1}{1 - q^{rk(x)}}$$

Algorithmic construction of $\rho(V)$

Apply the following piecewise linear transformations "from right to left" in $\Gamma(Q)$ to obtain ρ from $V \in \mathcal{C}_{Q,m}$.

- if W is a summand of V, replace $\rho_i(V)$ with $\rho_{i+1}(W) = \max_{W<U} \rho_i(U) + \mathrm{mult}(W)$,
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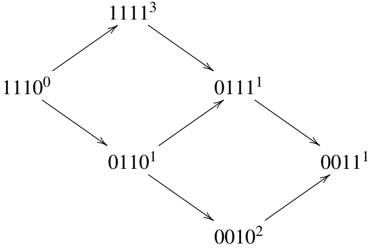
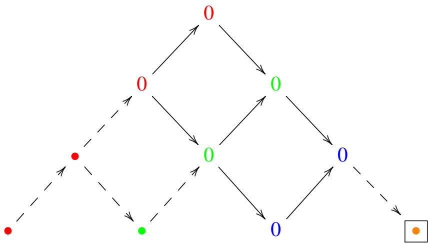


V



$\Gamma(Q)$

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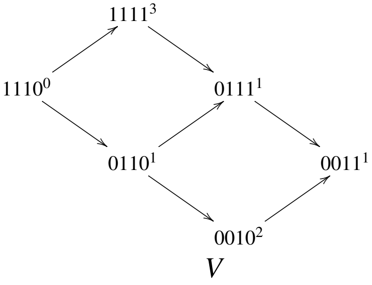

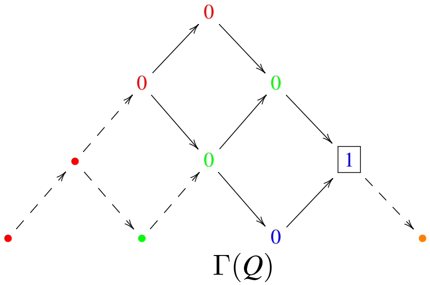

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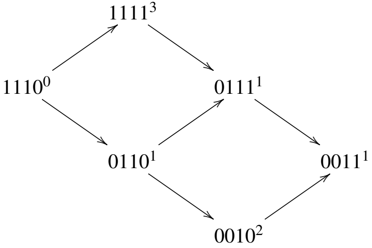

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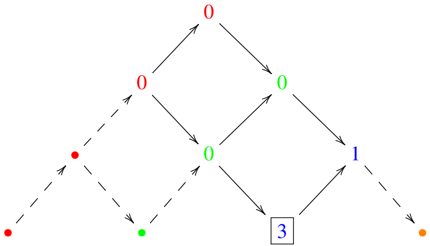

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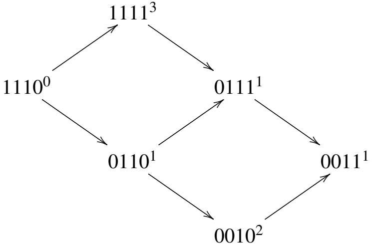

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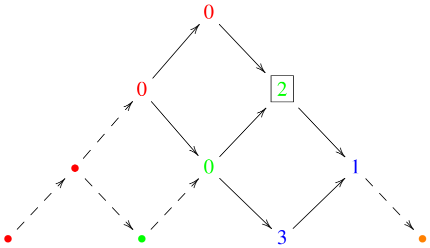

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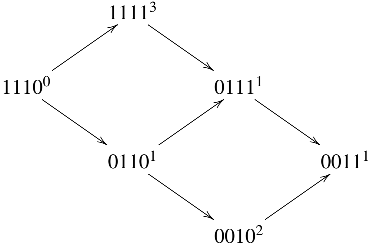

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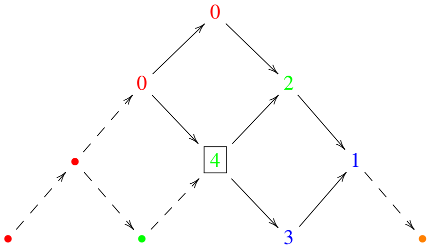

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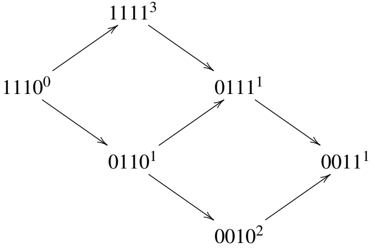

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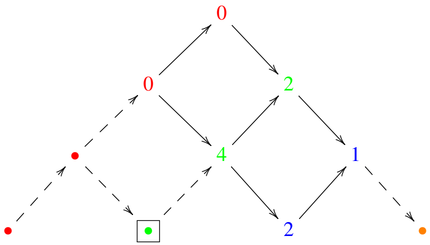

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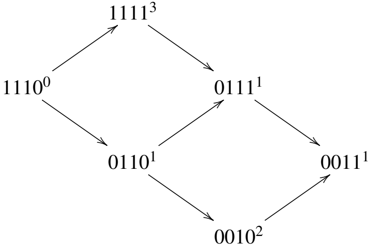

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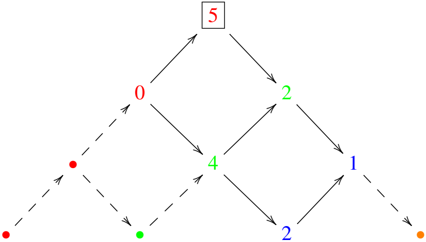

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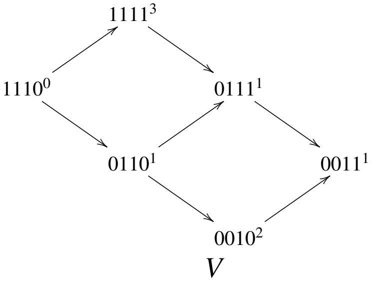

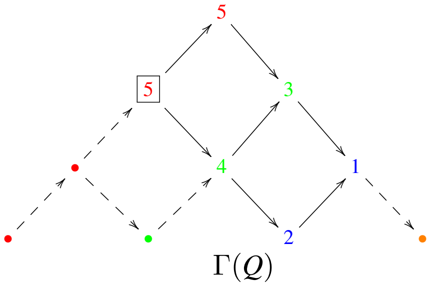

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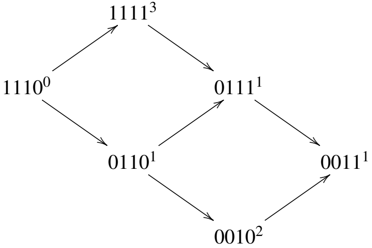

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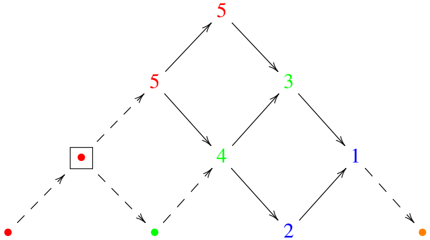

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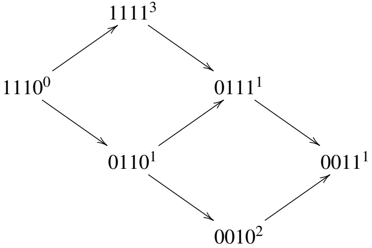

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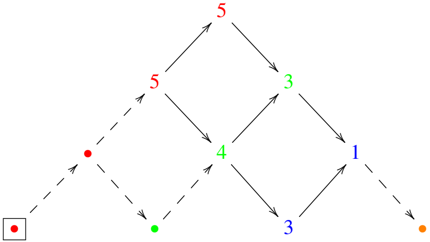
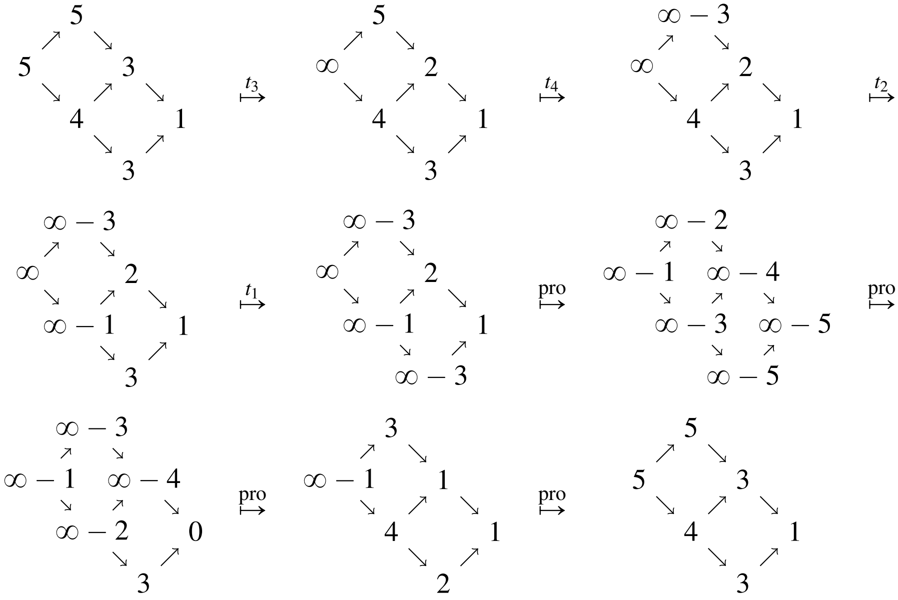


$\Gamma(Q)$

Periodicity for promotion $\text{pro} = t_1 t_2 t_4 t_3$



- earlier results in type A (Grinberg–Roby '15, Musiker–Roby '18)

Hvala!