From multiline queues to Macdonald polynomials

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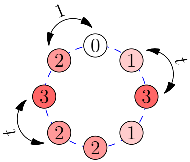
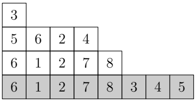
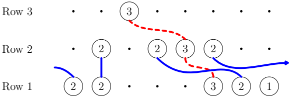
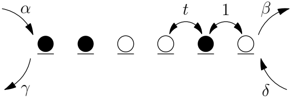


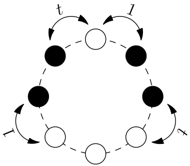


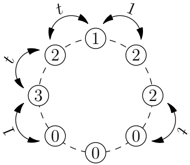


asymmetric simple exclusion process (ASEP)

- the ASEP is a particle process describing particles hopping on a finite 1D lattice: 1 particle per site, at each time step any two adjacent particles may swap with some probability, with possible interactions at the boundary





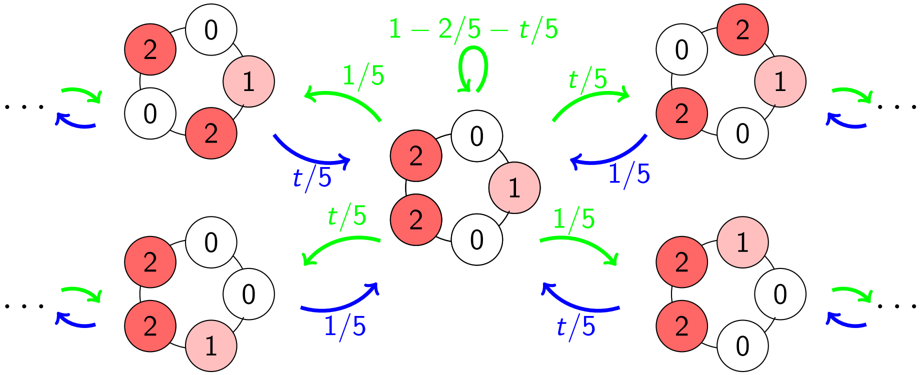


- **multispecies ASEP on a ring:** now we have particles of types $0, 1, \ldots, L$ with J_i particles of type i, represent the type by $\lambda = (L^{J_L}, \ldots, 1^{J_1}, 0^{J_0})$. (Here $\lambda = (3, 2, 2, 2, 1, 0, 0, 0)$)

  Markov chain with states that are rearrangements of the parts of λ, where possible transitions between states are swaps of adjacent particles:

$$X\,A\,B\,Y \xrightarrow{1} X\,B\,A\,Y$$

stationary probabilities



$$\Pr(2,0,1,0,2) = \frac{1}{Z}(3 + 7t + 7t^2 + 3t^3) \qquad \Pr(0,2,1,0,2) = \frac{1}{Z}(5 + 6t + 7t^2 + 2t^3)$$

$$\Pr(2,1,0,0,2) = \frac{1}{Z}(6 + 7t + 6t^2 + t^3) \qquad \Pr(2,0,0,1,2) = \frac{1}{Z}(1 + 6t + 7t^2 + 6t^3)$$

$$\Pr(2,1,2,0,0) = \frac{1}{Z}(3 + 7t + 7t^2 + 3t^3) \qquad \Pr(2,0,1,2,0) = \frac{1}{Z}(2 + 7t + 6t^2 + 5t^3)$$

$Z = \sum_\mu \tilde{Pr}(\mu)$ (partition function)

ASEP and Macdonald polynomials

symmetric Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$ defined by:

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\mu\lambda} m_\mu, \qquad \langle P_\lambda, P_\mu \rangle = 0 \text{ if } \lambda \neq \mu$$

- Schur functions s_λ at $q = t$
- Hall-Littlewood polynomials at $q = 0$
- Jack polynomials at $t = q^\alpha$ and $q \to 1$
- partition function of the ASEP on a ring at $x_1 = \cdots = x_n = q = 1$:

$$P_\lambda(1, \ldots, 1; 1, t) = \sum_\mu \tilde{\Pr}(\mu)$$

(Cantini-de Gier-Wheeler '15)

nonsymmetric Macdonald polynomials $E_\mu(\mathbf{x}; q, t)$

- E_μ are simultaneous eigenfunctions of certain products of Demazure-Luztig operators, which are generators for the affine Hecke algebra of type A_{n-1}:

$$(T_i - t)(T_i + 1) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ if } |i-j| > 1$$

$$T_i f = tf - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(f - s_i f)$$

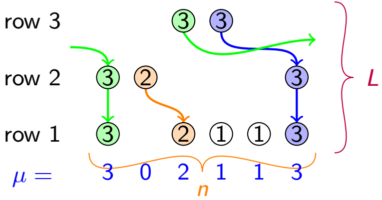
$$Y_i = T_i \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1}, \qquad Y_i E_\mu = \phi_i(\mu) E_\mu$$

- E_μ stabilize to P_λ, specialize to Demazure characters at $q = t = 0$, specialize to key polynomials at $q = t = \infty$.
- $E_\mu(1, \ldots, 1; 1, t) = \tilde{\mathrm{Pr}}(\mu)$ when μ is a partition

probabilities of the ASEP with **multiline queues**

Special case: $t = 0$ (Ferarri-Martin '05)

- A **multiline queue** for particles of types $0, 1, \ldots, L$ on an ASEP of n locations is a ball system on a cylinder of L rows and n columns
- Each ball picks the first available ball to pair with in the row below, weakly to its right
- The state of the multiline queue is read off Row 1



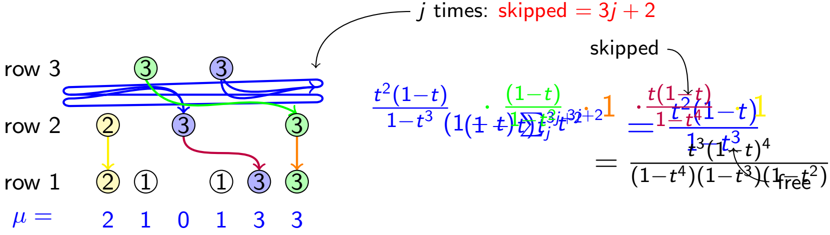
Theorem (Ferrari-Martin '05)

$$\Pr(\mu)(t = 0)$$

is proportional to the number of multiline queues with bottom row μ.

multiline queues for general t

- Combine a ball system with a queueing algorithm.
- Each ball chooses an available ball to pair with in the row below. t counts the number of available balls skipped: assign weight $t^{\text{total skipped}}(1-t)$.
- The weight of each non-trivial pairing is $t^{\text{skipped}}\frac{(1-t)}{1-t^{\text{free}}}$.
- The state of the multiline queue is read off Row 1.



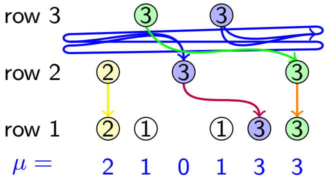
$$\text{wt}(M) = \prod_{\text{pairing}} t^{\text{skipped}}\frac{(1-t)}{1-t^{\text{free}}}$$

Theorem (Martin '18, Corteel-M-Williams '18)

$$\Pr(\mu) = \frac{1}{Z}\sum_{M\in \text{MLQ}(\mu)} \text{wt}(M)$$

putting the "q" in the queue

- Define the x-weight of a queue M to be $x^M = \prod_j x_j^{\#\text{ balls in col } j}$
- Each pairing (of type ℓ, from row r) that wraps around contributes $q^{\ell-r+1}$
- Weight for each pairing is $t^{\text{skipped}} q^{(\ell-r+1)\delta_{\text{wrap}}} \frac{1-t}{1-q^{\ell-r+1}t^{\text{free}}}$



$$x^M = x_1^2 x_2^2 x_3 x_4^2 x_5 x_6^2$$

$$\frac{qt^2(1-t)}{1-qt^3}(1-t)\sum_j t^{3j+2} q^{j+1} \cdot \frac{qt^2(1-t)}{1-qt^3}$$

$$\frac{qt^2(1-t)}{1-qt^3} \cdot \frac{(1-t)}{1-qt^2} \cdot 1 \cdot \frac{t(1-t)}{1-q^2t^4} \cdot 1$$

$$= \frac{qt^3(1-t)^4}{(1-q^2t^4)(1-qt^3)(1-qt^2)}$$

$$\operatorname{wt}(M)(\mathbf{x}; q, t) = x^M t^{\text{skipped}} \prod_{\text{pairings}} q^{(\ell-r+1)\delta_{\text{wrap}}} \frac{1-t}{1-q^{\ell-r+1}t^{\text{free}}}$$

Theorem (Corteel-M-Williams '18)

$$E_\mu(\mathbf{x}; q, t) = \sum_{M \in \mathrm{MLQ}(\mu)} \operatorname{wt}(M)(\mathbf{x}; q, t) \quad \text{when } \mu \text{ is a partition}$$

$$P_\lambda(\mathbf{x}; q, t) = \sum_{M \in \mathrm{MLQ}(\lambda)} \operatorname{wt}(M)(\mathbf{x}; q, t)$$

proof

We define $f_\mu(\mathbf{x}; q, t) = \sum_{M \in \mathrm{MLQ}(\mu)} \mathrm{wt}(M)$ and show that:

- $$T_i f_\mu = \begin{cases} f_{s_i \mu} \text{ if } \mu_i < \mu_{i+1} \\ t f_\mu \text{ if } \mu_i = \mu_{i+1} \end{cases}$$

- $$f_{\mu_1, \dots, \mu_n}(x_1, \dots, x_n) = q^{\mu_n} f_{\mu_n, \mu_1, \dots, \mu_{n-1}}(q x_n, x_1, \dots, x_{n-1})$$

(f_μ and E_μ are related by a triangular change of basis)

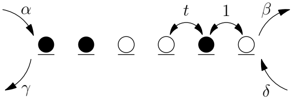
thus:

$$E_\mu = f_\mu \quad \text{when } \mu \text{ is a partition}$$

and

$$P_\lambda = \sum_\mu f_\mu$$

Koornwinder polynomials (Macdonald of type BC)



- Koornwinder polynomial $K_{(n-r,0,\ldots,0)}$ at $q = t$ can be computed from the partition function $Z_{n,r}(t; \alpha, \beta, \gamma, \delta)$ of the two-species ASEP with open boundaries (Corteel-Williams 2015, Cantini 2015)
- first combinatorial formula for certain special cases of Koornwinder polynomials using ASEP (Corteel-M-Williams 2016)
- Goal: compute nonsymmetric Kornwinder polynomials through multiline queues for the multispecies ASEP with open boundaries?