Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

FWF
Der Wissenschaftsfonds.

DD-finite functions in Sage

Computing beyond holonomic functions

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Outline

1. Installing the package
2. Using the package
3. Computing with the package
4. Proving with the package
5. Conclusions

How to install the package

Git repository

```
http://git.risc.jku.at/gitweb/?p=ajpastor/diff_defined_functions.git
```

Zip from webpage

```
https://www.dk-compmath.jku.at/Members/antonio/sage-package-dd_functions
```

PyPi (in process)

pip install dd_functions

- **Stable** version
- **Easy** to update

DD-finite Functions

Definition

Let $f \in K[[x]]$ and $R \subset K[[x]]$ a ring. We say that f is differentially definable over R if there exist $d \in \mathbb{N}$ and elements in R $r_0(x), \dots, r_d(x)$ such that:

$$r_d(x)f^{(d)}(x) + \dots + r_0(x)f(x) = 0.$$

D-finite examples

Elementary functions

Exponential (`Exp`), trigonometric (`Sin`, `Cos`),...

Special functions

Bessel functions (`BesselD`), hypergeometric functions (`HypergeometricFunction`),...

Combinatorial functions

Generating functions for holonomic sequences (Catalan numbers, Fibonacci sequence, etc)

DD-finite examples

Classic non-holonomic

Double exponential (e^{e^x-1}), tangent (`Tan`),...

Mathieu functions

DD-finite generalization of the sine and cosine

```
show(MathieuD(init=(1,1)))
```

$$\begin{cases} f''(x) + g_0(x)f(x) = 0 \\ \textit{where} \\ g_0(x) : \begin{cases} g_0^{(3)}(x) + (4)g_0'(x) = 0 \\ \hdashline g_0(0) = a - 2q, g_0'(0) = 0, g_0''(0) = 8q, g_0^{(3)}(0) = 0 \end{cases} \\ \hdashline f(0) = 1, f'(0) = 1, f''(0) = -a + 2q \end{cases}$$

Operations supported

Arithmetic operations

Addition (+, −), product (∗, /, ˆ)

Differential operations

Derivative (`derivative`), integration (`integrate`)

Composition

Using the standard `call` in Sage.

Extracting sequence

Ordinary generating functions

Method `getSequenceElement` allows to get the associated sequence.

Exponential generating functions

Method `getInitialValue` allows to get the associated exponential sequence.

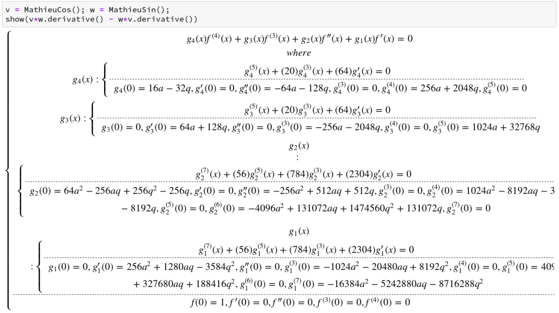
```
# Bell numbers
Exp(Exp(x)-1).getInitialValueList(10)
```

```
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
```

Proving DD-finite identities

Constant Wronskian

$$w''(x) - (a - 2q\cos(2x))w(x) = 0,$$
$$w_1'w_2 - w_2'w_1 = 1.$$



```
v = MathieuCos(); w = MathieuSin();
show(v*w.derivative() - w*v.derivative())
```

$$g_4(x)f^{(4)}(x) + g_3(x)f^{(3)}(x) + g_2(x)f''(x) + g_1(x)f'(x) = 0$$

where

$$g_4(x) : \left\{ \begin{array}{c} g_4^{(5)}(x) + (20)g_4^{(3)}(x) + (64)g_4'(x) = 0 \\ \hline g_4(0) = 16a - 32q, g_4'(0) = 0, g_4''(0) = -64a - 128q, g_4^{(3)}(0) = 0, g_4^{(4)}(0) = 256a + 2048q, g_4^{(5)}(0) = 0 \end{array} \right.$$

$$g_3(x) : \left\{ \begin{array}{c} g_3^{(5)}(x) + (20)g_3^{(3)}(x) + (64)g_3'(x) = 0 \\ \hline g_3(0) = 0, g_3'(0) = 64a + 128q, g_3''(0) = 0, g_3^{(3)}(0) = -256a - 2048q, g_3^{(4)}(0) = 0, g_3^{(5)}(0) = 1024a + 32768q \end{array} \right.$$

$$g_2(x)$$
$$:$$
$$\left\{ \begin{array}{c} g_2^{(7)}(x) + (56)g_2^{(5)}(x) + (784)g_2^{(3)}(x) + (2304)g_2'(x) = 0 \\ \hline g_2(0) = 64a^2 - 256aq + 256q^2 - 256q, g_2'(0) = 0, g_2''(0) = -256a^2 + 512aq + 512q, g_2^{(3)}(0) = 0, g_2^{(4)}(0) = 1024a^2 - 8192aq - 3 \\ - 8192q, g_2^{(5)}(0) = 0, g_2^{(6)}(0) = -4096a^2 + 131072aq + 1474560q^2 + 131072q, g_2^{(7)}(0) = 0 \end{array} \right.$$

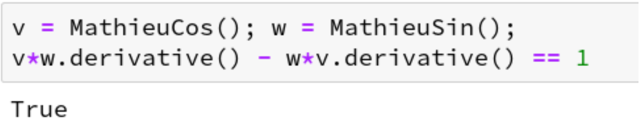
$$g_1(x)$$
$$: \left\{ \begin{array}{c} g_1^{(7)}(x) + (56)g_1^{(5)}(x) + (784)g_1^{(3)}(x) + (2304)g_1'(x) = 0 \\ \hline g_1(0) = 0, g_1'(0) = 256a^2 + 1280aq - 3584q^2, g_1''(0) = 0, g_1^{(3)}(0) = -1024a^2 - 20480aq + 8192q^2, g_1^{(4)}(0) = 0, g_1^{(5)}(0) = 409 \\ + 327680aq + 188416q^2, g_1^{(6)}(0) = 0, g_1^{(7)}(0) = -16384a^2 - 5242880aq - 8716288q^2 \end{array} \right.$$

$$f(0) = 1, f'(0) = 0, f''(0) = 0, f^{(3)}(0) = 0, f^{(4)}(0) = 0$$

Proving DD-finite identities

Constant Wronskian

$$w''(x) - (a - 2q\cos(2x))w(x) = 0,$$
$$w_1'w_2 - w_2'w_1 = 1.$$



```
v = MathieuCos(); w = MathieuSin();
v*w.derivative() - w*v.derivative() == 1
```

```
True
```

Conclusions

Features

- Sage package for DD-finite functions
- Arithmetic and differentil porperties implemented
- Composition implemented
- Zero recognition (equality) implemented

To be done

- Improve performance
- Getting more examples
- Polish the current implementation

Thank you!

Antonio Jiménez-Pastor

- https://www.dk-compmath.jku.at/people/antonio
- https://www.risc.jku.at/home/ajpastor

Sage package:

https://www.dk-compmath.jku.at/Members/antonio/
sage-package-dd_functions