New edge asymptotics of skew Young diagrams via free boundaries

Dan Betea

University of Bonn

joint work with J. Bouttier, P. Nejjar and M. Vuletić

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Outline

This talk contains stuff on

- partitions and tableaux
- the Plancherel (mostly) and uniform measures on Young diagrams
- main results on skew Young diagrams
- the beyond

and a few surprises.

Partitions

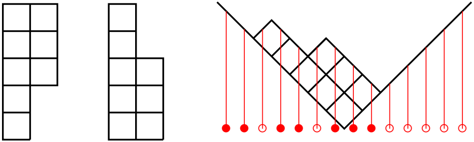


Figure: Partition (Young diagram) $\lambda = (2, 2, 2, 1, 1)$ (Frobenius coordinates $(1, 0|4, 1)$) in English, French and Russian notation, with associated Maya diagram (particle-hole representation). Size $|\lambda| = 8$, length $\ell(\lambda) = 5$.

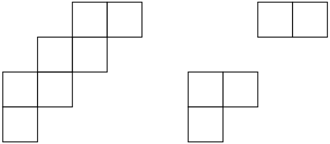


Figure: Skew partitions (Young diagrams) $(4, 3, 2, 1)/(2, 1)$ (but also $(5, 4, 3, 2, 1)/(5, 2, 1), \ldots$) and $(4, 4, 2, 1)/(2, 2)$ (but also $(6, 4, 4, 2, 1)/(6, 2, 2), \ldots$)

Counting tableaux

A standard (semi-standard) Young tableau SYT (SSYT) is a filling of a (possibly skew) Young diagram with numbers $1, 2, \dots$ strictly increasing down columns and rows (rows weakly increasing for semi-standard).

$$
\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&9\\\cline{1-3} 7\\\cline{1-1} 8\\\cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&3\\\cline{1-3} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}
\qquad
\begin{array}{cccc} & & \boxed{1} & \boxed{7}\\ & \boxed{3} & \boxed{4} & \\ \boxed{2} & \boxed{5} & & \\ \boxed{6} & & & \end{array}
\qquad
\begin{array}{cccc} & & \boxed{1} & \boxed{2}\\ & \boxed{1} & \boxed{3} & \\ \boxed{2} & \boxed{2} & & \\ \boxed{3} & & & \end{array}
$$

$$\dim \lambda := \text{number of SYTs of shape } \lambda,$$

$$\widetilde{\dim}\lambda := \text{number of SSYTs of shape } \lambda \text{ with entries from } 1 \dots n$$

and similarly for $\dim \lambda/\mu$, $\widetilde{\dim}\lambda/\mu$.

Two natural measures on partitions

- On partitions of n ($|\lambda| := \sum \lambda_i = n$): Plancherel vs. uniform

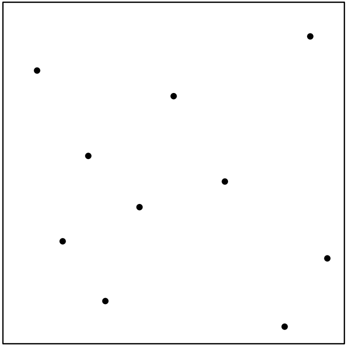
$$Prob(\lambda) = \frac{(\dim \lambda)^2}{n!} \qquad \text{vs.} \qquad Prob(\lambda) = \frac{1}{\#\{\text{partitions of } n\}}$$

- On all partitions: poissonized Plancherel vs. (grand canonical) uniform

$$Prob(\lambda) = e^{-\epsilon^2} \epsilon^{2|\lambda|} \frac{(\dim \lambda)^2}{(|\lambda|!)^2} \qquad \text{vs.} \qquad Prob(\lambda) = u^{|\lambda|} \prod_{i \geq 1} (1 - u^i)$$

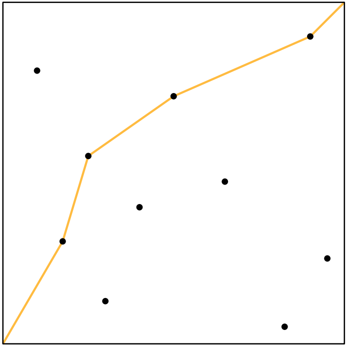
  with $\epsilon > 0, 1 > u > 0$ parameters.

Ulam's problem and Hammersley last passage percolation I



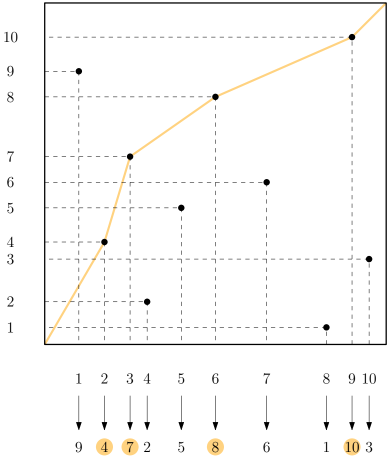
$PPP(\epsilon^2)$ in the unit square.

Ulam's problem and Hammersley last passage percolation II



Quantity of interest: L = longest up-right path from $(0,0)$ to $(1,1)$ $(= 4$ here).

Ulam's problem and Hammersley last passage percolation III



L is the length (any) of the longest increasing subsequence in a random permutation of S_N with $N \sim Poisson(\epsilon^2)$.

The poissonized Plancherel measure

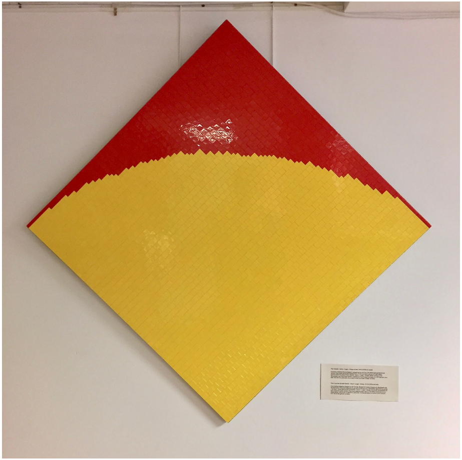
By the Robinson–Schensted–Knuth correspondence and Schensted's theorem, $L = \lambda_1$ in distribution where λ has the poissonized Plancherel measure:

$$\begin{aligned} Prob(\lambda) &= e^{-\epsilon^2} \epsilon^{2|\lambda|} \frac{(\dim \lambda)^2}{(|\lambda|!)^2} \\ &= e^{-\epsilon^2} s_\lambda(pl_\epsilon) s_\lambda(pl_\epsilon) \end{aligned}$$

(s is a Schur function, pl_ϵ the Plancherel specialization sending $p_1 \to \epsilon, p_i \to 0, i \geq 2$)

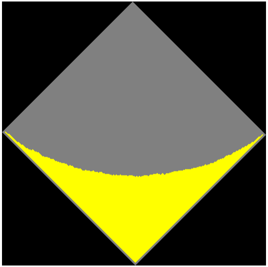
Interest: what happens to λ_1 as $\epsilon \to \infty$? (large PPP, large random permutation, ...)

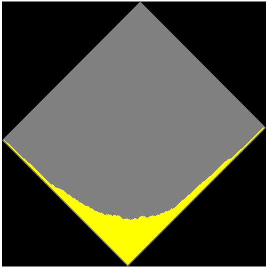
Limit shape



A Plancherel-random representation (partition!) of S_{2304} ($Prob(\lambda) = (\dim \lambda)^2/n!$, $n = 2304$), at IHP. The limit shape should be obvious (VerKer, LogShe 1977).

Limit shapes: Plancherel vs uniform





Random Plancherel (left) and uniform (right) partitions of $N = 10000$. The scale is different: $\sqrt{N}$ for Plancherel, $\sqrt{N} \log N$ for uniform.

The Baik–Deift–Johansson theorem and Tracy–Widom

Theorem (BaiDeiJoh 1999)

If λ is distributed as poissonized Plancherel, we have:

$$\lim_{\epsilon \to \infty} Prob\left(\frac{\lambda_1 - 2\epsilon}{\epsilon^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s) := \det(1 - Ai_2)_{L^2(s,\infty)}$$

with

$$Ai_2(x,y) := \int_0^\infty Ai(x+s)Ai(y+s)ds$$

and Ai the Airy function (solution of $y'' = xy$ decaying at ∞).

F_{GUE} is the Tracy–Widom GUE distribution. It is by (original) construction the extreme distribution of the largest eigenvalue of a random hermitian matrix with iid standard Gaussian entries as the size of the matrix goes to infinity.

The Erdős–Lehner theorem and Gumbel

Theorem (ErdLeh 1941)

For the uniform measure $Prob(\lambda) \propto u^{|\lambda|}$ we have:

$$\lim_{u \to 1^-} Prob\left(\lambda_1 < -\frac{\log(1-u)}{\log u} + \frac{\xi}{|\log u|}\right) = e^{-e^{-\xi}}.$$

The finite temperature Plancherel measure

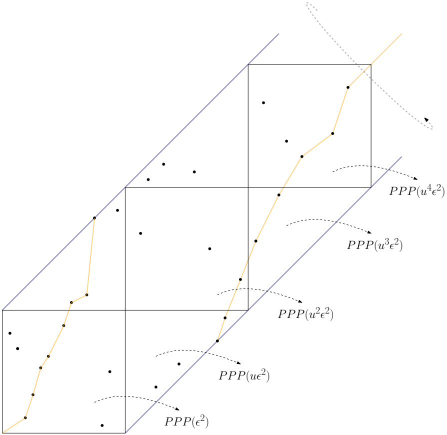
On pairs of partitions $\mu \subset \lambda \supset \mu$ consider the measure (Bor 06)

$$Prob(\mu, \lambda) \propto u^{|\mu|} \cdot \frac{\varepsilon^{|\lambda|-|\mu|} \dim^2(\lambda/\mu)}{(|\lambda/\mu|!)^2}$$

with $u = e^{-\beta}$, $\beta =$ inverse temperature.

- $u = 0$ yields the poissonized Plancherel measure
- $\varepsilon = 0$ yields the (grand canonical) uniform measure

What is in a part?



With L the longest up-right path in this cylindric geometry, in distribution, Schensted's theorem states that

$$\lambda_1 = L + \kappa_1$$

where κ is a uniform partition $Prob(\kappa) \propto u^{|\kappa|}$ independent of everything else.

The finite temperature Plancherel measure II

Theorem (B/Bouttier 2019)

Let $M = \frac{\sqrt{\varepsilon}}{1-u} \to \infty$ *and* $u = \exp(-\alpha M^{-1/3}) \to 1$. *Then*

$$\lim_{M\to\infty} Prob\left(\frac{\lambda_1 - 2M}{M^{1/3}} \leq s\right) = F^\alpha(s) := \det(1 - Ai^\alpha)_{L^2(s,\infty)}$$

with

$$Ai^\alpha(x,y) := \int_{-\infty}^{\infty} \frac{e^{\alpha s}}{1+e^{\alpha s}} \cdot Ai(x+s)Ai(y+s)ds$$

the finite temperature Airy kernel.

A word on the finite temperature Airy kernel

Ai^α is Johansson's (2007) *Airy kernel in finite temperature* (also appearing as the KPZ crossover kernel: SasSpo10 and AmiCorQua11, in random directed polymers BorCorFer11, cylindric OU processes LeDMajSch15):

$$Ai^\alpha(x,y) = \int_{-\infty}^{\infty} \frac{e^{\alpha s}}{1+e^{\alpha s}} Ai(x+s)Ai(y+s)ds$$

and interpolates between the Airy kernel and a diagonal exponential kernel:

$$\lim_{\alpha\to\infty} Ai^\alpha(x,y) = Ai_2(x,y),$$

$$\lim_{\alpha\to 0+} \frac{1}{\alpha} Ai^\alpha\left(\frac{x}{\alpha} - \frac{1}{2\alpha}\log(4\pi\alpha^3), \frac{y}{\alpha} - \frac{1}{2\alpha}\log(4\pi\alpha^3)\right) = e^{-x}\delta_{x,y}.$$

If $F^\alpha(s), F_{\mathrm{GUE}}(s)$, and $G(s)$ are the Fredholm determinants on (s,∞) of Ai^α, Ai_2 and $e^{-x}\delta_{x,y}$, then (Joh 2007)

$$\lim_{\alpha\to\infty} F^\alpha(s) = F_{\mathrm{GUE}}(s), \qquad \lim_{\alpha\to 0+} F^\alpha\left(\frac{s}{\alpha} - \frac{1}{2\alpha}\log(4\pi\alpha^3)\right) = G(s) = e^{-e^{-s}}.$$

It appeared in seemingly two different situations:

- random matrix models on the cylinder/in finite temperature (Joh, LeDMajSch, ...)
- the KPZ equation with wedge I.C. at finite time (SasSpo, AmiCorQua, ...)

Three limiting regimes for edge fluctuations

Theorem (B/Bouttier 2019)

With $u = e^{-r} \to 1$ *as* $r \to 0+$ *and* $\epsilon \to \infty$ *(or finite) we have:*

- $\epsilon r^2 \to 0+$ *leads to Gumbel behavior; thermal fluctuations win*
- $\epsilon r^2 \to \infty$ *leads to Tracy–Widom; quantum fluctuations win*
- $\epsilon r^2 \to \alpha \in (0, \infty)$ *leads to finite temperature Tracy–Widom* F^{α}*; equilibrium between thermal and quantum*

The stuff that's in the FPSAC abstract

Consider the following measures (oc = number of odd columns, n letters for $\widetilde{\dim}$):

$$\mathbb{M}^{\nearrow}(\mu,\lambda) \propto a_1^{oc(\mu)} a_2^{oc(\lambda)} \cdot u^{|\mu|} \cdot \frac{\epsilon^{|\lambda/\mu|} \dim(\lambda/\mu)}{|\lambda/\mu|!},$$

$$\mathbb{M}^{\nearrow\searrow}(\mu,\lambda,\nu) \propto a_1^{oc(\mu)} a_2^{oc(\lambda)} \cdot u^{|\mu|} v^{|\nu|} \cdot \frac{\epsilon^{|\lambda/\mu|+|\lambda/\nu|} \dim(\lambda/\mu) \dim(\lambda/\nu)}{|\lambda/\mu|! \cdot |\lambda/\nu|!},$$

$$\widetilde{\mathbb{M}}^{\nearrow}(\mu,\lambda) \propto a_1^{oc(\mu)} a_2^{oc(\lambda)} \cdot u^{|\mu|} \cdot q^{|\lambda/\mu|} \cdot \widetilde{\dim}(\lambda/\mu),$$

$$\widetilde{\mathbb{M}}^{\nearrow\searrow}(\mu,\lambda,\nu) \propto a_1^{oc(\mu)} a_2^{oc(\lambda)} \cdot u^{|\mu|} v^{|\nu|} \cdot q^{|\lambda/\mu|+|\lambda/\nu|} \cdot \widetilde{\dim}(\lambda/\mu)\widetilde{\dim}(\lambda/\nu).$$

They all interpolate between Plancherel-type ($u = 0$) and uniform ($\epsilon, q = 0$) measures.

What is in a part? ($\lambda_1 = L + \kappa_1$ via RSK)

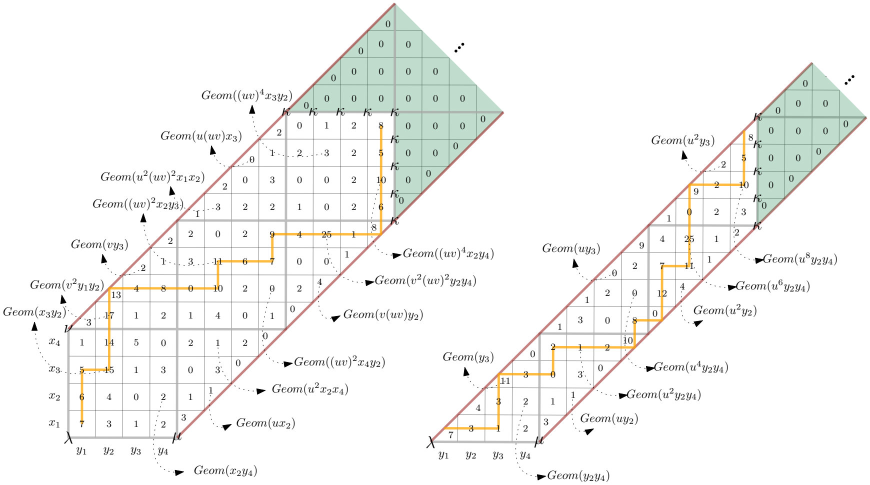


Figure: Longest up-right path in orange of length $L = 199$ (left) and $L = 130$ (right). $\widetilde{\mathbb{M}}^{\nearrow\searrow}(\mu, \lambda, \nu)$ (left) and $\widetilde{\mathbb{M}}^{\nearrow}(\mu, \lambda)$ (right); $x_i = y_i = q$; case $a_1 = a_2 = 0$ (for generic, multiply the parameters in the boundary triangles by a_1 and a_2 for the two different boundaries; κ is uniform with prob. $\propto (uv)^{|\kappa|}$ (left) and $\propto u^{|\kappa|}$ (right).

Main theorem: edge limits (SYT case)

Theorem (B/Bouttier/Nejjar/Vuletić FPSAC 2019)

Fix $\eta, \alpha_i, i = 1, 2$ *positive reals. Let* $M := \frac{\epsilon}{1-u^2} \to \infty$ *and set*

$$u = v = \exp(-\eta M^{-1/3}), \qquad a_i = u^{\alpha_i/\eta}, \quad i = 1, 2$$

all going to 1 as $M \to \infty$. *(In particular,* $\epsilon \sim M^{2/3} \to \infty$.*) We have:*

$$\lim_{M\to\infty} \mathbb{M}^{\nearrow} \left(\frac{\lambda_1 - 2M}{M^{1/3}} \leq s + \frac{1}{\eta} \log \frac{M^{1/3}}{\eta} \right) = F^{1;\alpha_1,\alpha_2;\eta}(s),$$

$$\lim_{M\to\infty} \mathbb{M}^{\nearrow\searrow} \left(\frac{\lambda_1 - 2M}{M^{1/3}} \leq s + \frac{1}{2\eta} \log \frac{M^{1/3}}{2\eta} \right) = F^{2;\alpha_1,\alpha_2;\eta}(s)$$

with the distributions $F^{\cdots}$ *explicit Fredholm pfaffians.*

Remark: This theorem generalizes celebrated results of Baik–Rains (2000) on longest increasing subsequences in symmetrized permutations, as well as the classical Baik–Deift–Johansson theorem.

Main theorem: edge limits (SSYT case on n letters)

Theorem (B/Bouttier/Nejjar/Vuletić FPSAC 2019)

Fix $\eta, \alpha_i, i = 1, 2$ positive reals. As $n \to \infty$ (n a positive integer), let

$$u = v = \exp(-\eta n^{-1/3}), \qquad a_i = u^{\alpha_i/\eta}, \quad i = 1, 2$$

all going to 1 and set $q = 1 - u^2 \to 0$. We have:

$$\lim_{n\to\infty} \widetilde{\mathbb{M}}^{\nearrow} \left(\frac{\lambda_1 - \chi n}{n^{1/3}} \leq s + \frac{1}{\eta} \log \frac{n^{1/3}}{\eta} \right) = F^{1;\alpha_1,\alpha_2;\eta}(s),$$

$$\lim_{n\to\infty} \widetilde{\mathbb{M}}^{\nearrow\searrow} \left(\frac{\lambda_1 - \chi n}{n^{1/3}} \leq s + \frac{1}{2\eta} \log \frac{n^{1/3}}{2\eta} \right) = F^{2;\alpha_1,\alpha_2;\eta}(s)$$

where $\chi = 2q \sum_{\ell \geq 0} \frac{u^{2\ell}}{1-u^{2\ell}q}$.

Limits to Tracy–Widom

Theorem (B/Bouttier/Nejjar/Vuletić FPSAC 2019)

We have:

$$\lim_{\eta\to\infty} F^{1;\alpha_1,\alpha_2;\eta}(s) = F^{\boxslash}(s;\alpha_2), \qquad \lim_{\eta\to\infty} F^{2;\alpha_1,\alpha_2;\eta}(s) = F_{\mathrm{GUE}}(s)$$

where F_{GUE} is the Tracy–Widom GUE distribution and $F^{\boxslash}(s;\alpha_2)$ is the Baik–Rains Tracy–Widom GOE/GSE crossover

$$F^{\boxslash}(s;0) = F_{\mathrm{GOE}}(s), \qquad F^{\boxslash}(s;\infty) = F_{\mathrm{GSE}}(s).$$

Remark: as $\eta \to 0$, the distributions should converge to Gumbel in the appropriate (so far unknown) scaling.

Defition of distribution functions

The distributions are Fredholm pfaffians $F^{k;\alpha_1,\alpha_2;\eta}(s) = \mathrm{pf}\left(J - A^{k;\alpha_1,\alpha_2;\eta}\right)_{L^2\left(s+\frac{\log 2}{k\cdot\eta},\infty\right)}$ for specific 2×2 matrix kernels A. For example:

$$A^{1;\alpha_1,\alpha_2;\eta}_{1,1}(x,y) = \int\int \Gamma\left(\frac{\zeta}{\eta},\frac{\omega}{\eta}\right)\gamma^{(1)}(\zeta)\gamma^{(1)}(\omega)\frac{\sin\frac{\pi(\zeta-\omega)}{2\eta}}{\sin\frac{\pi(\zeta+\omega)}{2\eta}}e^{\frac{\zeta^3}{3}-x\zeta+\frac{\omega^3}{3}-y\omega}\mathrm{d}_{\zeta\omega},$$

$$\begin{aligned}A^{1;\alpha_1,\alpha_2;\eta}_{1,2}(x,y) &= \int\int \Gamma\left(\frac{\zeta}{\eta},1-\frac{\omega}{\eta}\right)\frac{\gamma^{(1)}(\zeta)}{\gamma^{(1)}(\omega)}\frac{\sin\frac{\pi(\zeta+\omega)}{2\eta}}{\sin\frac{\pi(\zeta-\omega)}{2\eta}}e^{\frac{\zeta^3}{3}-x\zeta-\frac{\omega^3}{3}+y\omega}\frac{\mathrm{d}_{\zeta\omega}}{2\eta}\\ &= -A^{1;\alpha_1,\alpha_2;\eta}_{2,1}(y,x),\end{aligned}$$

$$\begin{aligned}A^{1;\alpha_1,\alpha_2;\eta}_{2,2}(x,y) &= \int\int \Gamma\left(1-\frac{\zeta}{\eta},1-\frac{\omega}{\eta}\right)\frac{1}{\gamma^{(1)}(\zeta)\gamma^{(1)}(\omega)}\frac{\sin\frac{\pi(\zeta-\omega)}{2\eta}}{\sin\frac{\pi(\zeta+\omega)}{2\eta}}e^{-\frac{\zeta^3}{3}+x\zeta-\frac{\omega^3}{3}+y\omega}\frac{\mathrm{d}_{\zeta\omega}}{4\eta^2}\\ &\quad -\mathrm{sgn}(x-y)\end{aligned}$$

where $\mathrm{d}_{\zeta\omega}=\frac{d\zeta d\omega}{(2\pi i)^2}$, $\gamma^{(1)}(\zeta) := \dfrac{\Gamma\left(\frac{1}{2}+\frac{\alpha_1-\zeta}{2\eta},1+\frac{\alpha_2-\zeta}{2\eta}\right)}{\Gamma\left(\frac{1}{2}+\frac{\alpha_1+\zeta}{2\eta},\frac{\alpha_2+\zeta}{2\eta}\right)}$, $\Gamma(a,b,c,\dots)=\Gamma(a)\Gamma(b)\Gamma(c)\cdots$ and where the contours are certain top-to-bottom vertical lines close enough to 0.

When $\alpha_1 = \alpha_2 = 0$ (no boundary parameters) things simplify

$$A^{1;\eta}_{1,1}(x,y) = \int\int \Gamma\left(1-\frac{\zeta}{\eta}, 1-\frac{\omega}{\eta}\right) \frac{\sin\frac{\pi(\zeta-\omega)}{2\eta}}{\sin\frac{\pi(\zeta+\omega)}{2\eta}} e^{\frac{\zeta^3}{3}-x\zeta+\frac{\omega^3}{3}-y\omega} \frac{\mathrm{d}\zeta\omega}{4},$$

$$A^{1;\eta}_{1,2}(x,y) = \int\int \Gamma\left(1-\frac{\zeta}{\eta}, \frac{\omega}{\eta}\right) \frac{\sin\frac{\pi(\zeta+\omega)}{2\eta}}{\sin\frac{\pi(\zeta-\omega)}{2\eta}} e^{\frac{\zeta^3}{3}-x\zeta-\frac{\omega^3}{3}+y\omega} \frac{\mathrm{d}\zeta\omega}{2\eta} = -A^{1;\eta}_{2,1}(y,x),$$

$$A^{1;\eta}_{2,2}(x,y) = \int\int \Gamma\left(\frac{\zeta}{\eta}, \frac{\omega}{\eta}\right) \frac{\sin\frac{\pi(\zeta-\omega)}{2\eta}}{\sin\frac{\pi(\zeta+\omega)}{2\eta}} e^{-\frac{\zeta^3}{3}+x\zeta-\frac{\omega^3}{3}+y\omega} \frac{\mathrm{d}\zeta\omega}{\eta^2}$$

$$+ \int \Gamma\left(\frac{\zeta}{\eta}\right) e^{-\frac{\zeta^3}{3}+x\zeta} \frac{\mathrm{d}\zeta}{\eta} - \int \Gamma\left(\frac{\omega}{\eta}\right) e^{-\frac{\omega^3}{3}+y\omega} \frac{\mathrm{d}\omega}{\eta} - \mathrm{sgn}(x-y);$$

$$A^{2;\eta}_{1,1}(x,y) = \int\int \Gamma\left(\frac{1}{2}-\frac{\zeta}{2\eta}, \frac{1}{2}-\frac{\omega}{2\eta}\right) \frac{\sin\frac{\pi(\zeta-\omega)}{4\eta}}{\cos\frac{\pi(\zeta+\omega)}{4\eta}} e^{\frac{\zeta^3}{3}-x\zeta+\frac{\omega^3}{3}-y\omega} \frac{\mathrm{d}\zeta\omega}{4\eta},$$

$$A^{2;\eta}_{1,2}(x,y) = \int\int \Gamma\left(\frac{1}{2}-\frac{\zeta}{2\eta}, \frac{1}{2}+\frac{\omega}{2\eta}\right) \frac{\cos\frac{\pi(\zeta+\omega)}{4\eta}}{\sin\frac{\pi(\zeta-\omega)}{4\eta}} e^{\frac{\zeta^3}{3}-x\zeta-\frac{\omega^3}{3}+y\omega} \frac{\mathrm{d}\zeta\omega}{4\eta} = -A^{2;\eta}_{2,1}(y,x),$$

$$A^{2;\eta}_{2,2}(x,y) = \int\int \Gamma\left(\frac{1}{2}+\frac{\zeta}{2\eta}, \frac{1}{2}+\frac{\omega}{2\eta}\right) \frac{\sin\frac{\pi(\zeta-\omega)}{4\eta}}{\cos\frac{\pi(\zeta+\omega)}{4\eta}} e^{-\frac{\zeta^3}{3}+x\zeta-\frac{\omega^3}{3}+y\omega} \frac{\mathrm{d}\zeta\omega}{4\eta}.$$

Proof

- pass to the grand canonical ensemble by introducing an independent (even) charge $2d$ from $Prob(d) \propto t^{2d}(uv)^{2d^2}$ shifting every part in every partition
- rewrite measures in terms of skew Schur functions, for example

$$\widetilde{\mathbb{M}}_{ext}^{\nearrow\searrow}(\mu,\lambda,\nu,d) \propto t^{2d}(uv)^{2d^2} \cdot a_1^{oc(\mu)} a_2^{oc(\lambda)} \cdot u^{|\mu|} v^{|\nu|} \cdot s_{\lambda/\mu}(q,\dots,q) s_{\lambda/\nu}(q,\dots,q)$$

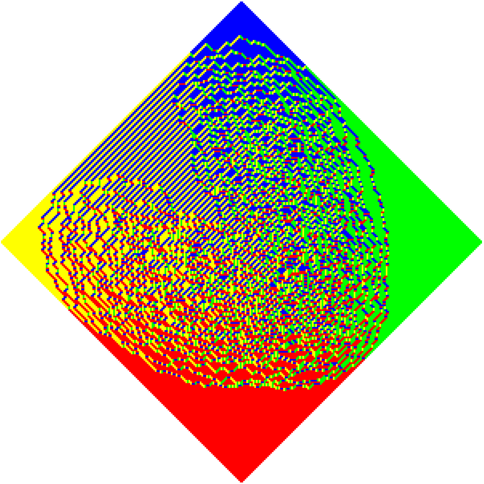
- rewrite in terms of lattice ($g\ell_\infty$ free) fermions and use new Wick lemma to obtain pfaffian correlations for the point process
- steepest descent analysis of correlation kernel
- remove charge at the end

Conclusion

Moral of the story: natural combinatorial measures on integer partitions lead to interesting asymptotic probabilistic behavior.

Future directions:

- Universality of the limiting distributions
- Connections to integrable hierarchies (i.e. the universal character hierarchy)
- Relation to (recent) work on asymptotics of dim λ/μ
- Connections to (asymptotic) representation theory (the Okounkov–Olshanski formula for dim λ/μ)



Thank you!