Steep-bounce zeta map in parabolic Cataland

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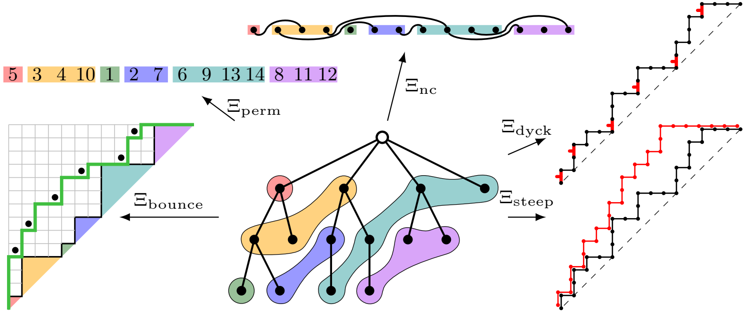
Summary



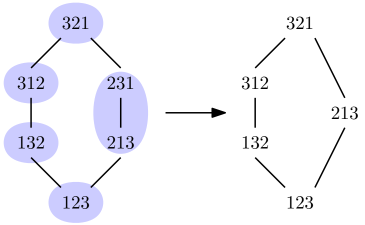
Parabolic Cataland

Catalan objects in action

$\mathfrak{S}_n$ as a Coxeter group generated by $s_i = (i, i+1)$

For $w \in \mathfrak{S}_n$, $\ell(w)$ = min. length of factorization of w into s_i's.

Weak order : w covered by w' iff $w' = ws_i$ and $\ell(w') = \ell(w) + 1$



Sylvester class: permutations with the same binary search tree

Representants: 231-avoiding permutations (A Catalan family!)

Restricted to 231-avoiding permutations = Tamari lattice.

Generalization to parabolic quotient of $\mathfrak{S}_n$

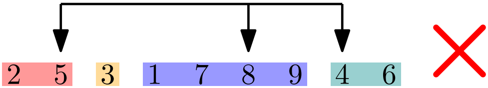
Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a composition of n.

Parabolic quotient : $\mathfrak{S}_n^\alpha = \mathfrak{S}_n / (\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$.

i	1	2	3	4	5	6	7	8	9
$\sigma(i)$	1	5	3	2	4	8	9	6	7

Increasing order in each block (here, $\alpha = (2, 1, 4, 2)$)

Also a notion of $(\alpha, 231)$-avoiding permutations



$\mathfrak{S}_n^\alpha(231)$: set of $(\alpha, 231)$-avoiding permutations

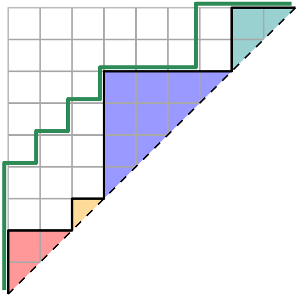
Weak order restricted to $\mathfrak{S}_n^\alpha(231)$ = Parabolic Tamari lattice (Mühle and Williams 2018+)

Parabolic Catalan objects

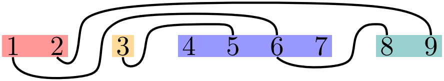
$(\alpha, 231)$-avoiding permutations

i	1	2	3	4	5	6	7	8	9
$\sigma(i)$	1	5	3	2	4	8	9	6	7

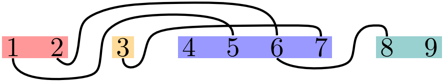
Bounce pairs



Parabolic non-crossing α-partition



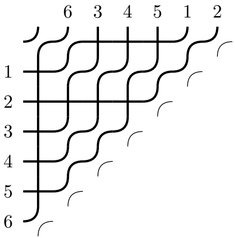
Parabolic non-nesting α-partition

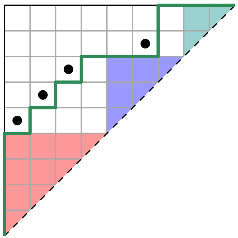


All in (somehow complicated) bijections! (Mühle and Williams, 2018+)

Detour to pipe dreams

Hopf algebra on pipe dreams (Bergeron, Ceballos et Pilaud, 2018+).





Proposition (Bergeron, Ceballos and Pilaud, 2018+)

Pipe dreams of size n whose permutation decomposes into identity permutations are in bijection with bounce pairs of order n.

Come to Cesar's talk on Wednesday!

Marked paths and steep pairs

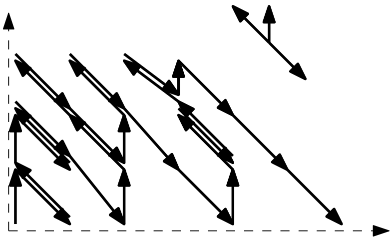
Observation by Bergeron, Ceballos and Pilaud and F. and Mühle:

Graded dimensions of a Hopf algebra on said pipe dreams:

$$1, 1, 3, 12, 57, 301, 1707, 10191, 63244, 404503, \ldots \text{(OEIS A151498)}$$

$=$ Walks in the quadrant: $\{(1,0),(1,-1),(-1,1)\}$, ending on x-axis

$=$ Number of parabolic Catalan objects of order n (summed over all α).

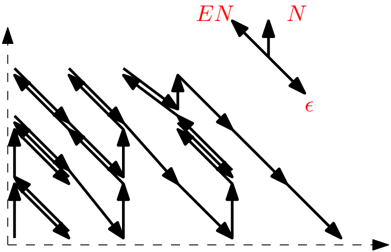


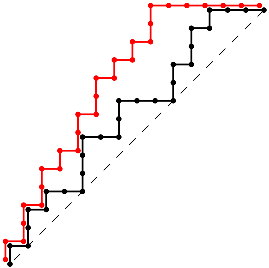
Considered in (Bousquet-Mélou and Mishna, 2010)

Counted in (Mishna and Rechnitzer, 2009)

Lattice paths and steep pairs

Steep pairs : 2 nested Dyck paths, the one above has no EE except at the end





Bijection:

- Path below: projection on y-axis
- Path above: $(0,1) \to N$, $(-1,1) \to EN$, $(1,-1) \to \epsilon$, padding of E

Steep-Bounce conjecture

Conjecture (Bergeron, Ceballos and Pilaud 2018+, Conjecture 2.2.8)

The following two sets are of the same size:

- *bounce pairs of order n with k blocks;*
- *steep pairs of order n with k east steps E on $y = n$.*

The cases $k = 1, 2, n-1, n$ already proved

Bijection?

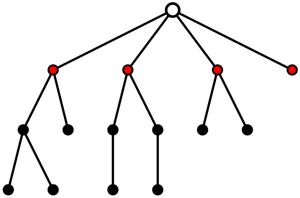
Left-aligned colored trees

- T : plane tree with n non-root nodes;
- $\alpha = (\alpha_1, \ldots, \alpha_k)$: composition of n

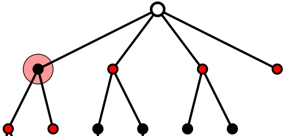
Active nodes : not yet colored, but parent is colored or is the root.

Coloring algorithm : For i from 1 to k,

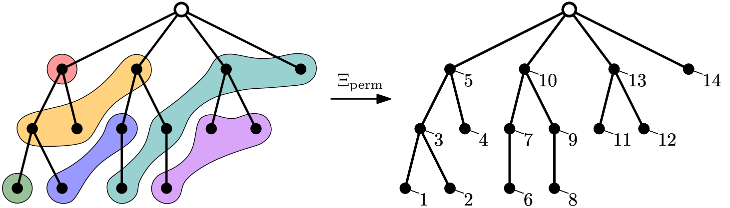
- If there are less than α_i active nodes, then fail;
- Otherwise, color the first α_i from left to right with color i.



$$\alpha = (1, 3, 1, 2, 4, 3) \vdash 14$$



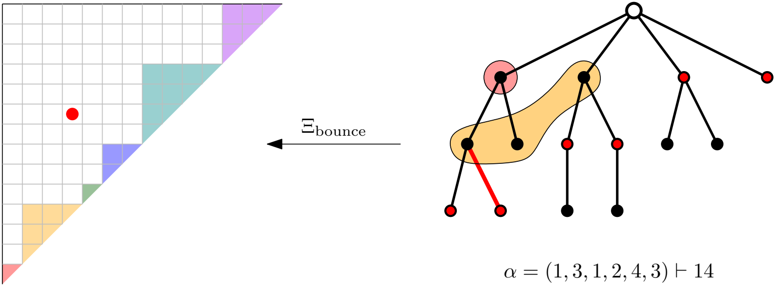
To permutations

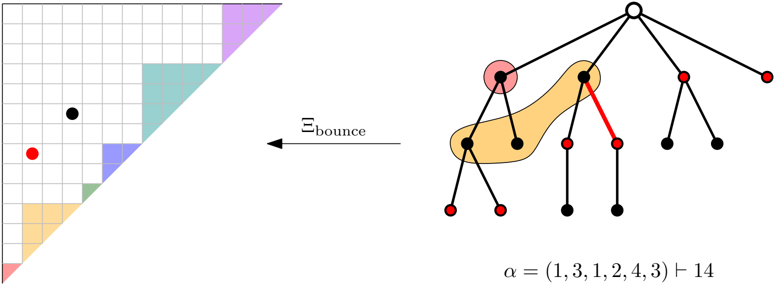


(T, α)

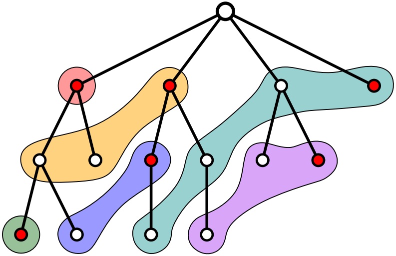
$\Xi_{\text{perm}}(T, \alpha) = 5 \mid 3\ 4\ 10 \mid 1 \mid 2\ 7 \mid 6\ 9\ 13\ 14 \mid 8\ 11\ 12 \in \mathfrak{S}_n^{\alpha}(231)$

To bounce pairs

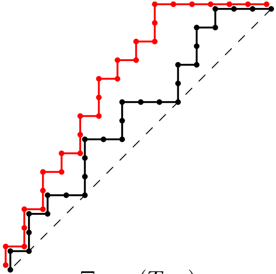




To steep pairs



(T, α)



$\Xi_{\text{steep}}(T, \alpha)$

- Lower path: depth-first search from right to left
- Upper path: red node $\to N$, white node $\to EN$

Steep-Bounce theorem

Theorem (Ceballos, F., Mühle 2018+)

There is a natural bijection Γ between the following two sets:

- *bounce pairs of order n with k blocks;*
- *steep pairs of order n with k each steps E on $y = n$.*

So we know how (hard it is) to count them.

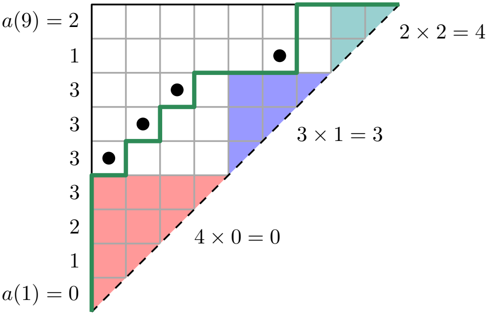
But there is more!

- Parabolic Tamari lattice: from Coxeter structure
- ν-Tamari lattice (Préville-Ratelle and Viennot 2014): from Dyck paths

Theorem (Ceballos, F., Mühle 2018+)

The parabolic Tamari lattice indexed by α is isomorphic to the ν-Tamari lattice with $\nu = N^{\alpha_1} E^{\alpha_1} \cdots N^{\alpha_k} E^{\alpha_k}$.

Detour to q, t-Catalan combinatorics



$\text{area}(D) = \sum_i a(i) = 18$

$\text{dinv}(D) = \#\{(i,j) \mid i < j, (a(i) = a(j) \vee a(i) = a(j) + 1\} = 13$

$\text{bounce}(D) = \sum_i (i-1)\alpha_i = 7$

Zeta map from diagonal harmonics

Theorem (Haglund and Haiman, see Haglund 2008)

By summing over all Dyck paths of order n, we have

$$\sum_D q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)} = \sum_D q^{\mathrm{dinv}(D)} t^{\mathrm{area}(D)}.$$

Each comes from a combinatorial description of the Hilbert series of the alternating component of the space of diagonal harmonics.

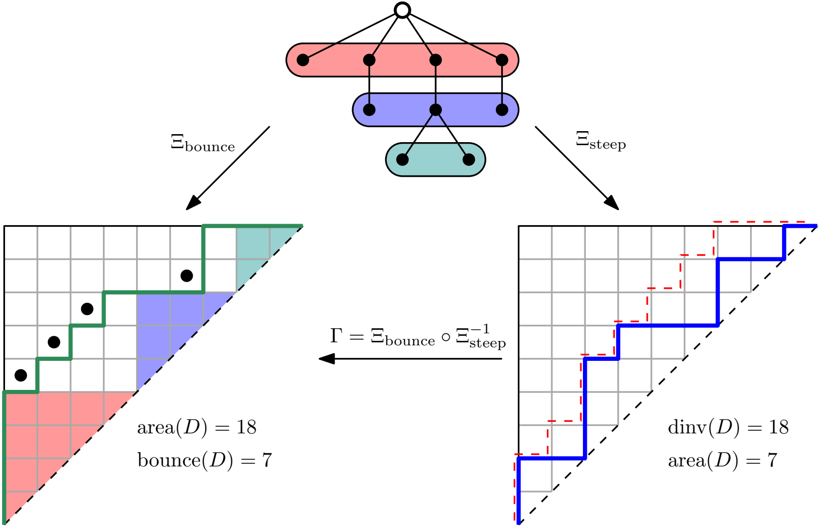
Theorem (Haglund 2008)

There is a bijection ζ on Dyck paths that transfers the pairs of statistics

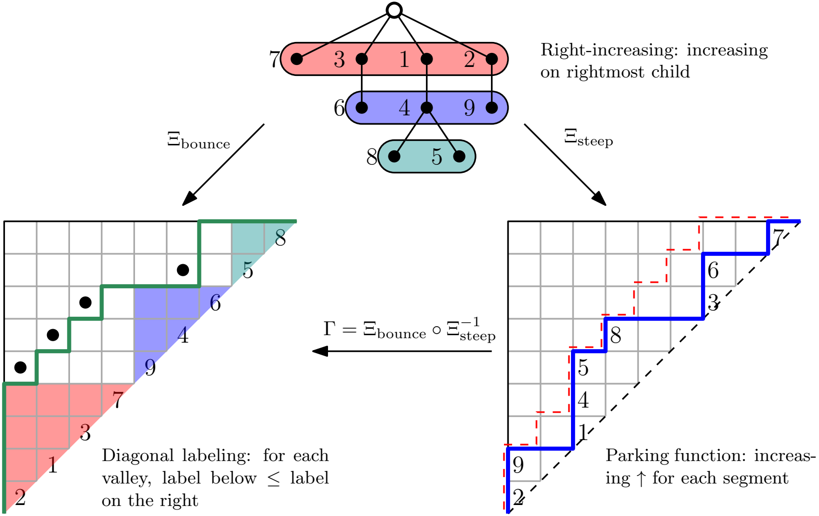
$$(\mathrm{dinv}, \mathrm{area}) \to (\mathrm{area}, \mathrm{bounce}).$$

Originally from (Andrews, Krattenthaler, Orsina and Papi, 2001) in the context of Borel subalgebras of $sl(n)$.

Our zeta map


Ξ_{bounce}
Ξ_{steep}
$\Gamma = \Xi_{\text{bounce}} \circ \Xi_{\text{steep}}^{-1}$
$\text{area}(D) = 18$
$\text{bounce}(D) = 7$
$\text{dinv}(D) = 18$
$\text{area}(D) = 7$

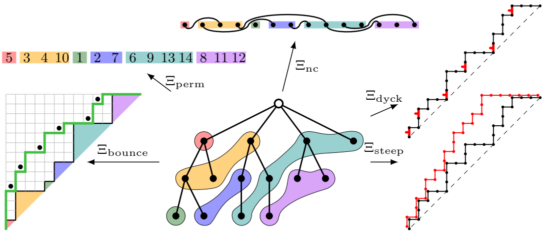
Our zeta map, labeled version



A generalization of the labeled zeta map (Haglund and Loehr, 2005).

Possible directions

- Many questions in enumeration (but possibly very difficult)
- Interesting special cases (See Henri's poster!)
- Other types?
- Implication in spaces of diagonal harmonics?
- *etc.*



Thank you for listening!