Queer supercrystals in SageMath

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Trac ticket: trac.sagemath.org/ticket/25918

based on joint work with Maria Gillespie and Graham Hawkes

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- Free, open-source mathematical software
- Based on Python (object-oriented)
- Interfaces to GAP, matplotlib, Numpy, R, SciPy, etc.
- Active contribution and maintenance by developers
- Extensive resources and code development for crystals

Queer supercrystals

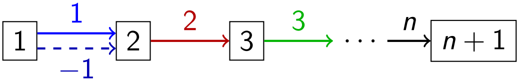
- Model tensor representations of $\mathfrak{q}(n+1)$
- Irreducible representations indexed by strict partitions λ
- Characters: Schur-P function P_λ
- Littlewood-Richardson rule:

$$\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \bigoplus_\nu g_{\lambda\mu}^{\nu} \mathcal{B}(\nu)$$

$$P_\lambda P_\mu = \sum_\nu g_{\lambda\mu}^{\nu} P_\nu$$

Standard $\mathfrak{q}(n+1)$ crystal

[Grantcharov, Jung, Kang, Kashiwara, Kim '10, '14]
Standard crystal $\mathcal{B}$ of type $\mathfrak{q}(n+1)$:



Let $2 \leq i \leq n$.

$$f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i}, \quad e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i},$$

where $s_{w_i} = s_2 s_3 \dots s_i s_1 s_2 \dots s_{i-1}$ and s_i is the reflection along the i-th string.

$$f_{-i'} = s_{w_0} f_{-i} s_{w_0}, \quad e_{-i'} = s_{w_0} e_{-i} s_{w_0},$$

where w_0 is the longest element in S_{n+1}.

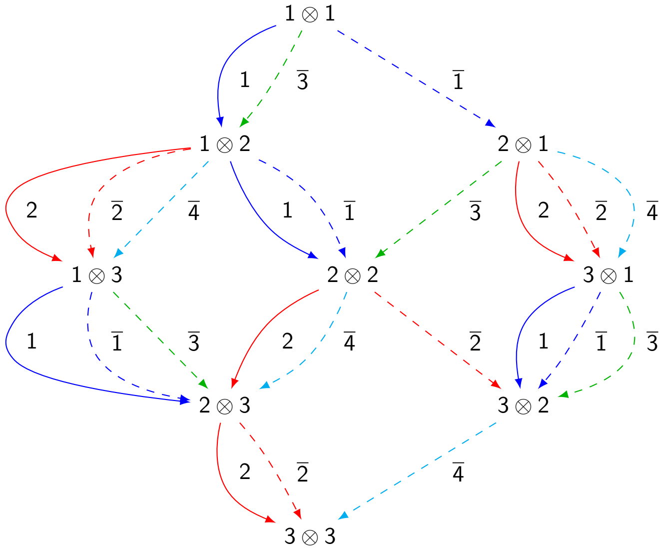
SageMath : Examples

```
sage: Q = crystals.Letters(['Q',3]); Q

The queer crystal of letters for q(3)

sage: T = tensor([Q]*6)
sage: T.index_set()

(-4, -3, -2, -1, 1, 2)

sage: [t for t in T
....:        if all(t.epsilon(i)==0
....:             for i in Q.index_set())]

[[1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 2, 1], [1, 1, 1, 2, 1, 1],
 [1, 1, 2, 1, 1, 1], [1, 1, 2, 1, 2, 1], [1, 1, 2, 2, 1, 1],
 [1, 2, 1, 1, 1, 1], [1, 2, 1, 1, 2, 1], [1, 2, 1, 2, 1, 1],
 [1, 2, 1, 3, 2, 1], [1, 2, 2, 1, 1, 1], [1, 2, 3, 1, 2, 1]]
```

SageMath : Examples

```
sage: Q = crystals.Letters(['Q',3])
sage: T = tensor([Q]*2)
sage: view(T)
```

SageMath : Examples



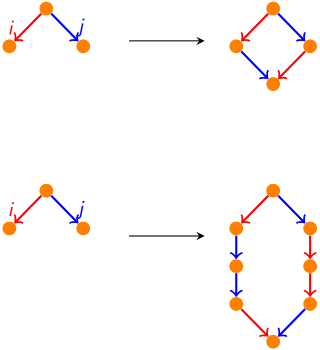
SageMath : Examples

```
sage: Q = crystals.Letters(['Q',3])
sage: T = tensor([Q]*2)
sage: view(T)
sage: latex(T)
```

```
\begin{tikzpicture}[>=latex,line join=bevel,]
%%
\node (node_8) at (142.8bp,287.0bp) [draw,draw=none] {$1 \otimes 1$};
  \node (node_7) at (162.8bp,147.0bp) [draw,draw=none] {$2 \otimes 2$};
  \node (node_6) at (102.8bp,77.0bp) [draw,draw=none] {$2 \otimes 3$};
  \node (node_5) at (242.8bp,217.0bp) [draw,draw=none] {$2 \otimes 1$};
  \node (node_4) at (282.8bp,147.0bp) [draw,draw=none] {$3 \otimes 1$};
  \node (node_3) at (142.8bp,7.0bp) [draw,draw=none] {$3 \otimes 3$};
  \node (node_2) at (42.797bp,147.0bp) [draw,draw=none] {$1 \otimes 3$};
  \node (node_1) at (242.8bp,77.0bp) [draw,draw=none] {$3 \otimes 2$};
  \node (node_0) at (102.8bp,217.0bp) [draw,draw=none] {$1 \otimes 2$};
... and more TikZ commands!
```

Stembridge axioms: Axioms

Main relations:

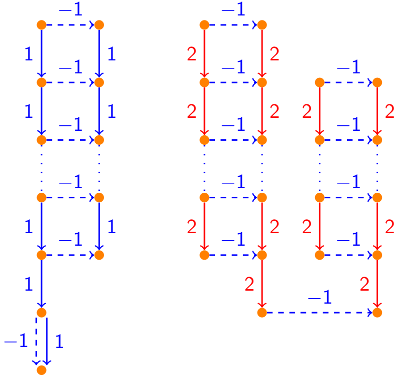


Dual axioms similarly hold.

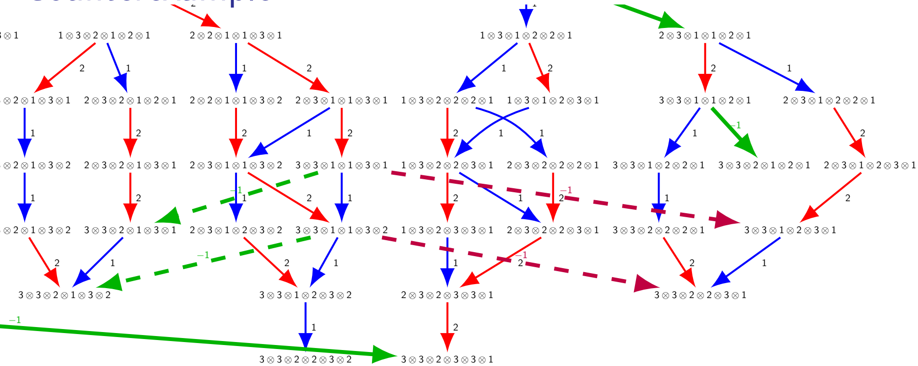
Characterization: Local queer axioms

Conjecture (Assaf, Oguz 2018)

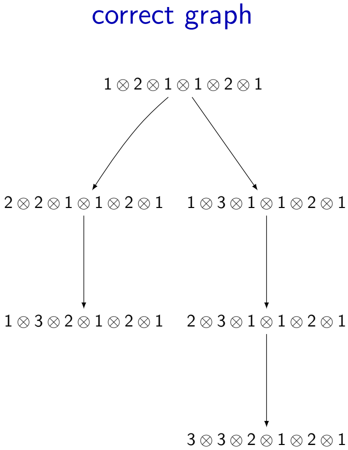
In addition to the Stembridge axioms, the following relations characterize type $\mathfrak{q}(n+1)$ crystals.

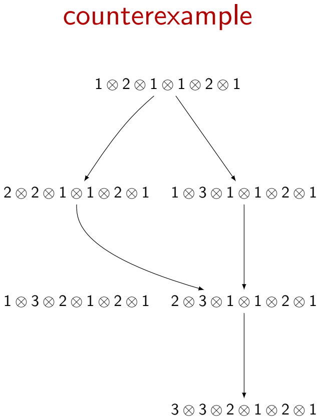


Counterexample



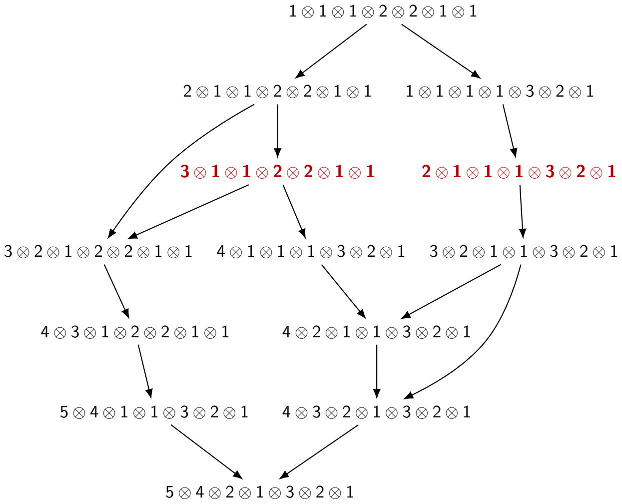
Graph on type A_n components: Counterexample





Graph on type A_n components: Another example

$$P_{52} = s_{52} + s_{511} + s_{43} + \mathbf{2s_{421}} + s_{4111} + s_{331} + s_{322} + 2s_{3211} + s_{31111} + s_{2221} + s_{22111}.$$



Characterization of queer supercrystals

Theorem (GHPS 2018)

Suppose that $\mathcal{C}$ is a connected abstract $\mathfrak{q}(n+1)$ crystals satisfying:

1. *$\mathcal{C}$ satisfies local queer axioms.*
2. *$G(\mathcal{C}) \cong G(\mathcal{D})$, where $\mathcal{D}$ is a connected component of $\mathcal{B}^{\otimes l}$.*
3. *$\mathcal{C}$ satisfies the connectivity axioms* **C1. - C3.**

Then as queer supercrystals, $\mathcal{C} \cong \mathcal{D}$.

Thank you!

I would be happy to give a more detailed private computer demonstration if desired!