Computer Algebra for Lattice Path Combinatorics

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Enumerative Combinatorics: science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

Computer Algebra: effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

Computer Algebra for Enumerative Combinatorics

Today: Algorithms for proving Theorems on Lattice Paths Combinatorics.

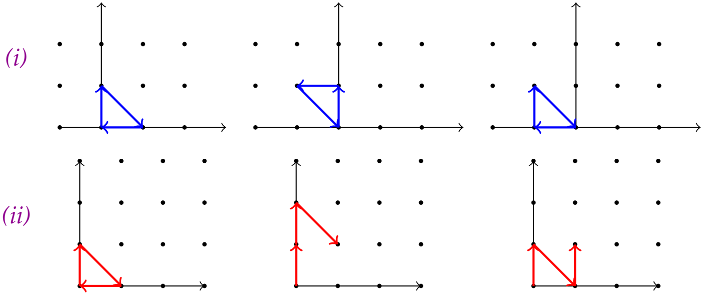
An (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of n-steps $\mathscr{S}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (*excursions*);

(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



Teaser 1: This "exercise" is non-trivial

Teaser 2: ... but it can be solved using Computer Algebra

Teaser 3: ... by two robust and efficient algorithmic techniques,
Guess-and-Prove and Creative Telescoping

Why care about counting walks?

Many objects can be encoded by walks:

- probability theory (voting, games of chance, branching processes, ...)
- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- operations research (queueing theory, ...)

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with a upsteps $\nearrow$ and b downsteps $\searrow$ that start at the origin and never touch the x-axis

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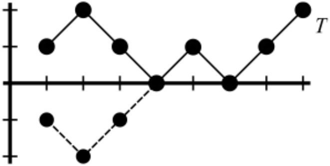
Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a-1$ upsteps $\nearrow$ and b downsteps $\searrow$ that start at $(1,1)$ and never touch the x-axis

Reflection principle [Aebly, 1923]: *paths in* $\mathbb{Z}^2$ *from* $(1,1)$ *to* $T(a+b, a-b)$ *that do touch the x-axis* are in bijection with *paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T



Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T) - (\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}$

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

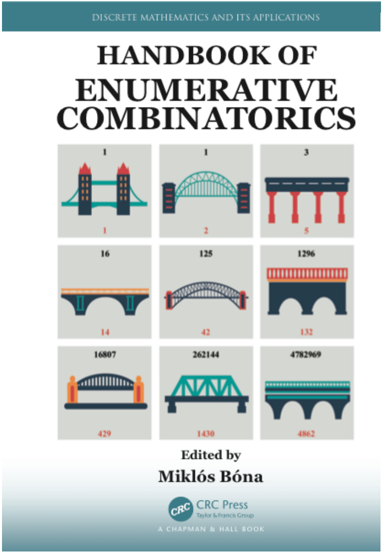
Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Bernardi, Bostan, Bousquet-Mélou, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Malyshev, Melczer, Miller, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Viennot, Wachtel, Wang, Wilf, D. Wilson, M. Wilson, Yatchak, Yeats, Zeilberger, …

etc.

~~**Specific question**~~
~~**Ad hoc solution**~~ ⟶ **Systematic approach**

. . . but it is still a very hot topic



Chapter 10

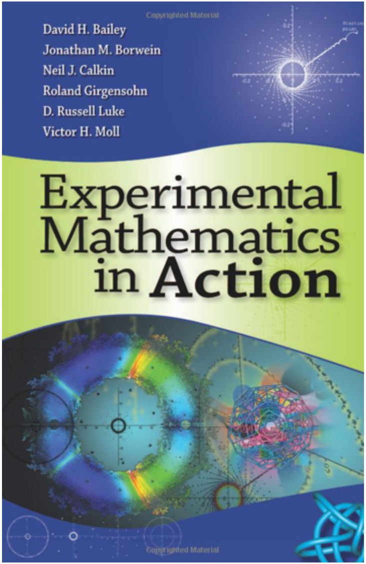
Lattice Path Enumeration

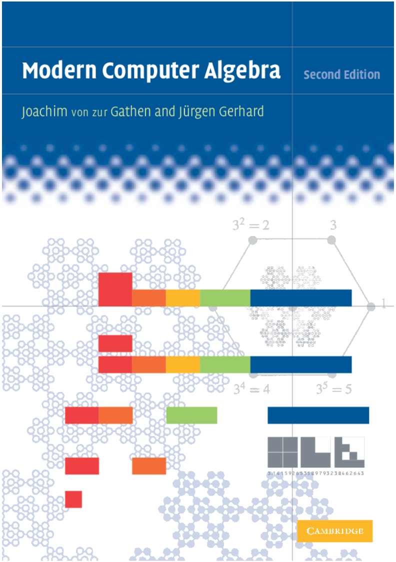
Christian Krattenthaler

Universität Wien

CONTENTS

Our approach: Experimental Mathematics using Computer Algebra





Lattice walks with small steps in the quarter plane

▷ Nearest-neighbor walks in the quarter plane:
$\mathscr{S}$-walks in $\mathbb{N}^2$: starting at $(0,0)$ and using steps in a *fixed* subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

▷ Counting sequence $q_{\mathscr{S}}(n)$: number of $\mathscr{S}$-walks of length n

▷ Generating function:

$$Q_{\mathscr{S}}(t) = \sum_{n=0}^{\infty} q_{\mathscr{S}}(n) t^n \in \mathbb{Z}[[t]]$$

Lattice walks with small steps in the quarter plane

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$\mathscr{S}$-walks in $\mathbb{N}^2$: starting at $(0,0)$ and using steps in a *fixed* subset $\mathscr{S}$ of

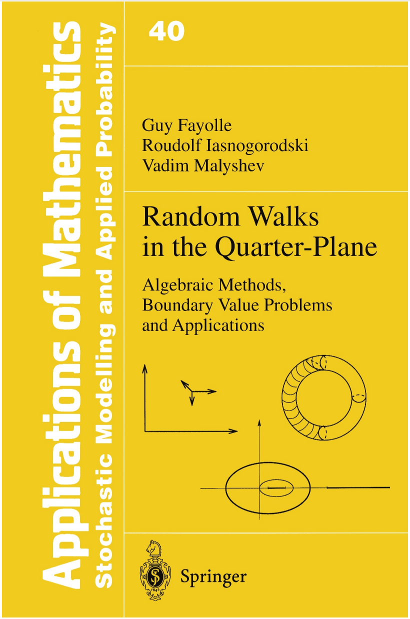
$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

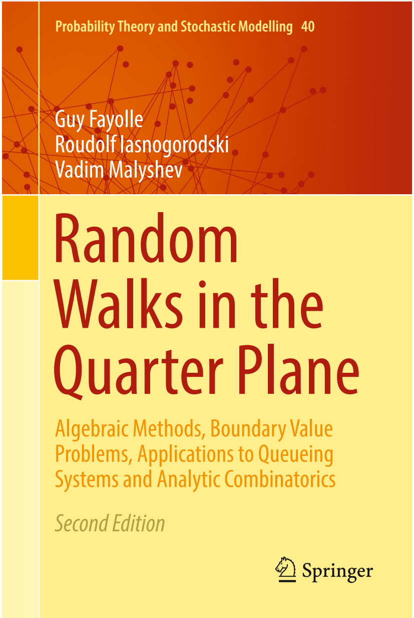
▷ Counting sequence $q_{\mathscr{S}}(i,j;n)$: number of walks of length n ending at (i,j)

▷ Complete generating function (with "catalytic " variables x, y):

$$Q_{\mathscr{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathscr{S}}(i,j;n) x^i y^j t^n \ \in \mathbb{Z}[[x,y,t]]$$

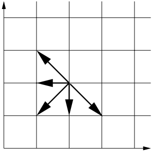
Entire books dedicated to small step walks in the quarter plane!



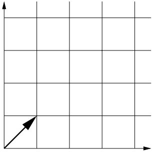


Small-step models of interest

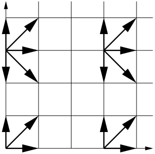
Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



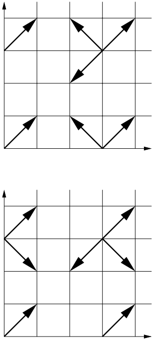
trivial,



simple,



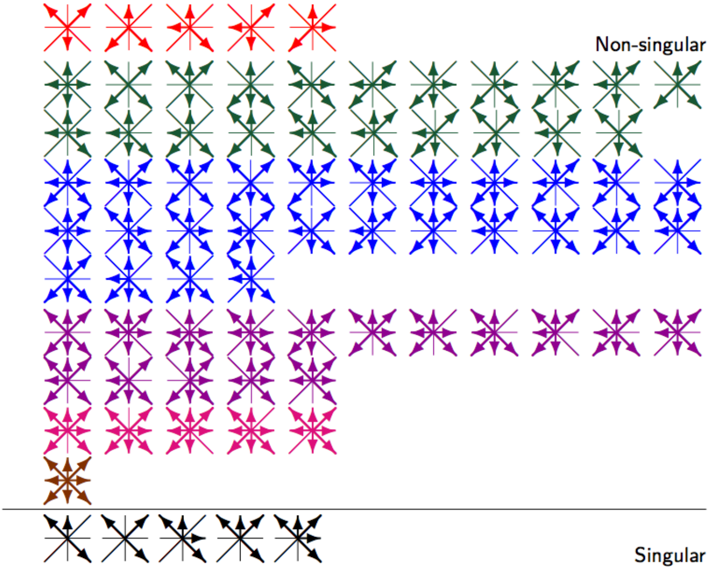
intrinsic to the half plane,



symmetrical.

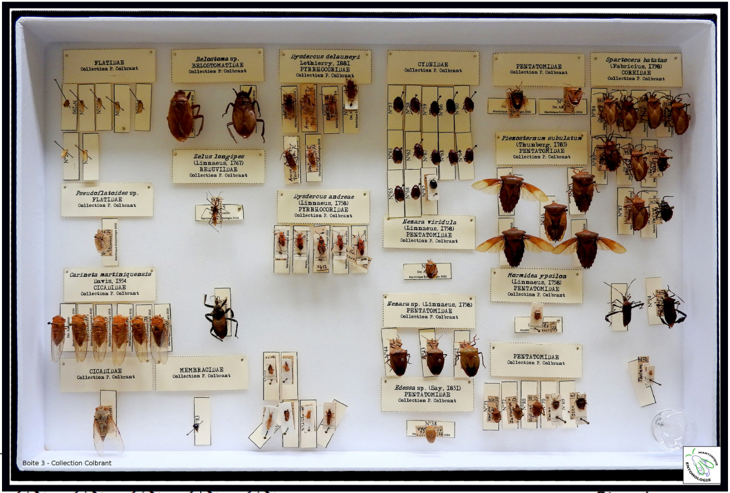
One is left with 79 interesting distinct models.

The 79 small steps models of interest



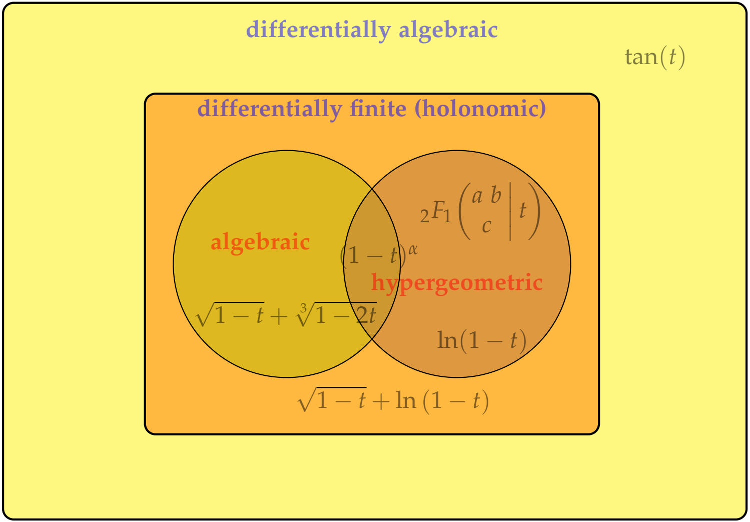
Task: classify their generating functions!

Non-singular



Singular

Classification criterion: properties of generating functions



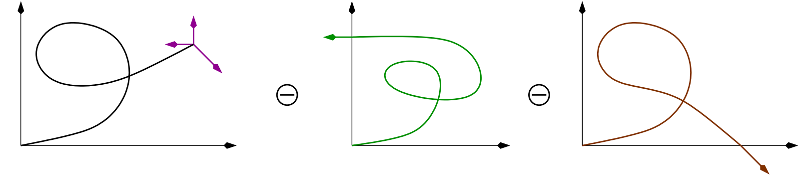
$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}\,\middle|\, t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1). $$

Algebraic reformulation of main task: solving a functional equation

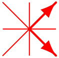
Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

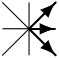
Recursive construction yields the *kernel equation*

$$\left(1 - t\left(y + \frac{1}{x} + x\frac{1}{y}\right)\right) xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



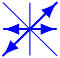
New task: Solve 79 such equations!

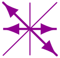
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

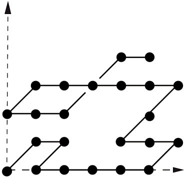
Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

Gessel walks (2000)

- $g(n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$
  1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...



Question: What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n)\, t^n \ ?$$

Theorem [B., Kauers, 2010] (former conjecture of Gessel's)

$(3n+1)\, g(2n) = (12n+2)\, g(2n-1)$ and $(n+1)\, g(2n+1) = (4n+2)\, g(2n)$

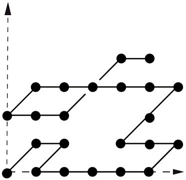
▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

Gessel walks (2000)

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Question: What is the nature of the generating function

$$G(x,y;t) = \sum_{i,j,n=0}^{\infty} g(i,j;n)\, x^i y^j t^n \ ?$$



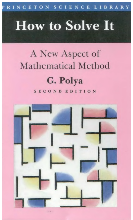
Theorem [B., Kauers, 2010]

$G(x,y;t)$ is an algebraic function†.

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

† Minimal polynomial $P(G(x,y;t); x, y, t) = 0$ has $> 10^{11}$ terms; $\approx 30\,\text{Gb}$ (6 DVDs!)

Guess-and-Prove



Guessing and Proving

George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

generate data → make conjectures → prove them

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

1. There are 2 ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

(I) Generate data:

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	$\longrightarrow \cdots$
1	3	6	10	15	21	28	$\longrightarrow \frac{(i+1)(i+2)}{2}$
1	2	3	4	5	6	7	$\longrightarrow i+1$
1	1	1	1	1	1	1	$\longrightarrow 1$

(II) Guess:

$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!\,j!}$$

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

(III) Prove: If $C_{i,j} \stackrel{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

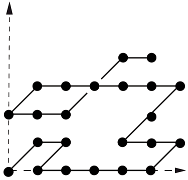
Thus $\boxed{B_{i,j} = C_{i,j}}$

Guess-and-Prove for Gessel walks

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Question: What is the nature of the generating function

$$G(x,y;t) = \sum_{i,j,n=0}^{\infty} g(i,j;n)\, x^i y^j t^n \text{ ?}$$



Answer: [B., Kauers, 2010] $G(x,y;t)$ is an algebraic function†.

Approach:

1. Generate data: compute G to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0;t)$ and $G(0,y;t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bits digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y;t);x,y,t)=0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (6 DVDs!)

A typical Guess-and-Prove algorithmic proof

Theorem ["Gessel excursions are algebraic"]

$$g(t) := G(0,0;\sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

Proof: First guess a polynomial $P(t,T)$ in $\mathbb{Q}[t,T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so (r_n) is P-recursive:

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
     pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t)):
> gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

Algorithmic classification of models with D-Finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size		OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Algorithmic classification of models with D-Finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	algebraic?	asymptotics		OEIS	$\mathscr{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$		$A=1+\sqrt{2}$, $B=1+\sqrt{3}$,	$C=1+\sqrt{6}$,	$\lambda=7+3\sqrt{6}$,	$\mu=\sqrt{\frac{4\sqrt{6}-1}{19}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

- ▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]
- ▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- ▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Models 1–19: proofs, explicit expressions and transcendence

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(t)$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(t)$ is transcendental, except for $\mathscr{S} =$ and $\mathscr{S} =$.

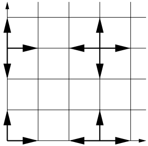
Example (King walks in the quarter plane, A151331)

$$Q(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\,\frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses: (1) kernel method + (2) creative telescoping.

(1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel $K(x,y;t) := 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is left invariant under the change of (x,y) into the elements of

$$\mathcal{G}_{\mathscr{S}} := \left\{ (x,y), (\tfrac{1}{x},y), (\tfrac{1}{x},\tfrac{1}{y}), (x,\tfrac{1}{y}) \right\}$$

Kernel equation:

$$\begin{aligned} K(x,y;t)xyQ(x,y;t) &= xy - txQ(x,0;t) - tyQ(0,y;t) \\ -K(x,y;t)\tfrac{1}{x}yQ(\tfrac{1}{x},y;t) &= -\tfrac{1}{x}y + t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) + tyQ(0,y;t) \\ K(x,y;t)\tfrac{1}{x}\tfrac{1}{y}Q(\tfrac{1}{x},\tfrac{1}{y};t) &= \tfrac{1}{x}\tfrac{1}{y} - t\tfrac{1}{x}Q(\tfrac{1}{x},0;t) - t\tfrac{1}{y}Q(0,\tfrac{1}{y};t) \\ -K(x,y;t)x\tfrac{1}{y}Q(x,\tfrac{1}{y};t) &= -x\tfrac{1}{y} + txQ(x,0;t) + t\tfrac{1}{y}Q(0,\tfrac{1}{y};t) \end{aligned}$$

Summing up and taking positive parts yields:

$$xy\,Q(x,y;t) = [x^{>}y^{>}]\frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$

▷ $Q(x,y;t)$ is D-finite by [Lipshitz, 1988]

▷ Creative Telescoping finds a differential equation for $Q(x,y;t)$

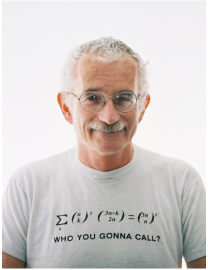
(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Apéry 1978]: $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5)\,A_n.$$

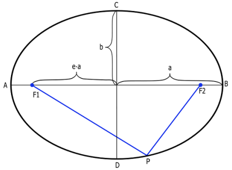
▷ Key fact used to prove that $\zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903\ldots$ is irrational.



"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oiint \frac{\mathrm{d}u\,\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$



Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}\right) =$$

$$\partial_u\left(-\frac{e(-1-u+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

$$+\,\partial_v\left(\frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

▷ Conclusion: $(e-e^3)\cdot p''(e) + (1-e^2)\cdot p'(e) + e\cdot p(e) = 0.$

Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, \mathbf{x})$ a rational function in e and $\mathbf{x} = x_1, \ldots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e)y = 0$ satisfied by $y(e) = \oint\!\!\oint R(e, \mathbf{x}) d\mathbf{x}$.
- **Complexity:** $\mathcal{O}(D^{8n+2})$, where $D = \deg R$.
- **Output size:** T has order $\leq D^n$ in ∂_e and degree $\leq D^{3n+2}$ in e.

- ▷ Avoids the (costly) computation of certificates, of size $\Omega(D^{n^2/2})$.
- ▷ Previous algorithms: complexity (at least) doubly exponential in n.
- ▷ Very efficient in practice.

A toy *transcendence proof*: blending Guess-and-Prove and CT

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

Proof:

1. Creative telescoping:

   $$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2+17n+5)A_n, \quad A_0 = 1,\ A_1 = 5$$

2. Conversion from recurrence to differential equation $L(f) = 0$, where

   $$L = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

3. Guess-and-Prove:

   compute least-order $L_f^{\min}$ in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $L_f^{\min}(f) = 0$

4. Basis of formal solutions of $L_f^{\min}$ at $t = 0$:

   $$\left\{1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2)\right\}$$

5. Conclusion: f is transcendental†

† f algebraic would imply a full basis of algebraic solutions for $L_f^{\min}$ [Tannery, 1875].

Summary: classification of walks with small steps in $\mathbb{N}^2$

$Q_{\mathscr{S}}$ is D-finite $\Longleftrightarrow$ a certain group $\mathcal{G}_{\mathscr{S}}$ is finite (!)

quadrant models $\mathscr{S}$: 79

$|\mathcal{G}_{\mathscr{S}}| < \infty$: 23

$|\mathcal{G}_{\mathscr{S}}| = \infty$: 56

OS $= 0$: 4 — Guess-and-Prove — **algebraic**

OS $\neq 0$: 19 — kernel + CT — **transcendental**

$\exists$ decoupling*: 9 — Tutte's invariants — **D-algebraic**

$\nexists$ decoupling: 47 — diff. Galois — **D-transcendental**

* $\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the kernel $K(x,y;t) = 0$.

▷ Many contributors (2010–2019): Bernardi, B., Bousquet-Mélou, Chyzak, Dreyfus, Hardouin, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Roques, Salvy, Singer

▷ Proofs use various tools: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.

Conclusion

- Enumerative Combinatorics and Computer Algebra enrich one another
- Classification of $Q(x, y; t)$ fully completed for 2D small step walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess-and-Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(x, y; t) \approx 30$Gb.

- Lack of "purely human" proofs for some results.
- Many beautiful open questions for 2D models with repeated or large steps, and in dimension > 2.

Beyond dimension 2: walks with small steps in $\mathbb{N}^3$

▷ $2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D

3D octant models $\mathscr{S}$ with ≤ 6 steps: 20804

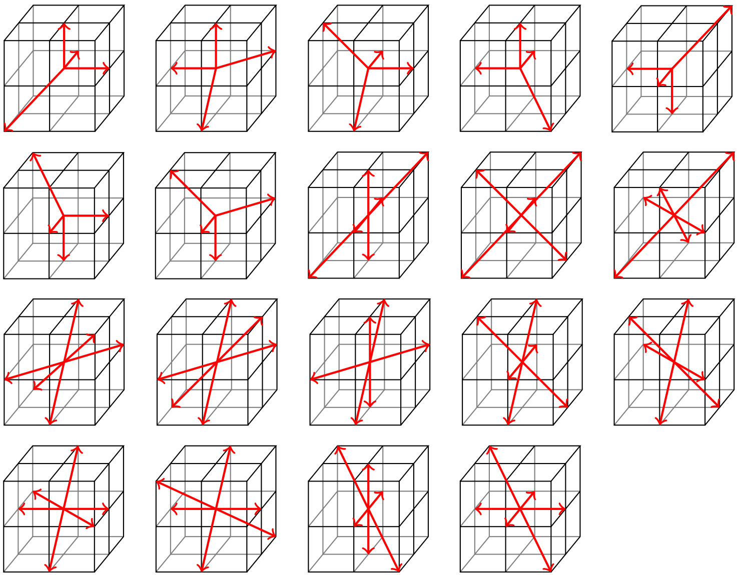
- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 170
  - orbit sum $\neq 0$: 108
    - Creative Telescoping
      - **D-finite**
  - orbit sum $= 0$: 62
    - 2D-reducible: 43
      - **D-finite**
    - not 2D-reducible: 19
      - **non-D-finite?**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 20634
  - **non-D-finite?**

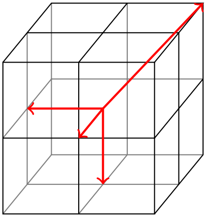
[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
completed by [Bacher, Kauers, Yatchak, 2016]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably no

19 mysterious 3D-models: finite $\mathcal{G}_{\mathscr{S}}$ and possibly non-D-finite $Q_{\mathscr{S}}$





Two different computations suggest:

$$k_{4n} = C \cdot 256^n / n^{\alpha}, \text{ for } \alpha = 3.3257570041744\ldots,$$

so excursions are very probably non-D-finite quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

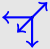
- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 240
  - $\mathrm{OS} \neq 0$: 431
    - **D-finite**
  - $\mathrm{OS} = 0$: 9
    - **D-finite?**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 12 870
  - α rational: 16
    - **non-D-finite?**
  - α irrational: 12 854
    - **non-D-finite**

[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness $\Longleftrightarrow$ finiteness of the group?

Answer: ?

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $Q(0,0;t^{1/2})$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1\left(\left. \begin{array}{cc} \frac{1}{6} & \frac{2}{3} \\ & 1 \end{array} \right| \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \sqrt{1-12t} \cdot {}_2F_1\left(\left. \begin{array}{cc} -\frac{1}{6} & \frac{2}{3} \\ & 1 \end{array} \right| \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $Q(0,0;t)$ equals

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

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