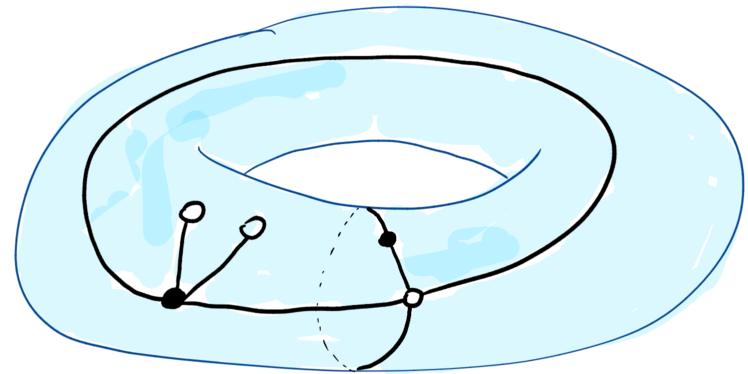
Spin characters and enumeration of maps

Sho MATSUMOTO

Piotr ŚNIADY



"exact formulas for asymptotic problems"

linear

V - linear space

linear representation is a homomorphism

$\rho : S_n \longrightarrow GL(V)$

symmetric group

spin

$P(V)$ - projective space

projective representation is a homomorphism

$\rho : S_n \longrightarrow PGL(V)$

symmetric group

projective representations of S_n

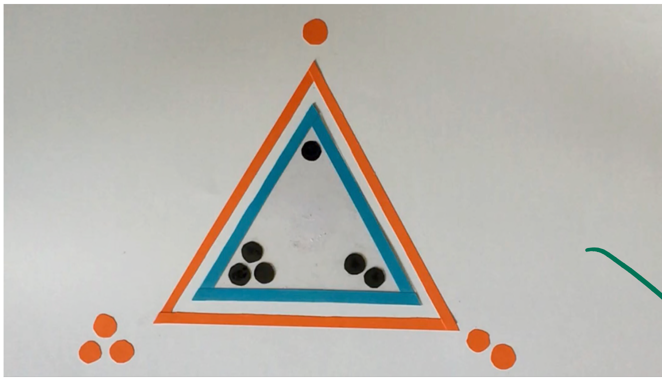
= linear representations of $\widetilde{S}_n$

symmetric group S_n

symmetries of $(n-1)$-simplex

generated by transpositions $t_1, \dots, t_{n-1}$

with $t_i = (i, i+1)$



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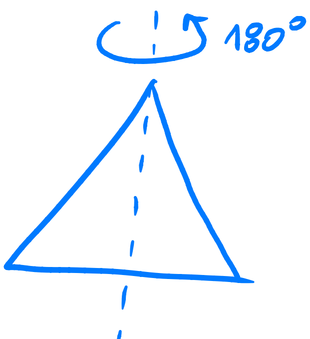
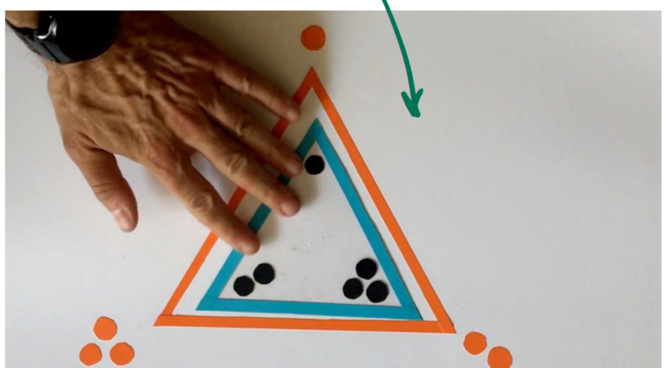




transposition $t_2 = (2,3)$

$t_2^2 = 1$

spin group $\widetilde{S}_n$

rotations of $(n-1)$-simplex $\subseteq \mathbb{R}^n$

with a ribbon attached

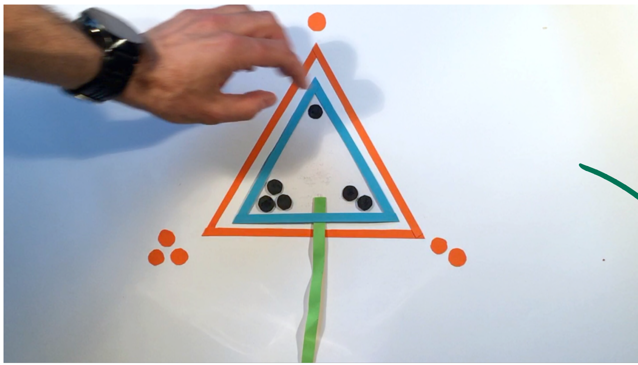
$|\widetilde{S}_n| = 2\,n!$

double cover of S_n

generated by

"transpositions"

$t_1, \dots, t_{n-1}$



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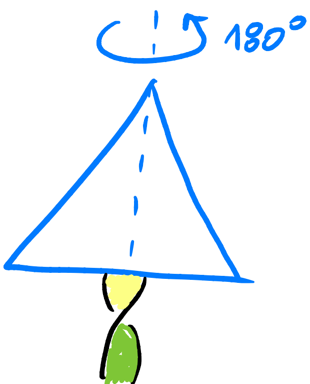
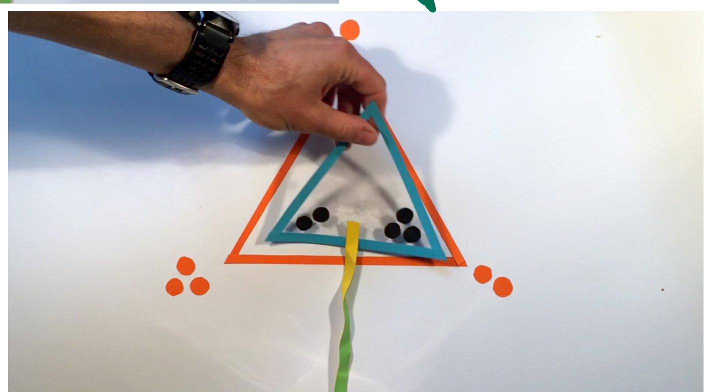




"transposition" $t_2 = (2,3)$

spin group $\tilde{S}_n$

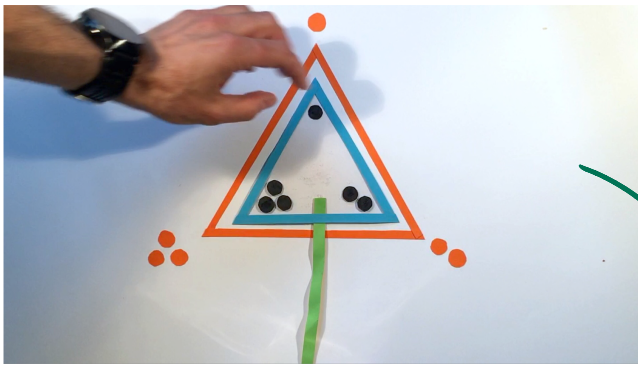
rotations of $(n-1)$-simplex $\subseteq \mathbb{R}^n$

with a ribbon attached

$|\tilde{S}_n| = 2\,n!$

double cover of S_n

generated by "transpositions" $t_1, \ldots, t_{n-1}$



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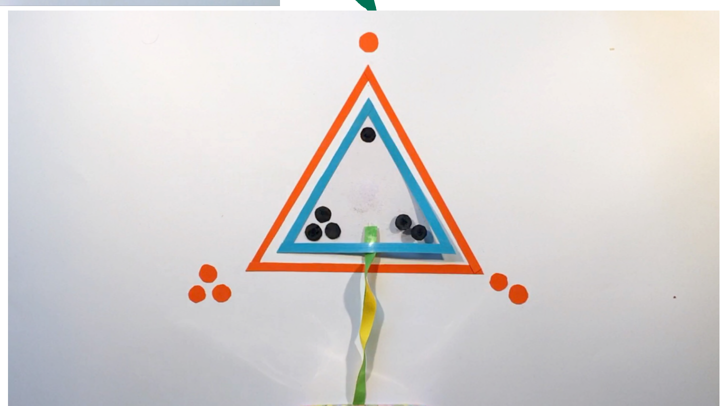
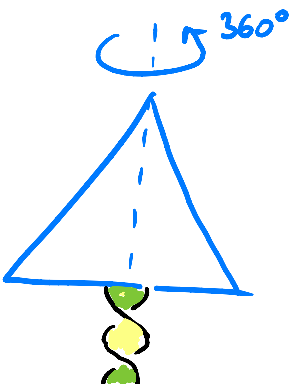




$t_2^2 = Z$

Z = 360° twist of the ribbon

spin group $\widetilde{S}_n$

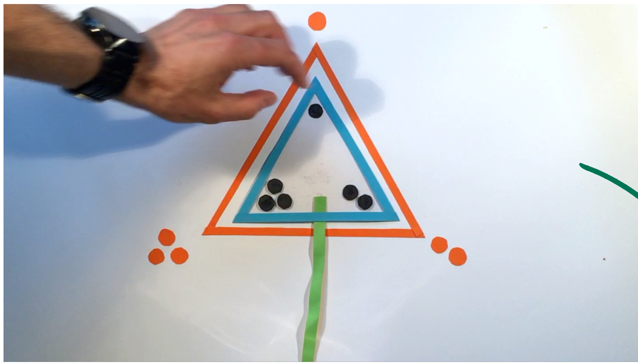
rotations of $(n-1)$-simplex $\subseteq \mathbb{R}^n$

with a ribbon attached

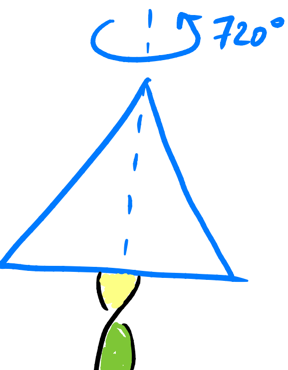
$|\widetilde{S}_n| = 2\,n!$

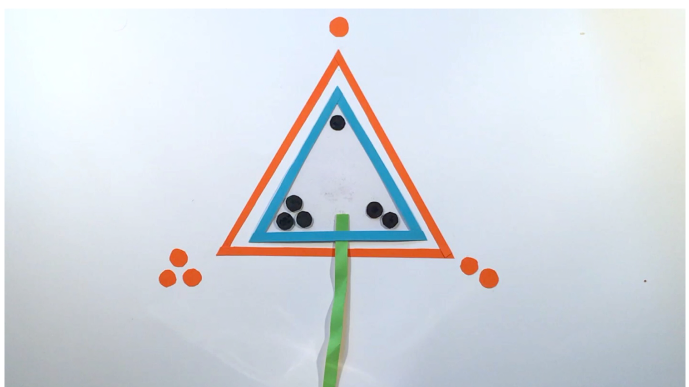
double cover of S_n

generated by "transpositions" $t_1, \dots, t_{n-1}$



[clever trick to untangle the ribbon]





$$t_2^4 = z^2 = 1$$

z = 360° twist of the ribbon

symmetric

conjugacy classes of the symmetric group S_n

are indexed by

partitions of n

$$\pi = (\pi_1 \geqslant \cdots \geqslant \pi_\ell)$$

$$\pi_1 + \cdots + \pi_\ell = n$$

spin

interesting* conjugacy classes of the spin group $\widetilde{S}_n$

are indexed by

odd partitions of n

$$\pi = (\underbrace{\pi_1 \geqslant \cdots \geqslant \pi_\ell}_{\in \{1,3,5,\dots\}})$$

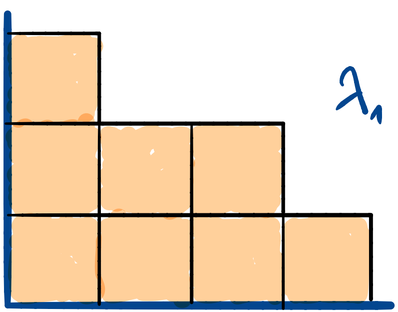
$$\pi_1 + \cdots + \pi_\ell = n$$

symmetric

irreducible representations of S_n

are indexed by

Young diagrams with n boxes



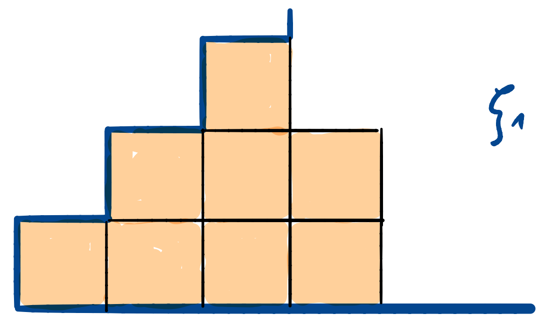
$\lambda_1 \geq \cdots \geq \lambda_\ell$

spin

irreducible representations of $\widetilde{S}_n$

are almost* indexed by

shifted Young diagrams with n boxes



$\xi_1 > \cdots > \xi_\ell$

symmetric

normalized irreducible character of S_n

"fix conjugacy class, function on all Young diagrams"

$$\mathrm{Ch}_{\pi}(\lambda) = \begin{cases} 0 & \text{if } n<k \\ n^{\downarrow k}\,\dfrac{\mathrm{Tr}\,\rho^{\lambda}(\pi, 1^{n-k})}{\mathrm{Tr}\,\rho^{\lambda}(1^{n})} & \text{if } n \geqslant k \end{cases}$$

$|\lambda| = n$ Young diagram

$|\pi| = k$ "conjugacy class"

spin

$\widetilde{S}_n$

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \left\{ \boxed{\text{CENSORED}} \right.$$

$|\xi| = n$ shifted Young diagram

$|\pi| = k$ odd partition

alternative definition

for any odd partition $\pi \vdash k$

Ch^{spin}_{π} is the unique symmetric function $F(x_1, \dots)$ s.t. :

- $F \in \mathbb{C}[p_1, p_3, p_5, \dots]$ "F is supersymmetric"

- $F(\xi) = 0$ for $\xi_1 > \dots > \xi_\ell$ $|\xi| < k$

- F is of degree k and

  $[\text{homogeneous top-degree part}]\, F = p_\pi$

alternative definition

→ Hall-Littlewood symmetric functions for $t=-1$

→ Schur P-functions

$$p_\pi = \sum_\xi X^\xi(\pi)\, P_\xi$$

$$\mathrm{Ch}^{\mathrm{spin}}_\pi(\xi) = \boxed{\text{MAGIC NORMALIZATION CENSORED}}\; X^\xi(\pi,1,\dots,1)$$

problem:

$$\mathrm{Ch}_\pi^{\mathrm{spin}}(\xi) \overset{?}{=} \sum_{\mathcal{G}} N_{\mathcal{G}}(\xi)$$

?

?

problem :

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) \overset{?}{=} \sum_{\mathcal{G}} N_{\mathcal{G}}(\xi)$$

?

graph on a surface

$$\sigma_1 = \underbrace{(1,5,4,2)}_{V}\ \underbrace{(3)}_{W}$$

$$\sigma_2 = \underbrace{(2,3,5)}_{\Pi}\ \underbrace{(1,4)}_{\Sigma}$$

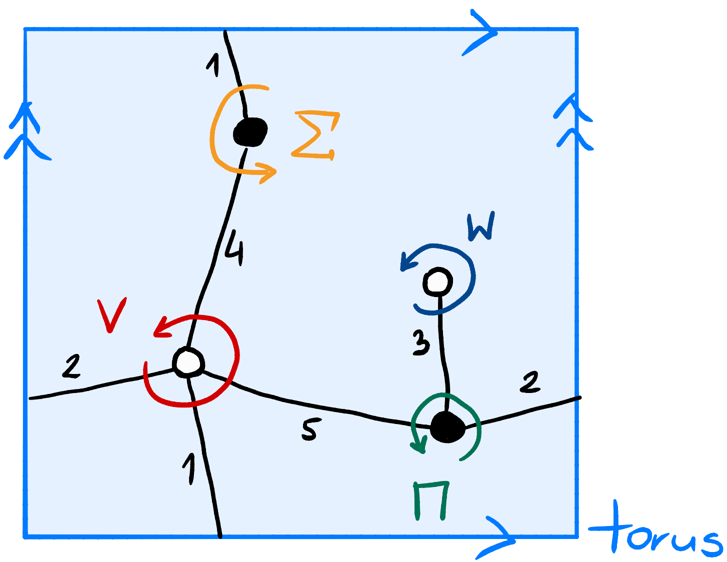
pair of permutations (σ_1, σ_2)

$$\pi = \sigma_1\,\sigma_2 = (1,2,3,4,5)$$

$$\updownarrow$$

bicolored graph G on a (not connected?) oriented surface with face-type π

"oriented map"



problem:

"number of colorings"

$$Ch^{spin}_{\pi}(\xi) \stackrel{?}{=} \sum_{G} N_{G}(\xi)$$

?

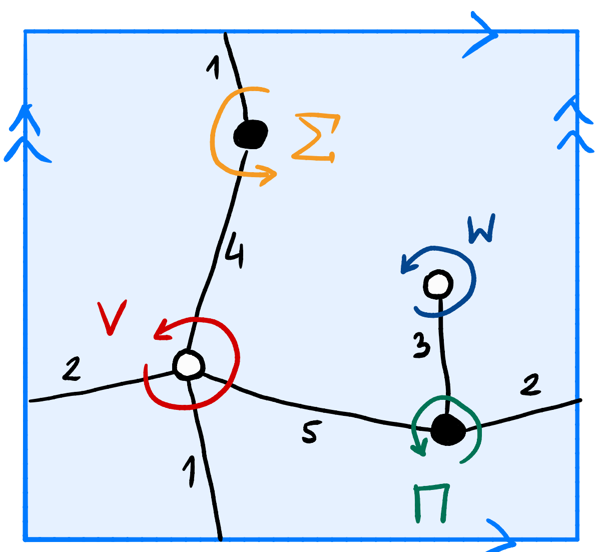
$$\sigma_1 = \underbrace{(1,5,4,2)}_{V}\underbrace{(3)}_{W}$$

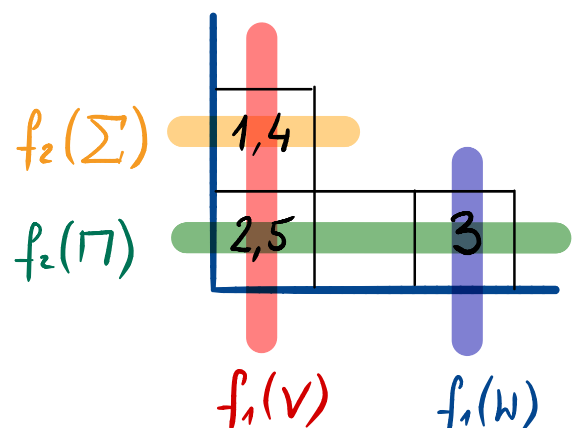
$$\sigma_2 = \underbrace{(2,3,5)}_{\Pi}\underbrace{(1,4)}_{\Sigma}$$

(f_1, f_2) is a **λ-coloring of (σ_1, σ_2)** if

- f_1 maps cycles of σ_1 to columns,
- f_2 maps cycles of σ_2 to rows,
- $\forall$ c_1 - cycle of σ_1, c_2 - cycle of σ_2

$$c_1 \cap c_2 \neq \emptyset \implies \big(f_1(c_1), f_2(c_2)\big) \in \lambda$$





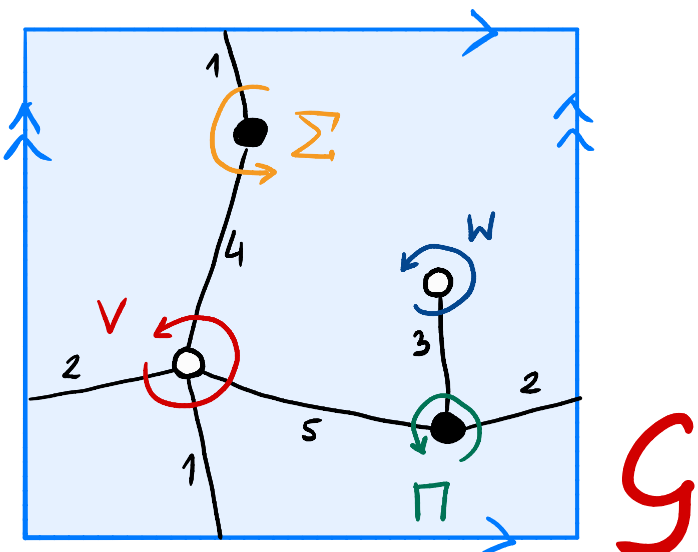
$$\sigma_1 = \underbrace{(1,5,4,2)}_{V}\underbrace{(3)}_{W}$$

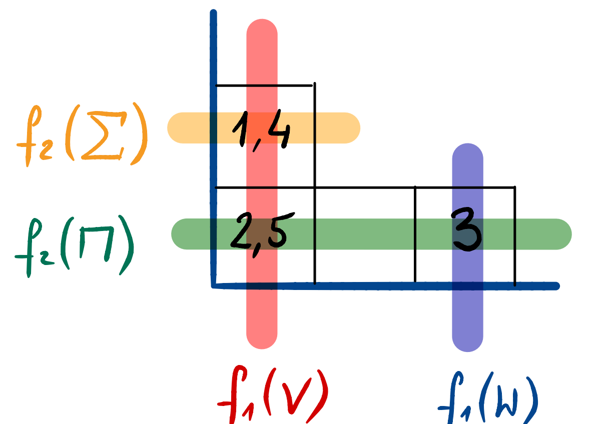
$$\sigma_2 = \underbrace{(2,3,5)}_{\Pi}\underbrace{(1,4)}_{\Sigma}$$

(f_1, f_2) is a λ-coloring of $\mathcal{G}$ if

- f_1 maps white vertices to columns,
- f_2 maps black vertices to rows,
- $\forall$ w - white vertex, b - black vertex

w, b connected by an edge $\Rightarrow$ $(f_1(c_1), f_2(c_2)) \in \lambda$





symmetric Stanley formula

for partition π $\quad |\pi| = k$

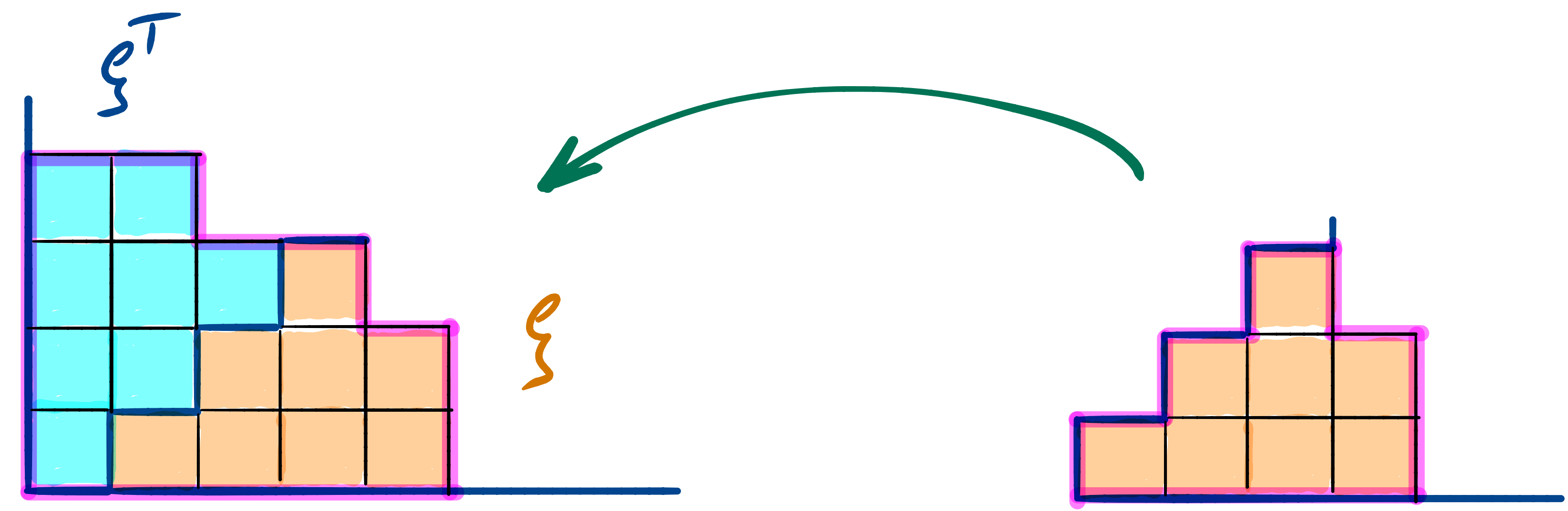
and Young diagram λ

$$\mathrm{Ch}_\pi(\lambda) = \underbrace{\sum_{\substack{\sigma_1,\sigma_2 \in S_k \\ \sigma_1\sigma_2 = \pi}}}_{\text{"sum over oriented maps with face-type } \pi\text{"}} (-1)^{\sigma_1} \underbrace{N_{\sigma_1,\sigma_2}(\lambda)}_{\text{number of } \lambda\text{-colorings of } (\sigma_1,\sigma_2)}$$

identify partition π with some permutation $\pi \in S_k$

symmetric

spin



Young diagram $\lambda = D(\xi)$

double

shifted Young diagram ξ

~~symmetric~~ spin Stanley formula for odd partition π $|\pi| = k$

and shifted Young diagram ξ

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \underbrace{\sum_{\substack{\sigma_1, \sigma_2 \in S_k \\ \sigma_1 \sigma_2 = \pi}}}_{\text{"sum over oriented maps with face-type } \pi\text{"}} \frac{1}{2^{|\sigma_1 \vee \sigma_2|}} (-1)^{\sigma_1} \underbrace{N_{\sigma_1, \sigma_2}(D(\xi))}_{\text{number of } D(\xi)\text{-colorings of } (\sigma_1, \sigma_2)}$$

number of orbits in $\{1, \dots, k\}$ under action of $\langle \sigma_1, \sigma_2 \rangle$ = # connected components of the map

two equivalent formulas

PREVIOUS SLIDE:

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \sum_{\substack{\text{oriented} \\ \text{maps with} \\ \text{face-type } \pi}} \frac{1}{2^{\#\text{connected components}}} (-1)^{n-\#\text{white vertices}} \mathcal{N}_{\mathcal{G}}\big(D(\xi)\big)$$

NEW:

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \frac{1}{2^{\ell(\pi)}} \sum_{\substack{\text{non-oriented} \\ \text{but orientable maps} \\ \text{with face-type } \pi}} (-1)^{n-\#\text{white vertices}} \mathcal{N}_{\mathcal{G}}\big(D(\xi)\big)$$

proof part 1

define $\widetilde{Ch}_\pi(\xi) := \frac{1}{2} Ch_\pi(D(\xi))$ "linear character exported to spin world"

---

fun fact

$a, b, c, \dots \in \{1, 3, 5, \dots\}$

symmetric spin

$$\widetilde{Ch}_a = Ch^{spin}_a$$

"split to at most two groups"

$$\widetilde{Ch}_{a,b} = Ch^{spin}_{a,b} + Ch^{spin}_a \cdot Ch^{spin}_b$$

$$\widetilde{Ch}_{a,b,c} = Ch^{spin}_{a,b,c} + Ch^{spin}_a \cdot Ch^{spin}_{b,c} + Ch^{spin}_b \, Ch^{spin}_{a,c} + Ch^{spin}_c \, Ch^{spin}_{a,b}$$

proof part 2

use Möbius invertion:

Hint: $(-1)^k \cdot (2k-1)!!$ for $k = \#$blocks

$$\mathrm{Ch}^{spin}_{a} = \widetilde{\mathrm{Ch}}_{a}$$

$$\mathrm{Ch}^{spin}_{a,b} = \widetilde{\mathrm{Ch}}_{a,b} - \widetilde{\mathrm{Ch}}_{a} \cdot \widetilde{\mathrm{Ch}}_{b}$$

$$\mathrm{Ch}^{spin}_{a,b,c} = \widetilde{\mathrm{Ch}}_{a,b,c} - \widetilde{\mathrm{Ch}}_{a} \widetilde{\mathrm{Ch}}_{b,c} - \widetilde{\mathrm{Ch}}_{b} \widetilde{\mathrm{Ch}}_{a,c} - \widetilde{\mathrm{Ch}}_{c} \widetilde{\mathrm{Ch}}_{a,b} + 3\, \widetilde{\mathrm{Ch}}_{a} \widetilde{\mathrm{Ch}}_{b} \widetilde{\mathrm{Ch}}_{c}$$

now apply symmetric Stanley formula & magic cancellations $\square$

Homework: find better proof

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \sum_{\substack{\sigma_1, \sigma_2 \in S_k \\ \sigma_1 \sigma_2 = \pi}} \frac{1}{2^{|\sigma_1 \vee \sigma_2|}} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2}(D(\xi))$$

Hint: $\mathrm{Ch}^{\mathrm{spin}}_{\pi}$ is the unique symmetric function $F(x_1, \dots)$ s.t.:

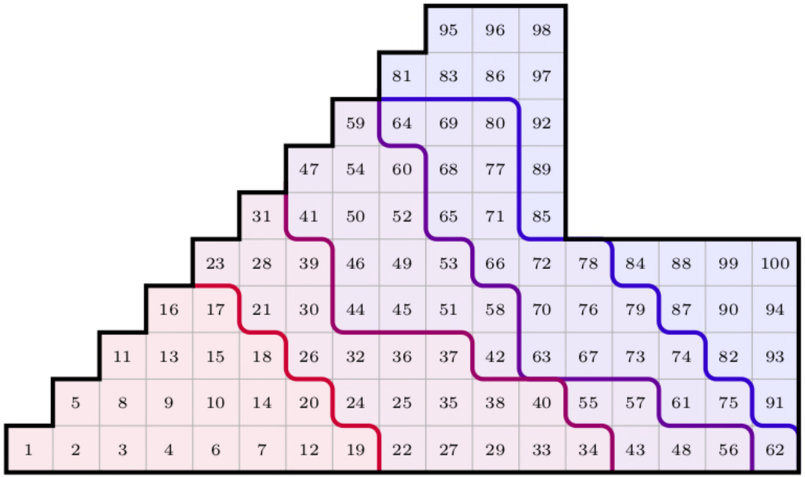
✓ • $F \in \mathbb{C}[p_1, p_3, p_5, \dots]$ "F is supersymmetric"

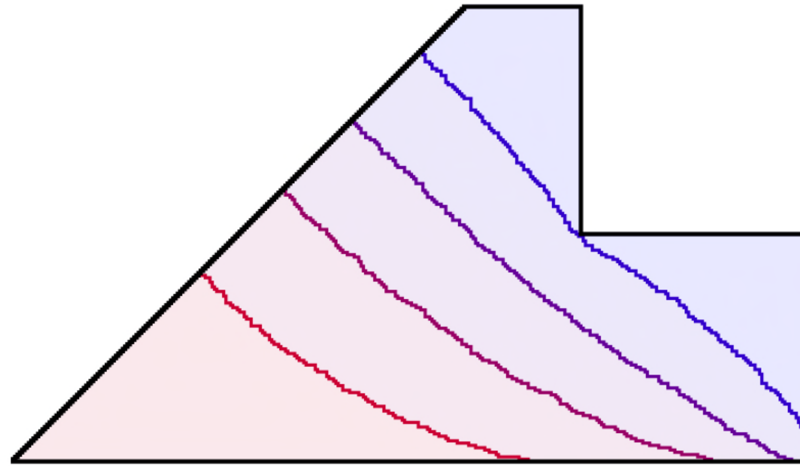
? ✗ • $F(\xi) = 0$ for $\xi_1 > \dots > \xi_\ell$ $|\xi| < k$

✓ • F is of degree k and

[homogeneous top-degree part] $F = p_\pi$

Application





random shifted Young diagrams

random shifted tableaux

Summer School on Algebraic Combinatorics

Cracow, Poland

July 6-10, 2020

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