Combinatorics of generalized exponents

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Representations of semisimple Lie algebras

Consider a complex semisimple Lie algebra $\mathfrak{g}$.

- $R = R^+ \sqcup R^-$ root system,
- P weight lattice,
- P^+ dominant weights,
- ω_i fundamental weights ($i \in I$),
- W Weyl group.

Type A_{n-1}:

- $\mathfrak{g} = \mathfrak{sl}_n$,
- weights are compositions,
- dominant weights are partitions (Young diagrams),
- $\omega_i = (1^i)$,
- $W = S_n$.

Representations of semisimple Lie algebras (cont.)

For a dominant weight $\lambda \in P^+$, let $V(\lambda)$ be the irreducible representation with highest weight λ, and $P(\lambda)$ its weights.

In classical types, a basis of $V(\lambda)$ is indexed by Kashiwara-Nakashima tableaux and King tableaux of shape λ.

Type A_{n-1}: semistandard Young tableaux (SSYT).

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \lambda = (4,2,1)\,,\ \mathrm{weight}(T) = (1,3,1,2)\,.$$

Lusztig's t-analogue of weight multiplicity

For $\mu \in P(\lambda)$, let $K_{\lambda,\mu}$ be the multiplicity of μ in $V(\lambda)$.
(In type A, this is the number of SSYT of shape λ, weight μ.)

Lusztig defined the t-analogue $K_{\lambda,\mu}(t)$, i.e., $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$, via

$$\frac{\sum_{w\in W} \operatorname{sgn}(w)\, x^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R^+}(1 - tx^{-\alpha})} = \sum_{\mu\in P(\lambda)} K_{\lambda,\mu}(t)\, x^{\mu}\,.$$

Importance of $K_{\lambda,\mu}(t)$

$K_{\lambda,\mu}(t)$, for λ, μ dominant, is also known as a Kostka-Foulkes polynomial.

This polynomial has remarkable properties:

- it is a special affine Kazhdan-Lusztig polynomial, so $K_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$;
- it records the Brylinski-Kostant filtration of the μ-weight space $V(\lambda)_\mu$;
- it is related to Hall-Littlewood polynomials (i.e., specializations of Macdonald polynomials at $q = 0$):

$$s_\lambda(x) = \sum_{\mu \in P^+} K_{\lambda,\mu}(t)\, P_\mu(x;t)\,,$$

where $s_\lambda(x)$ are the Weyl characters (Schur polynomials in type A).

Combinatorial formulas

In type A_{n-1}, $K_{\lambda,\mu}(t)$ is expressed combinatorially via the Lascoux-Schützenberger charge statistic on SSYT.

Finding combinatorial formulas beyond type A has been a long-standing problem.

Goal. The first such formula, for $K_{\lambda,0}(t)$ in type C_n ($\mathfrak{g} = \mathfrak{sp}_{2n}$). We also have: related formulas, applications, as well as the possibility to extend to all $K_{\lambda,\mu}(t)$ and types B, D.

Remark. The special case $\mu = 0$ is, in fact, the most complex one. Kostant called $K_{\lambda,0}(t)$ generalized exponents, as the classical ones are obtained when λ is the highest root.

Approach. Extend another combinatorial formula in type A, due to Lascoux-Leclerc-Thibon (LLT), which is based on Kashiwara's crystal graphs; our approach is simpler compared to LLT.

Kashiwara's crystal graphs

Encode irreducible representations $V(\lambda)$ of the corresponding quantum group $U_q(\mathfrak{g})$ as $q \to 0$.

Kashiwara (crystal) operators are modified versions of the Chevalley generators: e_i, f_i, $i \in I$.

Fact. $V(\lambda)$ has a crystal basis $B(\lambda)$: in the limit $q \to 0$ we have

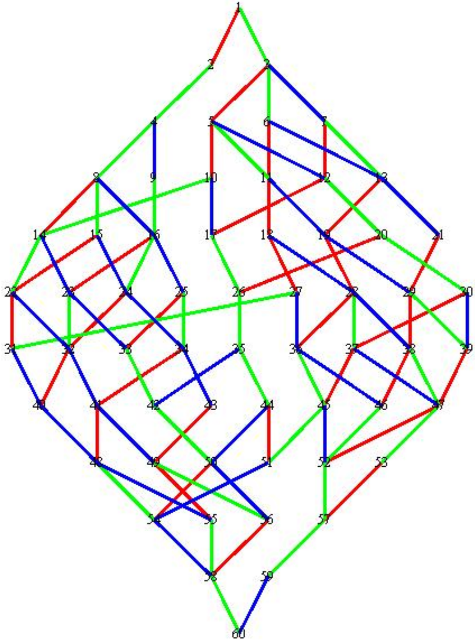
$$f_i, e_i \; : \; B(\lambda) \to B(\lambda) \sqcup \{\mathbf{0}\}\,,$$
$$f_i(b) = b' \quad \Longleftrightarrow \quad e_i(b') = b\,.$$

Encode as colored directed graph:

$$f_i(b) = b' \quad \Longleftrightarrow \quad b \xrightarrow{i} b'\,.$$

Fact. Classical crystals are realized as graphs on Kashiwara-Nakashima tableaux.

Example. $\mathfrak{g} = \mathfrak{sl}_4$, $\lambda = (3, 3, 1)$, blue: $\alpha_1 = \varepsilon_1 - \varepsilon_2$, green: $\alpha_2 = \varepsilon_2 - \varepsilon_3$, red: $\alpha_3 = \varepsilon_3 - \varepsilon_4$.



The LLT formula

Notation.

$$\varepsilon_i(b) = \max\{k \,:\, e_i^k(b) \neq \mathbf{0}\}\,, \quad \varphi_i(b) = \max\{k \,:\, f_i^k(b) \neq \mathbf{0}\}\,,$$

$$\boldsymbol{\varepsilon}(b) := \sum_{i \in I} \varepsilon_i(b)\omega_i\,, \quad |\boldsymbol{\varepsilon}(b)| = \sum_{i \in I} i\varepsilon_i(b)\,, \quad \boldsymbol{\varphi}(b)\,,\ |\boldsymbol{\varphi}(b)|\,.$$

Theorem. [Lascoux, Leclerc, Thibon] In type A_{n-1}, we have

$$K_{\lambda,0}(t) = \sum_{b \in B(\lambda)_0} t^{|\boldsymbol{\varepsilon}(b)|}\,.$$

There is a more involved formula for the other $K_{\lambda,\mu}(t)$.

Our approach to $K_{\lambda,0}(t)$ in classical types

Notation.

- $\mathcal{P}$ and $\mathcal{P}_n$ denote all partitions and partitions with at most n parts;
- $\mathcal{P}^{(2)}$ denotes partitions with all parts/rows even;
- $\mathcal{P}^{(1,1)}$ denotes partitions with all columns of even height;
- $c_\nu^\lambda(\mathfrak{sp}_{2n})$ is the branching coefficient for the restriction from $\mathfrak{gl}_{2n}$ to $\mathfrak{sp}_{2n}$, corresponding to the weights $\nu \in \mathcal{P}_{2n}$ and $\lambda \in \mathcal{P}_n$, respectively.

By classical results (Kostant, Hesselink, Littlewood), we derive in type C_n (and similarly in the other classical types):

$$\frac{K_{\lambda,0}^{C_n}(t)}{\prod_{i=1}^n (1-t^{2i})} = \sum_{\nu \in \mathcal{P}_{2n}^{(2)}} t^{|\nu|/2}\, c_\nu^\lambda(\mathfrak{sp}_{2n})\,.$$

Other ingredients

- the stable branching rule

$$c^{\lambda}_{\nu}(\mathfrak{sp}_{\infty}) = \sum_{\delta \in \mathcal{P}^{(1,1)}} c^{\nu}_{\lambda,\delta}\,,$$

  where $c^{\nu}_{\lambda,\delta}$ are the (type A) Littlewood-Richardson coefficients, giving the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\delta)$;

- the combinatorial formula for $c^{\nu}_{\lambda,\delta}$ in terms of the crystal:

$$c^{\nu}_{\lambda,\delta} = |LR^{\nu}_{\lambda,\delta}|\,,$$

  where

$$LR^{\nu}_{\lambda,\delta} = \{b \in B(\lambda) \,:\, \varepsilon(b) \leq \delta\,,\;\; \varphi(b) = \varepsilon(b) + \nu - \delta\}\,.$$

Immediate consequences

- new short proof of the LLT formula in type A;
- stable versions $K_{\lambda,0}^{X_\infty}(t)$ of $K_{\lambda,0}^{X_n}(t)$ when the rank n goes to ∞, for $X \in \{A, B, C, D\}$.

Remark. We have

$$K_{\lambda,0}^{B_\infty}(t) = K_{\lambda,0}^{D_\infty}(t)\,, \quad K_{\lambda,0}^{B_\infty}(t) = K_{\lambda',0}^{C_\infty}(t)\,.$$

Ingredients for finite rank: type C_n

- a nonstable stable branching rule expressing $c_\nu^\lambda(\mathfrak{sp}_{2n})$ outside the stable range $\nu \in \mathcal{P}_n$, namely when $\nu \in \mathcal{P}_{2n} \setminus \mathcal{P}_n$; based on recent work of J.-H. Kwon on his spin model for symplectic crystals;
- one of many versions of the combinatorial map expressing the symmetry of LR coefficients:

$$c^{\nu}_{\lambda,\delta} = c^{\nu'}_{\lambda',\delta'}.$$

The nonstable branching rule

Fix $\lambda \in \mathcal{P}_n$. Recall that when $\nu \in \mathcal{P}_n$ (stable case), we have

$$c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta \in \mathcal{P}_{2n}^{(1,1)}} c_{\lambda,\delta}^\nu ,$$

where $c_{\lambda,\delta}^\nu = |LR_{\lambda,\delta}^\nu| = |LR_{\lambda',\delta'}^{\nu'}|$.
But this fails for general $\nu \in \mathcal{P}_{2n}$.

Theorem. [Lecouvey, L.; based on Kwon] For $\nu \in \mathcal{P}_{2n}$, we have

$$c_\nu^\lambda(\mathfrak{sp}_{2n}) = \sum_{\delta \in \mathcal{P}_{2n}^{(1,1)}} \overline{c}_{\lambda,\delta}^\nu ,$$

where

$$\overline{c}_{\lambda,\delta}^\nu = |\{T \in LR_{\lambda',\delta'}^{\nu'} \,:\, r_i > \delta_{2i-1}^{\mathrm{rev}} = \delta_{2i}^{\mathrm{rev}}\}| ,$$

and $(r_1 \leq \ldots \leq r_p)$ is the first row of T.

The formula for $K^{C_n}_{\lambda,0}(t)$

Notation. $D_{2n}(\lambda)$ denotes the subset of distinguished vertices in $B_{2n}(\lambda)$ of type A_{2n-1}, that is, vertices b with

- $\varphi_i(b) = 0$ for any odd i,
- $\varepsilon_i(b)$ even for any odd i;
- flag condition: the entries in row i are $\geq 2i - 1$.

Main theorem. [Lecouvey, L.] We have

$$K^{C_n}_{\lambda,0}(t) = \sum_{b \in D_{2n}(\lambda)} t^{\left|\boldsymbol{\varepsilon}^*(b)+\mu_{b,n}\right|/2}.$$

where

$$\left|\boldsymbol{\varepsilon}^*(b) + \mu_{b,n}\right|/2 = \sum_{i=1}^{2n-1} (2n - i) \left\lceil \frac{\varepsilon_i(b)}{2} \right\rceil .$$

Another version of the formula

Goal. Express $K_{\lambda,0}^{C_n}(t)$ in terms a combinatorial set naturally indexing a basis of the 0-weight space $V(\lambda)_0$.

Definition. King tableaux are SSYT of a given shape λ in the alphabet $\{1 < \overline{1} < 2 < \overline{2} < \ldots < n < \overline{n}\}$ satisfying: the entries in row i are $\geq i$.

Fact. There is an easy bijection between $D_{2n}(\lambda)$ and King tableaux.

Applications of our formula for $K^{C_n}_{\lambda,0}(t)$

- $K^{C_{n+1}}_{\lambda,0}(t) - K^{C_n}_{\lambda,0}(t) \in \mathbb{Z}_{\geq 0}[t]$;
- $K^{C_n}_{\omega_{2p},0}(t) = K^{A_{n-1}}_{\gamma_p,0}(t^2)$, where $\gamma_p = (2^p, 1^{n-2p})$ (conjectured by Lecouvey);
- calculation of the smallest power in $K^{C_n}_{\lambda,0}(t)$.

Next goal

Extend our work from $K^{C_n}_{\lambda,0}(t)$ to all $K^{C_n}_{\lambda,\mu}(t)$.

Main idea. Extend the statistic on vertices of weight 0 to the whole crystal via an atomic decomposition of the crystal; see our poster.