Thrall's problem: cyclic sieving, necklaces, and branching rules

FPSAC 2019 in Ljubljana, Slovenia
July 2nd, 2019

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Based on joint work with
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arXiv:1808.06043
Published version in *Electron. J. Combin.* 25 (2018): [AS18a]
Slides: http://www.math.ucsd.edu/~jswanson/talks/2019_FPSAC.pdf

Outline

- We first apply the *cyclic sieving phenomenon* of Reiner–Stanton–White to prove Schur expansions due to Kraśkiewicz–Weyman related to *Thrall's problem*.
- The resulting argument is *remarkably simple and nearly bijective*. It is a rare example of the CSP being used *to prove other results*, rather than vice-versa.
- We then apply our approach to prove other results of Stembridge and Schocker.
- Guided by our experience, we *suggest a new approach* to Thrall's problem.

Thrall's problem

What is Thrall's problem?

Definition

Let...

- V be a finite-dimensional vector space over $\mathbb{C}$;
- $T(V) := \oplus_{n\geq 0} V^{\otimes n}$ be the *tensor algebra of V*;
- $\mathcal{L}(V)$ be the *free Lie algebra on V*, namely the Lie subalgebra of $T(V)$ generated by V;
- $\mathcal{L}_n(V) := \mathcal{L}(V) \cap V^{\otimes n}$ be the *nth Lie module*;
- $\mathfrak{U}(\mathcal{L}(V))$ be the *universal enveloping algebra* of $\mathcal{L}(V)$; and
- $\mathrm{Sym}^m(M)$ be the mth *symmetric power* of a vector space M.

Thrall's problem

By an appropriate version of the Poincaré–Birkhoff–Witt Theorem,

$$T(V) \cong \mathcal{U}(\mathcal{L}(V)) \cong \bigoplus_{\lambda=1^{m_1}2^{m_2}\cdots} \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots$$

as graded $\mathrm{GL}(V)$-modules.

Definition (Thrall [Thr42])

The *higher Lie module* associated to $\lambda = 1^{m_1}2^{m_2}\cdots$ is

$$\mathcal{L}_\lambda(V) := \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots .$$

Thus we have a *canonical* $\mathrm{GL}(V)$-module decomposition

$$T(V) \cong \oplus_{\lambda\in\mathrm{Par}}\mathcal{L}_\lambda(V).$$

Question (*Thrall's Problem*)

What are the irreducible decompositions of the $\mathcal{L}_\lambda(V)$?

Thrall's problem

$$\mathcal{L}_\lambda(V) := \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots .$$

- The Littlewood–Richardson rule reduces Thrall's problem to the *rectangular case* $\lambda = (a^b)$ with b rows of length a.
- In the rectangular case,

$$\mathcal{L}_{(a^b)}(V) = \mathrm{Sym}^b \, \mathcal{L}_a(V)$$

- In the one-row case,

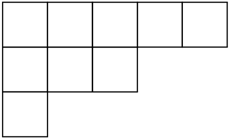
$$\mathcal{L}_{(a)}(V) = \mathcal{L}_a(V).$$

Kraśkiewicz–Weyman [KW01] solved Thrall's problem *in the one-row case*. We next describe their answer.

Partitions

Definition

A *partition* λ of n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum_i \lambda_i = n$. Partitions can be visualized by their *Ferrers diagram*

$\lambda = (5, 3, 1) \leftrightarrow$ 

Theorem

(Young, early 1900's) *The complex inequivalent irreducible representations S^λ of S_n are* **canonically** *indexed by partitions of n.*

Standard tableaux

Definition

A *standard Young tableau* (*SYT*) of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of λ with $1, 2, \ldots, n$ which **increases along rows** and **decreases down columns**.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1{\color{red}1} & 3{\color{red}3} & 6 & 7{\color{red}7} & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \in {\color{red}\mathrm{SYT}(\lambda)}$$

Descent set: $\{1, 3, 7\}$. Major index: $1 + 3 + 7 = 11$.

Definition

The *descent set* of $T \in \mathrm{SYT}(\lambda)$ is the set

$${\color{red}\mathrm{Des}(T)} := \{1 \leq i < n : i + 1 \text{ is in a lower row of } T \text{ than } i\}.$$

The *major index* of $T \in \mathrm{SYT}(\lambda)$ is ${\color{red}\mathrm{maj}(T)} := \sum_{i \in \mathrm{Des}(T)} i$.

Thrall's problem

Definition

Let

$$a_{\lambda,r} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\}.$$

Theorem (Kraśkiewicz–Weyman [KW01])

The multiplicity of the $\mathrm{GL}(V)$*-irreducible* V^λ *in* $\mathcal{L}_n(V)$ *is* $a_{\lambda,1}$*.*

Thrall's problem

Kraśkiewicz–Weyman's argument hinges on the following key formula:

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^r) = \chi^\lambda(\sigma_n^r) \tag{1}$$

for all $r \in \mathbb{Z}$, where:

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)},$$

- ω_n is any primitive nth root of unity,
- $\chi^\lambda(\sigma)$ is the character of S^λ at σ, and
- $\sigma_n = (1\ 2\ \cdots\ n) \in S_n$.

Their approach involves results of Lusztig and Stanley on coinvariant algebras and an intricate though beautiful argument involving ℓ-decomposable partitions. The key formula bears a striking resemblance to the *cyclic sieving phenomenon* of Reiner–Stanton–White, which we describe next.

Words

Definition

- A *word* is a sequence

$$w = w_1 w_2 \cdots w_n \qquad \text{s.t.} \qquad w_i \in \mathbb{Z}_{\geq 1}.$$

- W_n is the set of words of length n.
- The *content* of w is the weak composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_j = \#\{i : w_i = j\}$.
- W_α is the set of words of content α.

For example, $w = 412144 \in \mathrm{W}_{(2,1,0,3)} \subset \mathrm{W}_6$.

Major index on words

Definition (MacMahon, early 1900's)

The *descent set* of $w \in \mathrm{W}_n$ is

$$\mathrm{Des}(w) := \{1 \leq i \leq n-1 : w_i > w_{i+1}\}.$$

The *major index* is

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

For example,

$$\mathrm{Des}(412144) = \mathrm{Des}(4.12.144) = \{1, 3\}$$
$$\mathrm{maj}(412144) = 1 + 3 = 4.$$

Major index on words

Theorem (MacMahon [Mac])

The major index generating function on $\mathrm{W}_\alpha \subset \mathrm{W}_n$ *is*

$$\mathrm{W}_\alpha^{\mathrm{maj}}(q) := \sum_{w \in \mathrm{W}_\alpha} q^{\mathrm{maj}(w)} = \frac{[n]_q!}{\prod_{i \geq 1} [\alpha_i]_q!} = \binom{n}{\alpha}_q$$

where $[n]_q := (1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1}$ *and* $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$.

Major index on words

$$\mathrm{W}_{\alpha}^{\mathrm{maj}}(q) = \binom{n}{\alpha}_q$$

We have $\binom{n}{\alpha}_{q=1} = \binom{n}{\alpha} = \# \mathrm{W}_\alpha$.

Exercise

Let ω_d be any primitive dth root of unity. If $d \mid n$,

$$\binom{n}{\alpha}_{q=\omega_d} = \begin{cases} \binom{n/d}{\alpha_1/d, \alpha_2/d, \ldots} & \text{if } d \mid \alpha_1, \alpha_2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Question

What does $\binom{n}{\alpha}_{q=\omega_d}$ count?

Major index on words

Definition

Let $\sigma_n := (1\,2\,\cdots\,n) \in S_n$ be the standard n-cycle. Let $C_n := \langle \sigma_n \rangle$, which acts on each $\mathrm{W}_\alpha \subset \mathrm{W}_n$ by rotation.

Exercise

If $\sigma \in C_n$ has order $d \mid n$, then

$$\begin{aligned}\#\mathrm{W}_\alpha^\sigma &:= \#\{w \in \mathrm{W}_\alpha : \sigma(w) = w\} \\ &= \begin{cases} \binom{n/d}{\alpha_1/d, \alpha_2/d, \ldots} & \text{if } d \mid \alpha_1, \alpha_2, \ldots \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Corollary

For all $\sigma \in C_n$ of order $d \mid n$,

$$\mathrm{W}_\alpha^{\mathrm{maj}}(\omega_d) = \#\,\mathrm{W}_\alpha^\sigma\,.$$

The cyclic sieving phenomenon

Definition (Reiner–Stanton–White [RSW04])

Let X be a finite set on which a cyclic group C of order n acts and suppose $X(q) \in \mathbb{Z}[q]$. The triple $(X, C, X(q))$ *exhibits the cyclic sieving phenomenon* (*CSP*) if for all elements $\sigma_d \in C$ of order d,

$$X(\omega_d) = \#X^{\sigma_d}.$$

Remark

- $d = 1$ gives $X(1) = \#X$, so $X(q)$ *is a q-analogue* of $\#X$.
- $\#X^{\sigma_d} = \mathrm{Tr}_{\mathbb{C}\{X\}}(\sigma_d)$, so the CSP says that evaluations of $X(q)$ *encode the isomorphism type* of the C-action on X.
- $X(q)$ is uniquely determined *modulo $q^n - 1$*. If $\deg X(q) < n$, the kth coefficient of $X(q)$ is the number of elements of X whose *stabilizer has order dividing k*.

The cyclic sieving phenomenon

Theorem ([RSW04, Prop. 4.4])

The triple $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$ *exhibits the CSP.*

That is, maj is a *"universal" cyclic sieving statistic* on words W_n for the S_n-action in the following sense:

Corollary ([BER11, Prop. 3.1])

Let W *be a finite set of length n words closed under the* S_n*-action. Then, the triple*

$$(\mathrm{W}, C_n, \mathrm{W}^{\mathrm{maj}}(q))$$

exhibits the CSP.

Corollary

By "changing basis" from Schur functions and irreducible characters to homogeneous symmetric functions and induced trivial characters, Kraśkiewicz–Weyman's key formula (1) *holds:*

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^r) = \chi^\lambda(\sigma_n^r).$$

Schur–Weyl duality

To connect cyclic sieving to Thrall's problem, we require some standard $\mathrm{GL}(V)$-representation theory.

Definition

The *Schur character* of a $\mathrm{GL}(V)$-module E is

$$(\mathrm{ch}\, E)(x_1, \ldots, x_m) := \mathrm{Tr}_E(\mathrm{diag}(x_1, \ldots, x_m)),$$

where $m = \dim(V)$.

Definition

Let M be an S_n-module. The *Schur–Weyl dual* of M is the $\mathrm{GL}(V)$-module

$$E(M) := V^{\otimes n} \otimes_{\mathbb{C}S_n} M.$$

Theorem (*Schur–Weyl duality*)

For any S_n-module M,

$$\lim_{m \to \infty} \mathrm{ch}\, E(M) = \mathrm{ch}(M).$$

Thrall's problem and necklaces

Definition

- A *necklace* is a C_n-orbit $[w]$ of a word $w \in \mathrm{W}_n$, e.g.

$$[221221] = \{221221, 122122, 212212\}.$$

- [221] has trivial stabilizer so is *primitive*.
- [221221] is not primitive and has *frequency* 2 since it's made of two copies of a primitive word.

Thrall's problem and necklaces

Proposition (Klyachko [Kly74])

There is a weight space basis for $E(\exp(2\pi i/n)\uparrow_{C_n}^{S_n})$ *indexed by primitive necklaces of length* n *words.*

Theorem (Marshall Hall [Hal59, Lem. 11.2.1])

$\mathcal{L}_n$ *also has a weight space basis indexed by primitive necklaces.*

Corollary (Klyachko [Kly74])

The Schur–Weyl dual of $\exp(2\pi i/n)\uparrow_{C_n}^{S_n}$ *is* $\mathcal{L}_n$.

To apply cyclic sieving, we need generating functions over words, not primitive necklaces.

Thrall's problem and necklaces

Definition

Let

$\mathrm{NFD}_{n,r} := \{\underline{\text{n}}\text{ecklaces of length } n \text{ words with } \underline{\text{f}}\text{requency } \underline{\text{d}}\text{ividing } r\}.$

Hence $\mathrm{NFD}_{n,1}$ consists of primitive necklaces.

Proposition ([AS18a])

There is a weight space basis for $E(\exp(2\pi i r/n)\uparrow_{C_n}^{S_n})$ *indexed by* $\mathrm{NFD}_{n,r}$.

Corollary

We have

$$\sum_{r=1}^{n} q^r \operatorname{ch} \exp(2\pi i r/n)\uparrow_{C_n}^{S_n} = \sum_{r=1}^{n} q^r \, \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}).$$

However, as r varies, the $\mathrm{NFD}_{n,r}$ are *not disjoint*.

Flex

To fix this, we use the following.

Definition ([AS18b])

The statistic flex: $\mathrm{W}_n \to \mathbb{Z}_{\geq 0}$ is $\mathrm{flex}(w) := \mathrm{freq}(w) \cdot \mathrm{lex}(w)$ where $\mathrm{lex}(w)$ is the position at which w appears in the lexicographic order of its rotations, starting at 1.

Example

$\mathrm{flex}(221221) = 2 \cdot 3 = 6$ since 221221 is the concatenation of 2 copies of the primitive word 221 and 221221 is third in lexicographic order amongst its 3 cyclic rotations.

Lemma

We have

$$\sum_{r=1}^{n} q^r \, \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}) = \mathrm{W}_n^{\mathrm{cont;flex}}(\mathbf{x}; q).$$

Flex

Flex is a *"universal" cyclic sieving statistic* on words W_n for C_n-actions in the following sense:

Lemma ([AS18b])

Let W be a finite set of length n words closed under the C_n-action, where C_n acts by cyclic rotations. Then, the triple $(W, C_n, W^{\mathrm{flex}}(q))$ exhibits the CSP.

Corollary

We have

$$\mathrm{W}_n^{\mathrm{cont;flex}}(q) = \mathrm{W}_n^{\mathrm{cont;maj}_n}(q)$$

where $1 \leq \mathrm{maj}_n \leq n$ *is* maj *modulo* n.

Proving Kraśkiewicz–Weyman's theorem

We finally have the following *remarkably direct, largely bijective proof* of Kraśkiewicz–Weyman's result using cyclic sieving.

1. Using *Schur–Weyl duality* and *Hall's basis*, $\operatorname{ch}\mathcal{L}_n$ can be replaced by $\operatorname{ch}\exp(2\pi i/n)\uparrow_{C_n}^{S_n}$.
2. Using the *generalized Klyachko basis and* flex,
$$\sum_{r=1}^{n} q^r \operatorname{ch}\exp(2\pi ir/n)\uparrow_{C_n}^{S_n} = \mathrm{W}_n^{\mathrm{cont;flex}}(\mathbf{x};q).$$
3. Using *universal cyclic sieving* on words for S_n- or C_n-actions,
$$\mathrm{W}_n^{\mathrm{cont;flex}}(\mathbf{x};q) = \mathrm{W}_n^{\mathrm{cont;maj}_n}(\mathbf{x};q).$$
4. Using the *RSK algorithm* $w \mapsto (P,Q)$ where $\mathrm{Des}(w) = \mathrm{Des}(Q)$,
$$\mathrm{W}_n^{\mathrm{cont;maj}_n}(\mathbf{x};q) = \sum_{\substack{\lambda\vdash n\\ r\in[n]}} a_{\lambda,r}\, q^r s_\lambda(\mathbf{x}). \qquad \square$$

Kraśkiewicz–Weyman open problems

There are multiple published proofs of Kraśkiewicz–Weyman's theorem. However, none of them give a bijective explanation for the following symmetry:

Corollary

Let $\lambda \vdash n$*. Then* $\#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\}$ *depends only on* λ *and* $\gcd(n, r)$*.*

Open Problem

Find a *bijective proof* of the above symmetry.

Open Problem

Find a *content-preserving bijection* $\Phi \colon \mathrm{W}_n \to \mathrm{W}_n$ such that $\mathrm{maj}_n(w) = \mathrm{flex}(\Phi(w))$.

Such a bijection Φ would give a bijective proof of the identity

$$\sum_{\lambda \vdash n} a_{\lambda,r} s_\lambda(\mathbf{x}) = \sum_{\lambda \vdash n} a_{\lambda,\gcd(n,r)} s_\lambda(\mathbf{x}).$$

Kraśkiewicz–Weyman open problems

In [AS18b], we prove a *refinement* of the $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$ CSP involving the *cyclic descent type* of a word.

Question

Is there a refinement of Kraśkiewicz–Weyman's Schur expansion *involving cyclic descent types*?

The recent work of Adin, Elizalde, Huang, Reiner, Roichman on cyclic descent sets for standard tableaux may be relevant.

Cyclic group branching rules

Stembridge generalized Kraśkiewicz–Weyman's result to describe all branching rules for any $\langle\sigma\rangle \hookrightarrow S_n$ where σ is of cycle type ν and order ℓ:

Theorem (Stembridge [Ste89])

$$\sum_{r=1}^{\ell} q^r \operatorname{ch}(\exp(2\pi i r/\ell)\uparrow_{\langle\sigma\rangle}^{S_n}) = \sum_{\substack{\lambda\vdash n \\ T\in\mathrm{SYT}(\lambda)}} q^{\mathrm{maj}_\nu(T)} s_\lambda(\mathbf{x})$$

where maj_ν *is a generalization of* maj_n*.*

We give a cyclic sieving-based proof of Stembridge's result. The first step is a natural generalization of Klyachko's basis:

Proposition

$$\operatorname{ch} \exp(2\pi i r/\ell)\uparrow_{\langle\sigma\rangle}^{S_n} = \mathrm{OFD}_{n,r}^{\mathrm{cont}}(\mathbf{x})$$

where $\mathrm{OFD}_{n,r}$ *is the set of* $\langle\sigma\rangle$*-orbits with frequency (stabilizer order) dividing* r*.*

See the paper for more.

Higher Lie modules

Recall that $\mathcal{L}_{(a^b)} = \mathrm{Sym}^b \mathcal{L}_a$. Consequently,

$$\mathrm{ch}\,\mathcal{L}_{(a^b)} = h_b[\mathcal{L}_a].$$

Thus Thrall's problem is an instance of a *plethysm problem*. Such problems are notoriously difficult.

The preceding arguments and results strongly suggest the need to consider Thrall's problem in the larger context of *general branching rules*.

Higher Lie modules

One may show that $\mathcal{L}_{(a^b)}$ is the Schur–Weyl dual of a certain induced one-dimensional representation $\chi^{r,1}\uparrow_{C_a \wr S_b}^{S_{ab}}$ of the *wreath product* $C_a \wr S_b$. Here $C_a \wr S_b$ can be thought of as the subgroup of permutations on ab letters which permute the b size-a intervals in $[ab]$ amongst themselves and cyclically rotate each size-a interval independently.

Schocker [Sch03] gave a formula for the Schur expansion of $\operatorname{ch} \mathcal{L}_{(a^b)}$, though it involves many divisions and subtractions in general. We generalized Schocker's result to all induced *one-dimensional* representations of $C_a \wr S_b$ using cyclic sieving.

Higher Lie modules

Theorem (See [Sch03, Thm. 3.1])

For all $a, b \geq 1$ and $r = 1, \ldots, a$, we have

$$\operatorname{ch} \mathcal{L}^{r,1}_{(a^b)} = \sum_{\lambda \vdash ab} \left(\sum_{\nu \vdash b} \frac{1}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r*\nu) \mathbf{a}^{a*\nu}_{\lambda,\tau} \right) s_\lambda(\mathbf{x}) \qquad \text{and}$$

$$\operatorname{ch} \mathcal{L}^{r,\epsilon}_{(a^b)} = \sum_{\lambda \vdash ab} \left(\sum_{\nu \vdash b} \frac{(-1)^{b-\ell(\nu)}}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r*\nu) \mathbf{a}^{a*\nu}_{\lambda,\tau} \right) s_\lambda(\mathbf{x}),$$

where $\mathbf{maj}_{a*\nu}$ is a variation on maj*,*

$$\mathbf{a}^{a*\nu}_{\lambda,\tau} := \#\{Q \in \mathrm{SYT}(\lambda) : \mathbf{maj}_{a*\nu}(Q) = \tau\},$$

and $\mu_f(d, e)$ is a generalization of the classical Möbius function.

In our approach, the *subtractions and divisions* arise from the underlying combinatorics using *Möbius inversion and Burnside's lemma*, respectively.

A new approach

Our generalization of Schocker's formula involves considering only the one-dimensional representations of $C_a \wr S_b$, which may explain its failure to be cancellation-free.

The earlier statistics flex, maj_n, and maj_ν gave *monomial expansions* of the branching rules in question as generating functions on words. We have identified the monomial expansion for $C_a \wr S_b \hookrightarrow S_{ab}$ as a statistic generating function as follows.

Theorem

Fix integers $a, b \geq 1$. We have

$$\sum_{\underline{\lambda}} \dim S^{\underline{\lambda}} \cdot \mathrm{ch}\left(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}}\right) q^{\underline{\lambda}} = \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{flex}_a^b}(\mathbf{x}; q)$$

$$= \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{maj}_a^b}(\mathbf{x}; q)$$

where the sum is over all a-tuples $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(a)})$ of partitions with $\sum_{r=1}^{a} |\lambda^{(r)}| = b$ and the $q^{\underline{\lambda}}$ are independent indeterminates.

A new approach

The statistics flex_a^b and maj_a^b are somewhat involved. For flex_a^b:

1. Write $w \in \mathrm{W}_{ab}$ in the form $w = w^1 \cdots w^b$ where $w_j \in \mathrm{W}_a$.
2. Let $w^{(r)}$ denote the *subword* of w whose letters are those $w^j \in \mathrm{W}_a$ such that $\mathrm{flex}(w^j) = r$.
3. *Totally order* W_a lexicographically, so that RSK is well-defined for words with letters from W_a.
4. Set
$$\mathrm{flex}_a^b(w) := (\mathrm{sh}(w^{(1)}), \ldots, \mathrm{sh}(w^{(a)}))$$
where sh denotes the shape under RSK.

For maj_a^b, use maj_a instead of flex in step (2).

A new approach

Previously, we were able to simply use RSK to go from the *monomial to the Schur basis*, since maj_ν depends only on $Q(w)$. However, flex_a^b and maj_a^b do not have the corresponding property.

Open Problem

Fix $a, b \geq 1$. Find a statistic

$$\mathrm{mash}_a^b \colon \mathrm{W}_{ab} \to \{a\text{-tuples of partitions with total size } b\}$$

with the following properties.

(i) For all $\alpha \vDash ab$, maj_a^b (or equivalently flex_a^b) and mash_a^b are *equidistributed* on W_α.

(ii) If $v, w \in \mathrm{W}_{ab}$ satisfy $Q(v) = Q(w)$, then $\mathrm{mash}_a^b(v) = \mathrm{mash}_a^b(w)$.

A new approach

Finding such a statistic mash_a^b would *determine all branching rules* for $C_a \wr S_b \hookrightarrow S_{ab}$, in particularly *solving Thrall's problem*, as follows.

Corollary

Suppose mash_a^b *satisfies Properties (i) and (ii). Then*

$$\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}}) = \sum_{\nu \vdash ab} \frac{\#\{Q \in \mathrm{SYT}(\nu) : \mathrm{mash}_a^b(Q) = \underline{\lambda}\}}{\dim(S^{\underline{\lambda}})} s_\nu(\mathbf{x}),$$

where $\mathrm{mash}_a^b(Q) := \mathrm{mash}_a^b(w)$ *for any* $w \in \mathrm{W}_{ab}$ *with* $Q(w) = Q$.

A new approach

When $a = 1$ and $b = n$, $\mathrm{maj}_1^n(w)$ essentially reduces to $\mathrm{sh}(w)$, the *shape of w under RSK*. When $a = n$ and $b = 1$, $\mathrm{maj}_n^1(w)$ essentially reduces to $\mathrm{maj}_n(w)$. Both of these satisfy (i) and (ii). In this sense mash_a^b, *interpolates between the major index* maj_n *and the shape under RSK*, hence the name.

Question

Could a useful notion of *"group sieving"* for the wreath products $C_a \wr S_b$ be missing?

THANKS!

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