Combinatorics of cluster structures in Schubert varieties

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The set-up

Fix integers $0 < k < n$.

- $Gr_{k,n} := \{V \subseteq \mathbb{C}^n : \dim(V) = k\}$
- $V \in Gr_{k,n} \rightsquigarrow$ full rank $k \times n$ matrix A whose rows span V

$$\operatorname{span}(e_1 + 2e_2 + e_5, e_3 + 7e_4) \in Gr_{2,5} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 7 & 0 \end{bmatrix}$$

- $I \subseteq \{1, \ldots, n\}$ with $|I| = k$. The Plücker coordinate $P_I(V)$ is the maximal minor of A located in column set I.
- The *Schubert cell*
  $\Omega_I := \{V \in Gr_{k,n} : P_I(V) \neq 0,\ P_J(V) = 0 \text{ for } J < I\}$
  The *open Schubert variety* $X_I^\circ := \Omega_I \setminus \{V \in \Omega_I : P_I P_{I_2} \cdots P_{I_n} = 0\}$

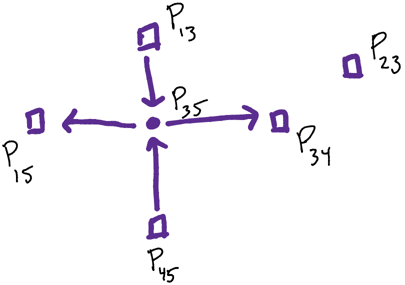
Running example: $X^\circ_{\{1,3\}} \subseteq Gr_{2,5}$

Cluster algebras, briefly

Introduced in (Fomin-Zelevinsky, '02)

A **seed** Σ: a quiver (directed graph with no loops or 2-cycles) with m vertices labeled by alg. indep. elements of a field of rational functions in m variables.

- **mutable** vertices (labeled by **cluster variables** $x_1, \ldots, x_r$) and **frozen** vertices (labeled by **frozen variables** $x_{r+1}, \ldots, x_m$)



Mutate at any mutable vertex (changing the label of that vertex and the arrows in its neighborhood) to obtain another seed.

$\mathcal{A}(\Sigma) = \mathbb{C}[x_{r+1}^{\pm 1}, \ldots, x_m^{\pm 1}][X]$, where X is the set of all cluster variables obtainable from Σ by a sequence of mutations.

Motivation

Theorem (Scott '06)

$\mathbb{C}[\widehat{Gr_{k,n}}]$ *is a cluster algebra with seeds (consisting entirely of Plücker coordinates) given by Postnikov's* **plabic graphs** *for* $Gr_{k,n}$ [a]*.*

[a] $\widehat{Gr_{k,n}}$ is the affine cone over $Gr_{k,n}$ wrt Plücker embedding.

- (Oh-Postnikov-Speyer '15): plabic graphs give *all* seeds in this cluster algebra that consist entirely of Plücker coordinates.

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Conjecture (Muller–Speyer '16)

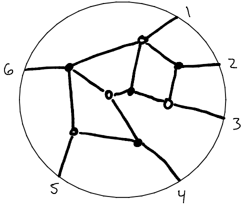
Scott's result holds if you replace $Gr_{k,n}$ with an open positroid variety.

Main result

Theorem (SSW '19)

$\mathbb{C}[\widehat{X_I^\circ}]$ *is a cluster algebra, with seeds (consisting entirely of Plücker coordinates) given by plabic graphs for* X_I°.[a]

[a] $\widehat{X_I^\circ}$ is the affine cone over X_I° wrt Plücker embedding.

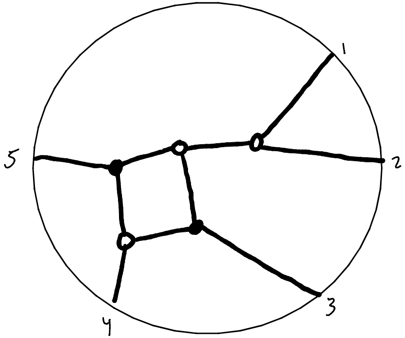


- We use a result of (Leclerc '16), who shows that coordinate rings of many Richardson varieties in the flag variety are cluster algebras.
- More general result for open "skew Schubert" varieties, where seeds for the cluster structure are given by **generalized** plabic graphs.

Postnikov's plabic graphs

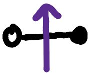
A (reduced) plabic graph of type (k, n) is a planar graph embedded in a disk with

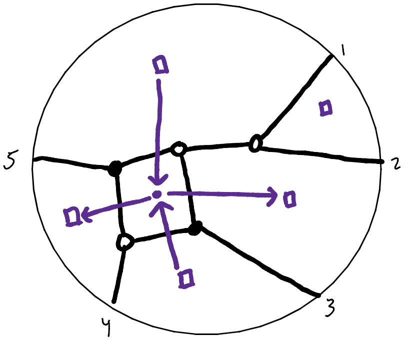
- n boundary vertices labeled $1, \dots, n$ clockwise.
- Internal vertices colored white and black.
- Boundary vertices are adjacent to a unique internal vertex (+ more technical conditions).



Let G be a reduced plabic graph of type (k, n). To get the dual quiver $Q(G)$

1. Put a frozen vertex in each boundary face of G and a mutable vertex in each internal face.
2. Add arrows across properly colored edges so you "see white vertex on the left."

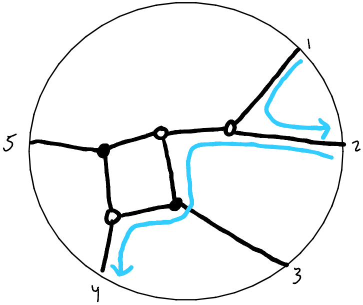




Variables from plabic graphs

A **trip** in G is a walk from boundary vertex to boundary vertex that

- turns maximally left at white vertices
- turns maximally right at black vertices

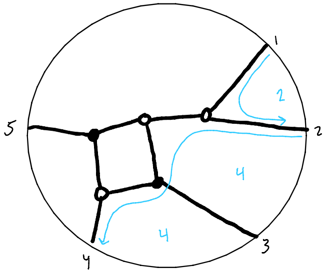


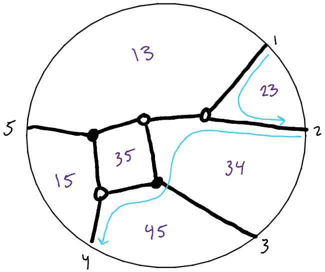
Aside: The trip permutation of this graph is

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 5 & 1 & 3 \end{array}$$

Face labels

If the trip T ends at j, put a j in all faces of G to the left of T. Do this for all trips.





Fact:(Postnikov '06) All faces of G will be labeled by subsets of the same size (which is k).

To get cluster variables, we interpret each face label as a Plücker coordinate.

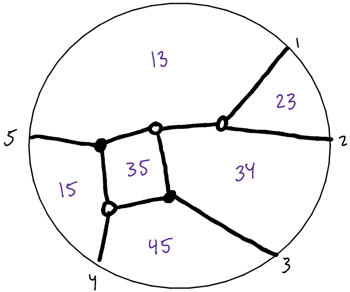
Which Schubert variety is it for?

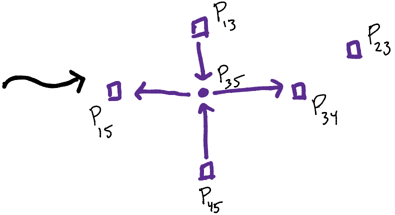
- Each reduced plabic graph corresponds to a unique positroid variety, determined by its trip permutation.
- The plabic graphs for X_I° have trip permutation

$$\pi_I = j_1 j_2 \dots j_{n-k} i_1 i_2 \dots i_k$$

where $I = \{i_1 < i_2 < \cdots < i_k\}$ and $\{1, \dots, n\} \setminus I = \{j_1 < j_2 < \cdots < j_{n-k}\}$.

To summarize





The trip permutation of G is 24513, so this is a seed for $X^{\circ}_{\{1,3\}}$.

Applications

Theorem

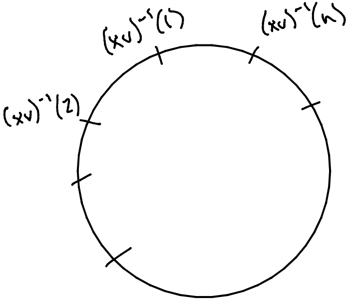
Let G be a reduced plabic graph corresponding to X_I°, and let $(\mathbf{x}, Q(G))$ be the associated seed. Then $\mathcal{A}(\mathbf{x}, Q(G)) = \mathbb{C}[\widehat{X_I^\circ}]$.

Corollaries:

- Classification of when $\mathcal{A}(\mathbf{x}, Q(G))$ is finite type. All types (ADE) occur.
- From (Muller '13) and (Muller-Speyer '16), $\mathcal{A}(\mathbf{x}, Q(G))$ is locally acyclic, so it's locally a complete intersection and equal to its upper cluster algebra
- From (Ford-Serhiyenko '18), $\mathcal{A}(\mathbf{x}, Q(G))$ has green-to-red sequence, so satisfies the EGM property of (GHKK '18) and has a canonical basis of θ-functions parameterized by g-vectors.

Skew-Schubert case

- Indexed by pairs $I = \{i_1, \ldots, i_k\}, J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$ with $i_s \leq j_s$ for all s.
- Coordinate rings of open skew-Schubert varieties $X^\circ_{I,J}$ are cluster algebras. Seeds are given by **generalized** plabic graphs with boundary

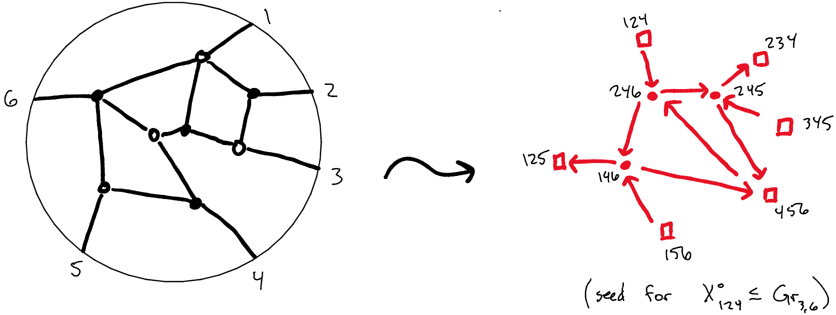


where x and v are permutations obtained from I and J.

Some questions

- What about other positroid varieties?
  - After conversations with Khrystyna and me, (Galashin-Lam '19) proved similar result for arbitrary positroids, using our proof strategy for one inclusion.
- Relation between cluster structure from generalized plabic graphs and cluster structure from normal plabic graphs? (ongoing work with C. Fraser)
- For the open Schubert varieties, there are seeds consisting entirely of Plückers that do *not* come from plabic graphs. Can we find an analogous combinatorial object that gives these seeds?

Thank you!



Cluster structures in Schubert varieties in the Grassmannian
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