Semistable reduction in characteristic 0

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The KMW theorem

A *lattice polytope* is a polytope in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$.

A *unimodular triangulation* is a triangulation of a lattice polytope into lattice simplices all of whose volumes are $1/d!$. (Equivalently, the edge vectors of each simplex generate $\mathbb{Z}^d$ as a lattice.)

In general, lattice polytopes may not have unimodular triangulations when $d \geq 3$. However, we have the following celebrated result of Knudsen, Mumford, Waterman:

Theorem (KMW 1973)

For any lattice polytope P, there is a positive integer c such that cP has a unimodular triangulation.

Unimodular triangulations

Is there a constant c_d such that for all d-dimensional lattice polytopes P, $c_d P$ has a unimodular triangulation?

Given a lattice polytope P, is there a constant c_0 such that cP has a unimodular triangulation for all $c \geq c_0$?

Do parallelepipeds have unimodular triangulations?

Do smooth polytopes have unimodular triangulations?

What is semistable reduction? (KKMS)

Resolution of singularities is a classic problem in algebraic geometry where one tries to replace a variety X with a related variety X' that is non-singular.

- For toric varieties, this corresponds to subdividing cones of the corresponding fan into smooth cones.

Semistable reduction is a relative analogue of this problem, where one tries to replace a family of varieties $f : X \to B$ with a related family $f' : X' \to B'$ which is "as smooth as possible".

- The most well-known appearance of the problem is Kempf, Knudsen, Mumford, Saint-Donat (1973), where a strong version is proven for $\dim B = 1$ and characteristic 0.
- The core of the proof is the afformentioned KMW theorem on unimodular triangulations.

What is semistable reduction? (Abromovich-Karu)

A "best possible" version of semistable reduction in characteristic 0 for all $\dim(B)$ was proposed by Abromovich and Karu (2000).

They proved a weak version of their conjecture, and Karu (2000) proved the conjecture for $\dim(X) - \dim(B) \leq 3$.

They reduce the problem to a combinatorial problem that generalizes the KKMS result on unimodular triangulations. Here we restate and solve the combinatorial problem.

Maps of polytopes

Given two lattice polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$, a *map* between P and Q is a homomorphism $f : \mathbb{Z}^m \to \mathbb{Z}^n$, extended linearly to $f : \mathbb{R}^m \to \mathbb{R}^n$, such that $f(P) \subset Q$.

If $f : \mathbb{Z}^m \to \mathbb{Z}^n$ is surjective and $f(P) = Q$, then f is a *projection* of polytopes.

Theorem (Adiprasito-L-Temkin)

Given a projection of polytopes $f : P \to Q$, where Q is a unimodular simplex, there exists a positive integer c and regular unimodular triangulations X and Y of cP and cQ, respectively, such that f projects every simplex of X onto a simplex of Y.

The case where Q is a point is the KMW theorem.

Cayley polytopes

A *Cayley polytope* is a polytope P along with a projection $P \to \Delta$, where Δ is a simplex, such that every vertex of P maps to a vertex of Δ.

Alternatively, a Cayley polytope is a polytope isomorphic to

$$\operatorname{conv}(P_1 \times \{e_1\}, P_2 \times \{e_2\}, \ldots, P_n \times \{e_n\})$$

where $P_1, \ldots, P_n \subset \mathbb{R}^d$ are polytopes and $\{e_1, \ldots, e_n\}$ are the vertices of an $(n-1)$-simplex.

We write the above polytope as $\mathcal{C}(P_1, \ldots, P_n)$, and call this the *Cayley sum* of $P_1, \ldots, P_n$.

Polysimplices

A *polysimplex* is a polytope of the form $\sum \sigma_i$, where $\{\sigma_i\}$ is a set of affinely independent simplices and the sum is Minkowski sum.

In this talk we will deal with Cayley polytopes of the form $\mathcal{C}(\Sigma_1, \ldots, \Sigma_m)$, where the Σ_i are polysimplices.

Remark: A polysimplex can also be rewritten as a Cayley polytope of this form.

Main lemma

Lemma

Let $\{\sigma_j\}_{j=1}^n$ be a set of affinely independent simplices, and let A be an $m \times n$ matrix of nonnegative integers. Then

$$\mathcal{C}\left(\sum_{j=1}^n A_{1j}\sigma_j, \sum_{j=1}^n A_{2j}\sigma_j, \ldots, \sum_{j=1}^n A_{mj}\sigma_j\right)$$

has a triangulation where each simplex has the same normalized volume as $\sigma := \mathcal{C}(\sigma_1, \ldots, \sigma_n)$. Moreover, suppose σ is not unimodular, $A_{ij} = 0$ or $A_{ij} \geq \dim \sigma_j$ for all i, j, and

$$\operatorname{support} A_1 \supseteq \operatorname{support} A_2 \supseteq \cdots \supseteq \operatorname{support} A_m,$$

where A_i denotes the i-th row of A. Then there is a triangulation where each simplex has normalized volume less than that of σ.

Lemma $\implies$ Theorem

Theorem (Adiprasito-L-Temkin)

Given a projection of polytopes $f : P \to Q$, where Q is a unimodular simplex, there exists a positive integer c and regular unimodular triangulations X and Y of cP and cQ, respectively, such that f projects every simplex of X onto a simplex of Y.

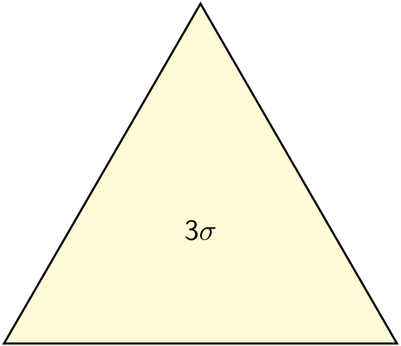
Proof.

By triangulating P, we can assume P is a simplex. Let $\{e_1, \ldots, e_n\}$ be the vertices of Q, and $\sigma_i = f^{-1}(e_i)$. Then $P = \mathcal{C}(\sigma_1, \ldots, \sigma_n)$.

For $c \geq \dim Q$, construct a unimodular triangulation of cQ so that for every simplex τ of the triangulation, the vertices of τ can be ordered $v_1, \ldots, v_n$ so that if v_i is contained in a face of cQ, then $v_{i+1}, \ldots, v_n$ are also contained in that face.

Then $f^{-1}(\tau)$ is a Cayley polytope satisfying the conditions of the Lemma, so we can triangulate it with simplices of volume less than P. Repeat with the simplices of this triangulation. $\square$

Proof of Lemma (Part 1)



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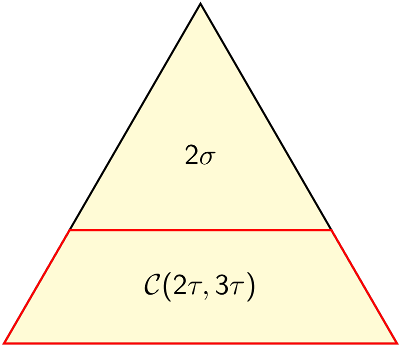

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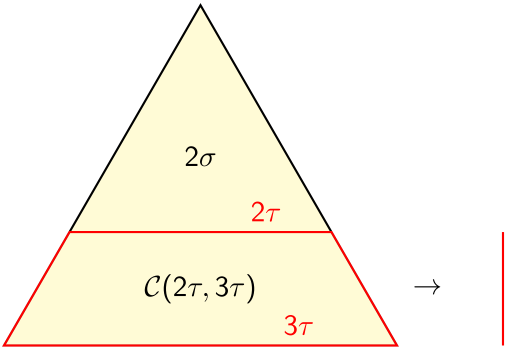

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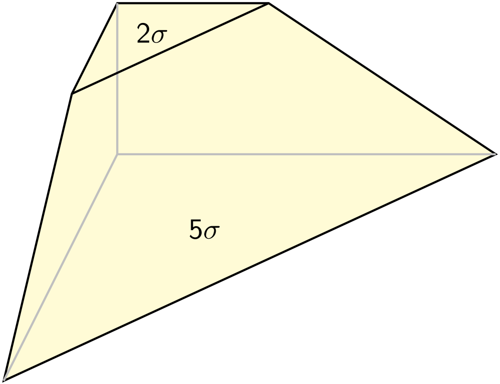
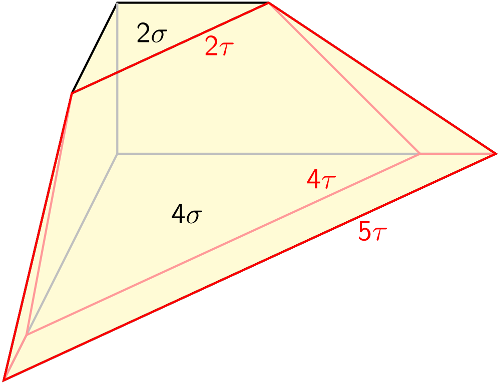


$\mathcal{C}(2\sigma, 5\sigma)$

Proof of Lemma (Part 1)



$\mathcal{C}(2\sigma, 4\sigma)$
$\mathcal{C}(2\tau, 4\tau, 5\tau)$

Main lemma

Lemma

Let $\{\sigma_j\}_{j=1}^n$ be a set of affinely independent simplices, and let A be an $m \times n$ matrix of nonnegative integers. Then

$$\mathcal{C}\left(\sum_{j=1}^{n} A_{1j}\sigma_j, \sum_{j=1}^{n} A_{2j}\sigma_j, \ldots, \sum_{j=1}^{n} A_{mj}\sigma_j\right)$$

has a triangulation where each simplex has the same normalized volume as $\sigma := \mathcal{C}(\sigma_1, \ldots, \sigma_n)$. Moreover, suppose σ is not unimodular, $A_{ij} = 0$ or $A_{ij} \geq \dim \sigma_j$ for all i, j, and

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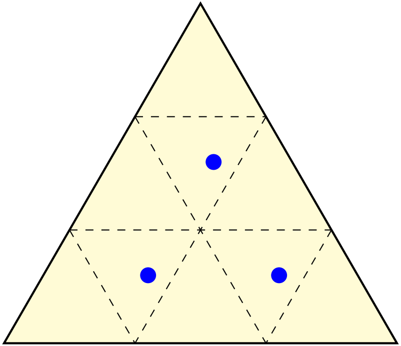
where A_i denotes the i-th row of A. Then there is a triangulation where each simplex has normalized volume less than that of σ.

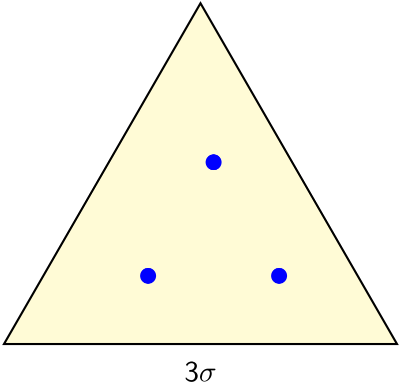
Proof of Lemma (Part 2)

Given a full-dimensional lattice polysimplex $P \subset \mathbb{Z}^d$, let L_P denote the lattice generated by its edges. A nonzero element of $\mathbb{Z}^d / L_P$ is called a *Waterman point* or *box point* of P.

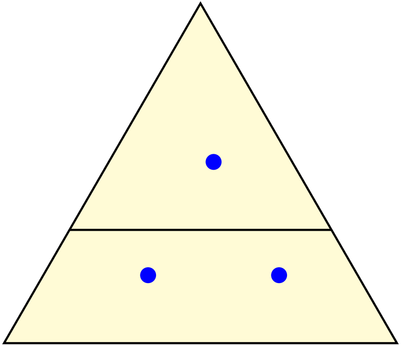


Representatives of a single box point of σ in contained in 3σ.

Proof of Lemma (Part 2)



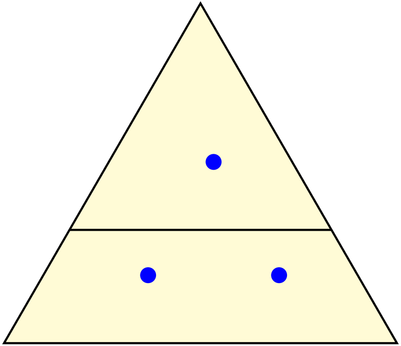
Proof of Lemma (Part 2)



2σ

$\mathcal{C}(2\tau, 3\tau)$

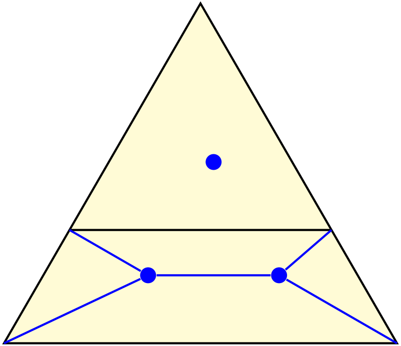
Proof of Lemma (Part 2)



2σ

$\mathcal{C}(2\tau, 3\tau)$

Proof of Lemma (Part 2)



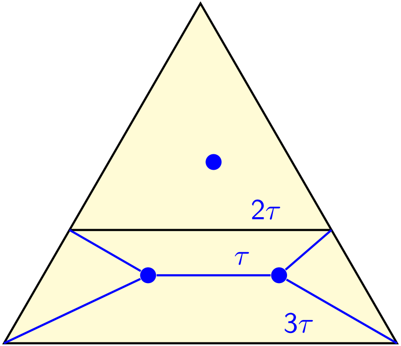
2σ

$\mathcal{C}(2\tau, 3\tau)$

$\mathcal{C}(\tau, 2\tau)$ $\mathcal{C}(\tau, 3\tau)$

$\mathcal{C}(\rho_1, 2\rho_1, 3\rho_1)$ $\mathcal{C}(\rho_2, 2\rho_2, 3\rho_2)$

Proof of Lemma (Part 2)



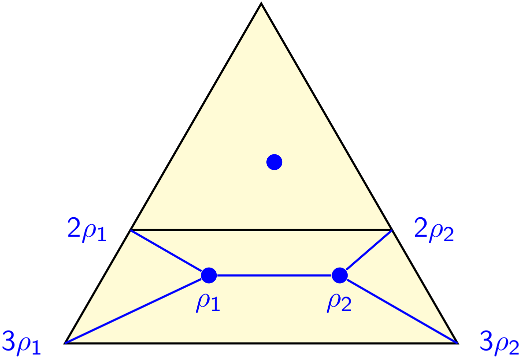
2σ

$\mathcal{C}(2\tau, 3\tau)$

$\mathcal{C}(\tau, 2\tau) \quad \mathcal{C}(\tau, 3\tau)$

$\mathcal{C}(\rho_1, 2\rho_1, 3\rho_1) \quad \mathcal{C}(\rho_2, 2\rho_2, 3\rho_2)$

Proof of Lemma (Part 2)



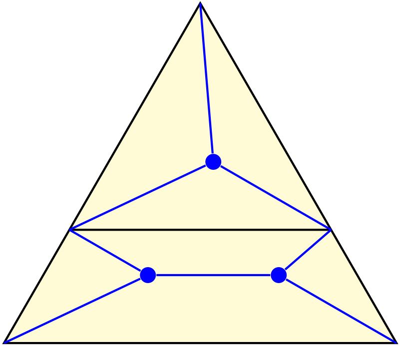
$$2\sigma$$

$$\mathcal{C}(2\tau, 3\tau)$$

$$\mathcal{C}(\tau, 2\tau) \quad \mathcal{C}(\tau, 3\tau)$$

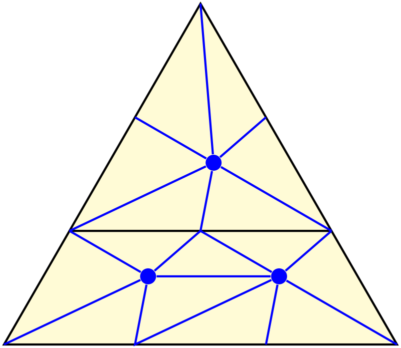
$$\mathcal{C}(\rho_1, 2\rho_1, 3\rho_1) \quad \mathcal{C}(\rho_2, 2\rho_2, 3\rho_2)$$

Proof of Lemma (Part 2)



$\mathcal{C}(\mathrm{pt}, \mathrm{face})$

Proof of Lemma (Part 2)



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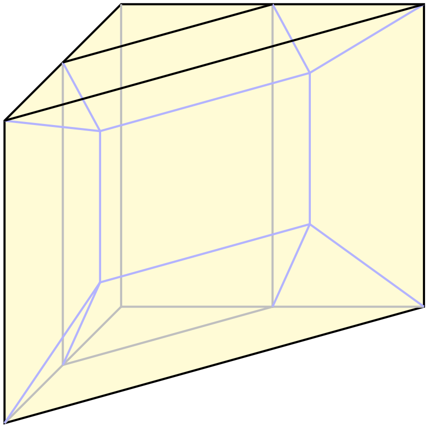

A note on functoriality

To guarantee that subdivisions of smaller pieces glue together properly, we want to prove that our construction is *functorial*. In other words, our construction should be a rule that assigns to each polytope $P = \mathcal{C}(\Sigma_1, \ldots, \Sigma_m)$ a triangulation $T(P)$ of P, so that if F is a face of P, then the restriction of $T(P)$ to F is $T(F)$.

We need to assume that all polytopes have an ordering on their vertices, and be consistent with this ordering throughout.

For the proof of Part 2 of the lemma, we also need to prove certain subdivision steps are confluent with each other—we use the diamond lemma to prove this.

Thank you!