On enumerating factorizations in reflection groups.

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The number of reduced reflection factorizations of c

Theorem (Hurwitz, 1892)

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Theorem (Deligne-Arnol'd-Bessis)

For a well-generated, complex reflection group W and a Coxeter element c, there are $\dfrac{h^n n!}{|W|}$ (minimal length) reflection factorizations $t_1 \cdots t_n = c$ where $h = |c|$.

Arbitrary length reflection factorizations of c

If $\mathcal{R}$ denotes the set of reflections of W, we write

$$\mathrm{Fact}_{W,c}(N) := \#\{(t_1, \cdots, t_N) \in \mathcal{R}^n \mid t_1 \cdots t_N = c\}.$$

Now, consider the exponential generating function:

$$\mathrm{FAC}_{S_n,c}(t) = \sum_{N \geq 0} \mathrm{Fact}_{S_n,c}(N) \frac{t^N}{N!}.$$

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Theorem (Jackson, '88)

If $c = (12\cdots n) \in S_n$, then

$$\text{FAC}_{S_n,c}(t) = \frac{e^{t\binom{n}{2}}}{n!}\left(1 - e^{-tn}\right)^{n-1}.$$

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Notice that

$$\left[\frac{t^{n-1}}{(n-1)!}\right] \text{FAC}_{S_n,c}(t) = \frac{1}{n!} \cdot (n)^{n-1} \cdot (n-1)! = n^{n-2}.$$

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There must be an example we can do by hand?

Let $C_2 := \{\mathrm{Id}, c\}$ be the group of order 2 and $\mathcal{R} = \{c\}$. Then,

$$\mathrm{FAC}_{C_2,c}(t) \quad = \quad t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$$

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with ξ a n-th root of unity.

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Remark (Hurwitz 1901)

Exponential generating functions that enumerate factorizations of the form $a_1\cdots a_N = g$, where all a_i's belong to a set C closed under conjugation, **are finite** *(weighted)* **sums of** *(scaled)* **exponentials**.

Complex reflection groups and regular elements

A *finite* subgroup $G \leq GL_n(V)$ is called a *complex reflection group* if it is generated by pseudo-reflections. There are $\mathbb{C}$-linear maps t that fix a hyperplane (i.e. $\text{codim}(V^t) = 1$).

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Definition

An element $g \in W$ is called ζ-regular if it has a ζ-eigenvector $\vec{v}$ that lies in no reflection hyperplane.
In particular, a *Coxeter element* is defined as a $e^{2\pi i/h}$-regular element for $h = (|\mathcal{R}| + |\mathcal{A}|)/n$.

You already know this definition of Coxeter elements

Example

1. In S_n, the regular elements are $(12\cdots n)$, $(12\cdots n-1)(n)$, and their powers. Indeed, $(\zeta^{n-1}, \zeta^{n-2}, \cdots, 1)$ with $\zeta = e^{2\pi i/n}$ is an eigenvector for $(12\cdots n)$.

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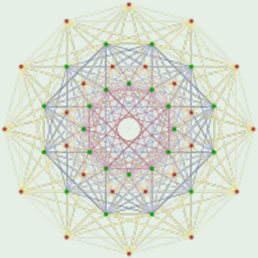
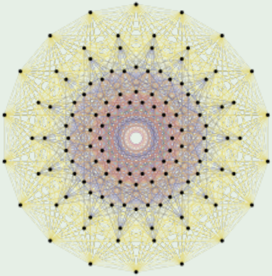
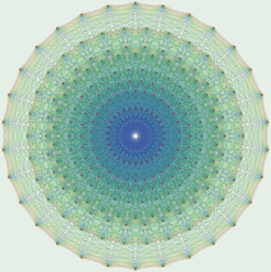
2. For real reflection groups:

The Chapuy-Stump proof

$$\mathsf{FAC}_{W,c}(t) = \frac{1}{|W|} \sum_{\chi \in \widehat{W}} \chi(1) \cdot \chi(c^{-1}) \cdot \exp(t \cdot \frac{\chi(\mathfrak{R})}{\chi(1)})$$

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Ingredients to calculate the above sum:

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3. All complex reflection groups can be described as permutation groups on a set of *roots*. GAP can then produce their character tables.

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Remark

The fact that there is no uniform construction of the irreducible characters Irr(W) *makes it is very difficult to have a uniform proof.*

A uniform argument; the decaf version

Definition

Given a character $\chi \in \widehat{W}$, we define the Coxeter number c_χ as the normalized trace of $\sum_{t \in \mathcal{R}} (\mathbf{1} - t)$. That is,

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Lemma

For a cpx reflection group W and a **regular** *element $g \in W$, the total contribution in (1) of those characters $\chi \in \widehat{W}$ for which c_χ is not a multiple of $|g|$ is 0.*

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[Just a whiff of coffee]There is a cyclic permutation on the characters, induced by a galois action on the corresponding Hecke characters, that cancels out the contributions in each non-singleton orbit.

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Remark

We write $l_R(g)$ *for the reflection length of* g*, i.e. the smallest number* k *of (quasi-)reflections* t_i *needed to write* $g = t_1 \cdots t_k$.

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$$\mathrm{FAC}_{W,g}(t) = 0 + 0 \cdot \frac{t}{1} + 0 \cdot \frac{t^2}{2!} + \cdots + 0 \cdot \frac{t^{l_R(g)-1}}{(l_R(g)-1)!} + (something) \cdot \frac{t^{l_R(g)}}{l_R(g)!} + \cdots .$$

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For a complex reflection group W, and a regular element $g \in W$:

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When W is a complex reflection group and $g \in W$ a regular element, then

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Can anyone guess what is happening? (AKA Why a duck?)

Example

Below are the polynomials $\tilde{\Phi}(X)$ for $W = S_n$, $n = 4 \cdots 6$ and all regular classes

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   1. (1234) : $(1 - X)^3$

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$$\mathcal{H}(G_{26}) \; = \; \langle \; \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u} \mid \quad \boldsymbol{stst} = \boldsymbol{tsts}, \; \boldsymbol{su} = \boldsymbol{us}, \; \boldsymbol{tut} = \boldsymbol{utu},$$

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Definition

We consider the 1-parameter specialization $\{x_0, y_0\} \to x$, $x_1 \to -1$, and $y_1 \to \xi$, $y_2 \to \xi^2$ with $\xi^3 = 1$. Then, for some $y^N = x$, $K(y)\mathcal{H}_x(W)$ is split.

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Definition (Malle's Permutation Ψ)

We write Ψ for the permutation of the irreducible modules of $\mathcal{H}_x(W)$ induced by the galois conjugation $y \to e^{2\pi i/N} \cdot y \;\in \mathrm{Gal}\big(K(y)/K(x)\big)$.

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1. There is a special element $\boldsymbol{\pi} \in P(W) = \pi_1(V^{\mathrm{reg}}, x_0)$ called *full twist*, central in the braid group $B(W)$. It is the geometric circle $[0,1] \ni t \to e^{2\pi i t} \cdot x_0$.

The proof of the technical lemma ♨ arxiv:1811.06566

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4. If w is a regular element of order d and χ any character we have:
$$\Psi(\chi)(w) = \exp\left(2\pi i\cdot\frac{lc_\chi}{d}\right)\cdot\chi(w)$$

$$\mathrm{FAC}_{W,g}(t) = \frac{e^{t|\mathcal{R}|}}{|W|} \cdot \sum_{\chi\in\widehat{W},\ |g|\,|\,c_\chi} \chi(1)\cdot\chi(g^{-1})\cdot\exp(-t\cdot c_\chi).$$

1. There is a special element $\boldsymbol{\pi} \in P(W) = \pi_1(V^{\text{reg}}, x_0)$ called *full twist*, central in the braid group $B(W)$. It is the geometric circle $[0,1] \ni t \to e^{2\pi i t}\cdot x_0$.
2. Every ζ-regular element w, with $\zeta = e^{2\pi i l/d}$, lifts to a d-th root of $\boldsymbol{\pi}^l$. (i.e. there exists $\boldsymbol{w} \in B(W)$ with $\boldsymbol{w}^d = \boldsymbol{\pi}^l$ and $\boldsymbol{w} \to w$ under $B(W) \to W$)
3. [Broue-Michel] The value of a character χ_x that corresponds to $\chi \in \widehat{W}$ (after Tits' deformation theorem) is given on roots of the full twist by:
$$\chi_x(T_{\boldsymbol{w}}) = \chi(w)\cdot x^{(|\mathcal{R}|+|\mathcal{A}|-c_\chi)l/d}.$$
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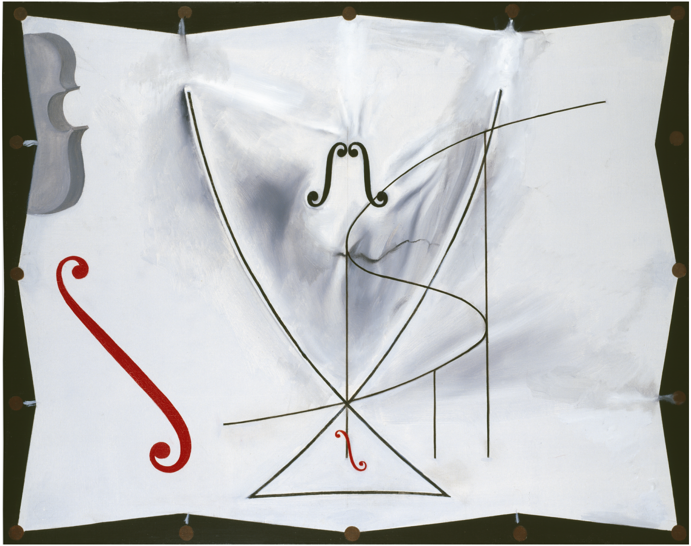
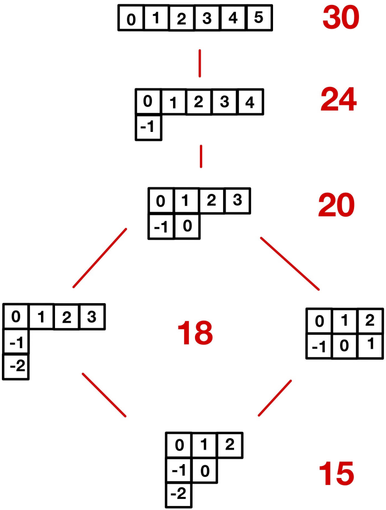
5. If $k = \dfrac{d}{\gcd(c_\chi, d)} \neq 1$, we have $\displaystyle\sum_{i=1}^{k}\Psi^k(\chi)(w) = 0$.

Thank you!



What do the c_χ look like?



Fake degree palindromicity ☕

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Definition

The fake degree $P_\chi(q) := \sum q^{e_i(\chi)}$ of a character $\chi \in \widehat{W}$ is a polynomial that records the *exponents* $e_i(\chi)$ of χ. These are the degrees of the graded components of $\mathrm{co}(W)$ that contain copies of χ.

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Theorem (Beynon-Lusztig, Malle, Opdam)

The fake degrees $P_\chi(q)$ satisfy the following palindromicity property:

$$P_\chi(q) = q^{c_\chi} P_{\Psi(\chi^*)}(q^{-1}),$$

where c_χ are the Coxeter numbers and Ψ is Malle's permutation on $\mathrm{Irr}(W)$.