Spanning line configurations

Brendan Pawlowski
(joint with Brendon Rhoades)

University of Southern California

July 3, 2019

Coinvariant algebras

Type A_{n-1} coinvariant algebra:

$$R_n := \mathbb{Z}[\mathbf{x}]/(e_1, \ldots, e_n),$$

e_d = sum of degree d squarefree monomials in $\mathbf{x} := \{x_1, \ldots, x_n\}$.

- R_n is a free $\mathbb{Z}$-module of rank $n!$
- As S_n-modules, $R_n \otimes \mathbb{Q} \simeq \mathbb{Q}[S_n]$.
- $R_n \simeq H^*(\mathrm{Fl}(n))$, the integral cohomology ring of the complete flag variety $\mathrm{Fl}(n)$.
- Basis of *Schubert polynomials* coming from $H^*(\mathrm{Fl}(n), \mathbb{Z})$ picture.

Generalized coinvariant algebras

Haglund, Rhoades, and Shimozono: for $1 \leq k \leq n$, define

$$R_{n,k} := \mathbb{Z}[\mathbf{x}]/(x_1^k, \ldots, x_n^k, e_{n-k+1}, \ldots, e_n).$$

Example

- $R_{n,n} = \mathbb{Z}[\mathbf{x}]/(e_1, \ldots, e_n, x_1^n, \ldots, x_n^n) = R_n$ free of rank $n!$.
- $R_{n,1} = \mathbb{Z}[\mathbf{x}]/(e_n, x_1^1, \ldots, x_n^1) = \mathbb{Z}$ free of rank 1.

Let $\mathcal{OSP}_{n,k} := \{$partitions of $\{1, \ldots, n\}$ into k *ordered* blocks$\}$.

- $R_{n,k}$ is free of rank $\#\mathcal{OSP}_{n,k}$.
- As S_n-modules, $R_{n,k} \otimes \mathbb{Q} \simeq \mathbb{Q}[\mathcal{OSP}_{n,k}]$.

Schubert basis?

- Haglund, Rhoades, and Shimozono give several bases of $R_{n,k}$, but not a Schubert-like basis.
- Is $R_{n,k} \simeq H^*(X)$ for some nice X?
- $R_{n,k}$ isn't usually rank-symmetric, so such an X can't be a compact smooth manifold!

Why $H^*(\mathrm{Fl}(n)) \simeq R_n$?

Recall $\mathrm{Fl}(n)$ is the space of chains of linear subspaces

$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n.$$

Define $X_n = \{(\ell_1, \ldots, \ell_n) \in (\mathbb{P}^{n-1})^n : \ell_1, \ldots, \ell_n \text{ span } \mathbb{C}^n\}$, e.g.

$$\text{columns of } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in X_2 \text{ but not } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

- Have a map $X_n \to \mathrm{Fl}(n)$:

$$(\ell_1, \ldots, \ell_n) \mapsto \ell_1 \subseteq \ell_1 \oplus \ell_2 \subseteq \cdots$$

- This is a homotopy equivalence! $\implies H^*(Fl(n)) \simeq H^*(X_n)$.
- Con: X_n isn't a compact manifold / projective variety.
- Pro: S_n acts on X_n by permuting lines, which induces the S_n-action on R_n; no such S_n-action on $\mathrm{Fl}(n)$ is evident.

Why $H^*(X_n) \simeq R_n$?

- A rank m vector bundle E over a space X assigns an m-dimensional (complex) vector space to each point of X.
- Example: a *trivial* vector bundle assigns the same vector space to each point.
- Example: the rank 1 *tautological bundle* L_i assigns to the point $(\ell_1, \ldots, \ell_n) \in X_n$ the vector space ℓ_i.

Why $H^*(X_n) \simeq R_n$?

Given E a vector bundle over X:

- Have *Chern classes* $c_d(E) \in H^{2d}(X)$ and the *total Chern class* $c(E) = \sum_{d \geq 0} c_d(E) = 1 + c_1(E) + \cdots + c_{\mathrm{rank}(E)}(E) \in H^*(X)$.
- If E is trivial then $c(E) = 1$.
- Whitney sum formula: $c(E \oplus F) = c(E)c(F)$ and $c(E/F) = c(E)/c(F)$.

Recall the (rank 1) tautological bundles $L_1, \ldots, L_n$ over X_n.

- Set $x_i = c_1(L_i)$, so $c(L_i) = 1 + x_i$.
- $L_1 \oplus \cdots \oplus L_n = \mathbb{C}^n$ is trivial on X_n, so

$$1 = c(L_1 \oplus \cdots \oplus L_n) = \prod_i c(L_i) = \prod_i (1 + x_i) = \sum_{d \geq 0} e_d(x).$$

- Gives an S_n-equivariant map $R_n = \mathbb{Z}[\mathbf{x}]/(e_1, \ldots, e_n) \to H^*(X_n)$; in fact an isomorphism.

The moduli space of spanning line configurations

For $1 \leq k \leq n$, define

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) \in (\mathbb{P}^{k-1})^n : \sum_i \ell_i = \mathbb{C}^k\}.$$

Example:

columns of $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \in X_{4,3}$; not columns of $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Theorem (Pawlowski and Rhoades)

$H^*(X_{n,k}) \simeq R_{n,k}$ as rings with S_n-action.

Example

- $X_{n,n} = X_n$
- $X_{n,1} = (\mathbb{P}^0)^n = \{\mathrm{pt}\}$

Why $H^*(X_{n,k}) \simeq R_{n,k}$?

Chern class arguments give an S_n-equivariant map $R_{n,k} = \mathbb{Z}[\mathbf{x}]/(e_{n-k+1}, \ldots, e_n, x_1^k, \ldots, x_n^k) \to H^*(X_{n,k})$; turns out to be an isomorphism.

- ~~Set~~ $\cancel{S = L_1 \oplus \cdots \oplus L_n}$, ~~so~~ $\cancel{c(S) = \prod_i c(L_i) = \sum_d e_d(x)}$.
- ~~Have a short exact sequence~~ $\cancel{0 \to M \to S \to L_1 + \cdots + L_n = \mathbb{C}^k \to 0}$.
- ~~Whitney formula:~~ $\cancel{1 = c(\mathbb{C}^k) = c(S/M) = c(S)/c(M)}$.
- ~~So~~ $\cancel{c(S) = \sum_d e_d(x) = c(M)}$, ~~which vanishes above degree~~ $\cancel{\mathrm{rank}(M) = n - k}$.
- ~~Get an~~ $\cancel{S_n}$~~-equivariant map~~ $\cancel{R_{n,k} = \mathbb{Z}[\mathbf{x}]/(e_{n-k+1}, \ldots, e_n, x_1^k, \ldots, x_n^k) \to H^*(X_{n,k})}$; ~~turns out to be an isomorphism.~~

The Schubert decomposition of X_n (or $\mathrm{Fl}(n)$)

The (i, j) entry of the *rank table* of a matrix A is the rank of the upper-left $i \times j$ corner of A:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has rank table } \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 1 & 2 & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

rank table of $\ell_\bullet \in (\mathbb{P}^{k-1})^n :=$ rank table of matrix with columns $\ell_\bullet$.

- For a permutation matrix $w \in S_n$, the set of $\ell_\bullet \in X_n$ with the same rank table as w is a *Schubert cell* C_w.
- Example: $\ell_\bullet \in C_{213}$ iff the rank table of $\ell_\bullet$ is the one shown above.
- Fact: $X_n = \bigsqcup_{w \in S_n} C_w$.

The Schubert decomposition of X_n (or $\mathrm{Fl}(n)$)

Affine paving of a variety Z: sequence of closed subvarieties $Z = Z_0 \supseteq \cdots \supseteq Z_m = \emptyset$ with $Z_i \setminus Z_{i+1}$ isomorphic to a disjoint union of affine spaces, the *cells* of the paving.

- The Schubert cells are the cells of an affine paving of X_n.
- The closed *Schubert variety* $\overline{C}_w$ determines a cohomology class $[\overline{C}_w] \in H^*(X_n)$.
- Affine paving $\implies H^*(X_n)$ is free on the $n!$ classes $[\overline{C}_w]$.
- Under the iso. $R_n \simeq H^*(X_n)$, the *Schubert polynomial* $\mathfrak{S}_w$ of Lascoux and Schützenberger maps to $[\overline{C}_w]$.

An affine paving of $X_{n,k}$

Each $\ell_\bullet \in X_{n,k}$ is in a "Schubert cell" C_w labeled by a length n word w on $[k] := \{1, \ldots, k\}$ determined as follows:

- Say the lex minimal linearly independent subtuple of $\ell_\bullet$ occurs in positions J; this subtuple is in some Schubert cell $C_v \subseteq X_k$:

$$k = 2, n = 3 : \ell_\bullet = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \in C_{21} \subseteq X_2, \quad J = \{1, 3\}$$

- Fill the positions of w in J with the letters of v: $v = 21 \rightsquigarrow w = 2?1$.
- If $j \notin J$, then $\ell_j \subseteq \ell_1 + \cdots + \ell_i$ for some $i < j$
- Set $w_j = w_i$ for the minimal such i: $w = 221$.

Let C_w be the set of $\ell_\bullet \subseteq X_{n,k}$ with word w, e.g. $\ell_\bullet \in C_{221}$ above.

An affine paving of $X_{n,k}$

For which w is C_w nonempty?

- A word $w = w_1 \cdots w_n$ is a *Fubini word (packed word)* if $\{w_1, \ldots, w_n\} = [k]$ for some k.
- Example: 31323 is Fubini but not 3133.
- Length n Fubini words on $[k] \longleftrightarrow$ ordered set partitions of $[n]$ into k blocks, e.g. $31323 \longleftrightarrow 2|4|135$.

Theorem (Pawlowski and Rhoades)

The sets C_w as w runs over the length n Fubini words on $[k]$ are the cells of an affine paving of $X_{n,k}$.

- Corollary: $H^*(X_{n,k})$ is free of rank $\#\mathcal{OSP}(n,k)$.
- The classes $[\overline{C}_w]$ are represented by certain *permuted* Schubert polynomials $\mathfrak{S}_v(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

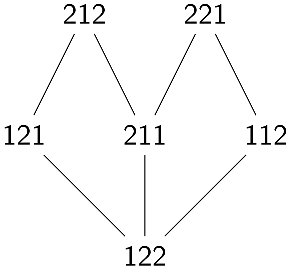
Further directions

- Rhoades: fix a composition $(d_1, \ldots, d_n)$, consider the space of tuples $(V_1, \ldots, V_n)$ with $\dim V_i = d_i$ and $V_1 + \cdots + V_n = \mathbb{C}^k$.
- Rhoades and Wilson: require linear independence of some of the lines in $\ell_\bullet \in X_{n,k} \rightsquigarrow$ r-Stirling numbers
- Pawlowski, Ramos, and Rhoades (in progress): representation stability for the S_n-modules $H^*(X_{n,k}) \simeq R_{n,k}$.

Questions

Define the *Bruhat order* on Fubini words by

$$v \leq w \iff C_w \supseteq \overline{C}_v.$$



- When $n = k$ this is the usual strong Bruhat order on S_n
- How to describe covering relations in general?
- Our affine paving of $X_{n,k}$ is *not* a CW decomposition: $C_w \cap \overline{C}_v \neq \emptyset$ need not imply $C_w \subseteq \overline{C}_v$.

Questions

- $\Lambda :=$ ring of symmetric functions over $\mathbb{Z}$
- The *Delta conjecture* of Haglund, Remmel, and Wilson predicts a combinatorial formula for $\Delta'_{e_{k-1}} e_n \in \Lambda \otimes \mathbb{Q}(q,t)$.
- Haglund, Rhoades, Shimozono: The graded Frobenius characteristic $\mathrm{grFrob}(R_{n,k} \otimes \mathbb{Q})$ is $\Delta'_{e_{k-1}} e_n \Big|_{t=0}$ (up to a twist).

Does the $H^*(X_{n,k})$ picture help here?

- HRS give an explicit expansion $\mathrm{grFrob}(R_{n,k} \otimes \mathbb{Q}) = \sum g_\lambda(q) Q'_\lambda$ in terms of dual Hall-Littlewood functions Q'_λ.
- Q'_λ is also (essentially) the graded Frobenius characteristic of H^*(Springer fiber indexed by λ).
- Can $X_{n,k}$ be decomposed using Springer fibers in a way that explains the expansion $\mathrm{grFrob}(R_{n,k} \otimes \mathbb{Q}) = \sum g_\lambda(q) Q'_\lambda$?
- Is there an "$X_{n,k}$ version" of Springer fibers?