Restricting Schubert classes to symplectic Grassmannians using self-dual puzzles

Iva Halacheva (The University of Melbourne)

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Joint work with: A. Knutson (Cornell), P. Zinn-Justin (Melbourne)

Grassmannians and Schubert calculus

Grassmannians We are interested in the cohomology pullback of

$$Gr(k,2n) := \{V \subseteq \mathbb{C}^{2n} \mid \dim V = k\} \cong GL_{2n}/P$$

$$\uparrow \iota$$

$$SpGr(k,2n) := \{V \subseteq \mathbb{C}^{2n} \mid \dim V = k,\ V \subseteq V^{\perp}\} \cong Sp_{2n}/(P \cap Sp_{2n})$$

General setup: partial flag varieties

- G an algebraic group over $\mathbb{C}$
- fix a pinning $T \subset B \subset G$, $W = W_G = N(T)/T$,
- $P \supset B$ be a parabolic, then G/P is a smooth projective variety,
- T-fixed points: $(G/P)^T \cong W/W_P \cong W_P\backslash W$.

For G of type $A/B/C/D$ and P maximal, G/P is a **Grassmannian**.

Our case: Take $\sigma : GL_{2n} \to GL_{2n}$, $X \mapsto J^{-1}(X^{-1})^{\mathrm{tr}}J$, (involution) where J = antidiagonal matrix with n (-1)'s in NE, n 1's in SW. Take $Sp_{2n} = GL_{2n}^{\sigma}$, $P = P_{GL_{2n}}$ parabolic of type $(k, 2n-k)$, $(k < n)$.

Schubert classes For $\pi \in W_P\backslash W$, the corresp. **Schubert class** is

$$S_\pi := \left[\overline{B_-\pi^{-1}P/P}\right] \in H_T^*(G/P).$$

Then $\{S_\pi\}_{\pi \in W_P\backslash W}$ freely generate $H_T^*(G/P)$ as an $H_T^*(\mathrm{pt})$-module.

Classical question: Determine the structure constants,

$$S_\lambda \cdot S_\mu = \sum_\nu c_{\lambda\mu}^\nu S_\nu$$

Note: if $G/P = Gr(k,n)$, then (in H^*, not H_T^*) the $c_{\lambda\mu}^\nu$ are the *Littlewood-Richardson coefficients* for $GL_k : V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$

E.g. In $Gr(2,4)$, $(H_T^*(pt) \cong \mathbb{Z}[y_1,y_2,y_3,y_4])$:

$$S_{\square} \cdot S_{\square} = S_{\square\square} + S_{\substack{\square\\\square}} + (y_2 - y_3)S_{\square} \qquad (\text{in } H_T^*)$$

Cohomology rings Consider the *involution* $\lambda \mapsto \overline{\lambda}$ reversing λ and switching $0 \leftrightarrow 1$. For $\widetilde{\iota}(\nu) := (\nu\overline{\nu}$ with 10's turned into 1's), we have

$$\begin{array}{ccc} \{\nu \in (10)^{n-k}\{0,1\}^k\} \cong (SpGr(k,2n))^{T^n} & \xrightarrow{f_2} & SpGr(k,2n) \\ \downarrow \widetilde{\iota} & & \downarrow \iota \\ \{\lambda \in 0^k1^{2n-k}\} \cong (Gr(k,2n))^{T^n} & \xrightarrow{f_1} & Gr(k,2n) \end{array}$$

This, together with the inclusion $T^n = T^\sigma \hookrightarrow T = T^{2n}$, gives

$$\begin{array}{ccc} H_{T^n}^*(SpGr(k,2n)^{T^n}) & \xleftarrow{f_2^*} & H_{T^n}^*(SpGr(k,2n)) \\ \uparrow (\widetilde{\iota})^* & & \uparrow \iota^* \\ H_{T^n}^*(Gr(k,2n)^{T^n}) & \xleftarrow{f_1^*} & H_{T^n}^*(Gr(k,2n)) \end{array}$$

and since each f_i^* is injective (Kirwan), we can compute on the left.

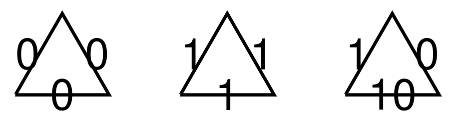
Main question: $\boxed{\iota^*(S_\lambda) = \sum_\nu c_\nu^\lambda S_\nu}$ $\quad c_\nu^\lambda = ??$

- Pragacz '00: (building on work of Stembridge) positive tableau formulæ for $H^*(Gr(n,2n))$
- Coşkun '11: positive geometric rule for $H^*(Gr(k,2n))$

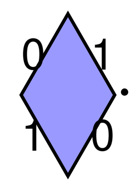
Puzzles

Note: We interchangeably consider binary strings $\pi \in 0^k1^{2n-k}$ (i.e. in $W_P\backslash W$) and $\pi^{-1} \in W/W_P$.

Puzzles A puzzle of size $2n$, $\triangle_{\nu}^{\lambda\,\mu}$, for $\lambda,\mu,\nu \in 0^k1^{2n-k}$ is a tiling by the puzzle pieces



, their rotations, and the equivariant piece



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Theorem (Knutson-Tao '03, many extensions since)

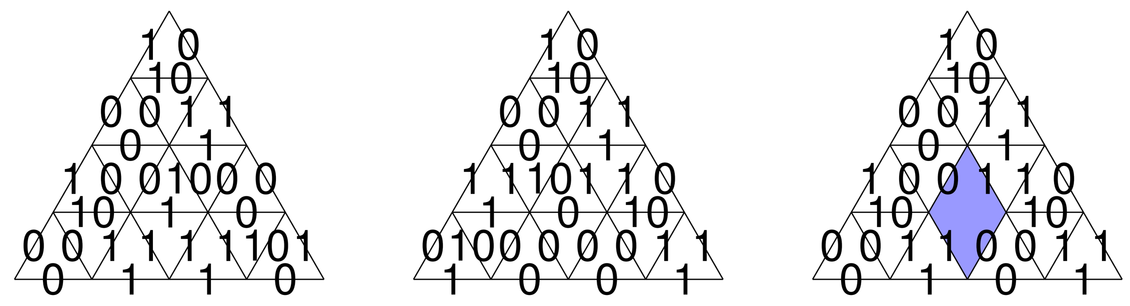
For $\lambda,\mu \in 0^k1^{2n-k}$, the product of S_λ and S_μ in $H_T^*(Gr(k,2n))$ is given by

$$S_\lambda \cdot S_\mu = \sum_{\nu \in 0^k1^{2n-k}} \left(\sum_{\mathbf{P}} \left\{ v(\mathbf{P}) \ : \ \text{puzzles } \mathbf{P} \text{ with boundary } \triangle_{\nu}^{\lambda\,\mu} \right\} \right) S_\nu$$

where $v(\mathbf{P}) = \prod_{p \in \mathbf{P}} v(p)$, and for the individual pieces

- $v(\triangle) = 1$,
- $v(\lozenge) = y_i - y_j \in \mathbb{Z}[y_1,\ldots,y_{2n}] \cong H_T^*(pt)$, if the rhombus sides face the i-th and j-th positions at the bottom of the puzzle.

Example: $S_{0101} \cdot S_{0101} = S_{0110} + S_{1001} + (y_2 - y_3)S_{0101}$



Grassmann duality

There is a ring isomorphism (from a homeo. of Grassmannians):

$$H_T^*(Gr(k,2n)) \cong H_T^*(Gr(2n-k,2n)), \quad S_\lambda \mapsto S_{\overline{\lambda}}$$

$$S_\lambda \cdot S_\mu \leftrightarrow S_{\overline{\mu}} \cdot S_{\overline{\lambda}}$$

$\triangle_{\nu}^{\lambda\,\mu} \xleftrightarrow{Dual} \triangle_{\overline{\nu}}^{\overline{\mu}\,\overline{\lambda}} :=$ reflect though vertical axis and swap 0 and 1

Question: What do self-dual puzzles count?

Main Theorem

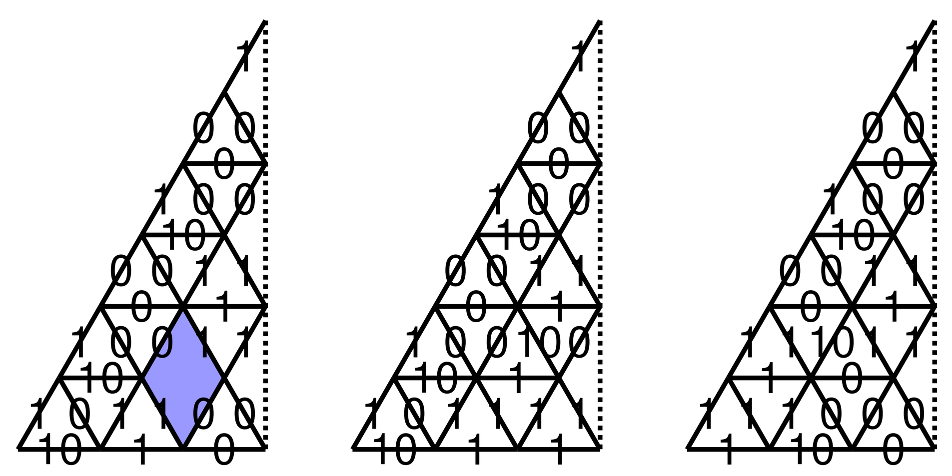
Theorem (H-Knutson-Zinn-Justin '18)

For every $S_\lambda \in H_{T^n}^*(Gr(k,2n))$, $\lambda \in 0^k1^{2n-k}$, we get in $H_{T^n}^*(SpGr(k,2n))$

$$\iota^*(S_\lambda) = \sum_{\nu \in (10)^{n-k}\{0,1\}^k} \left(\sum_{\mathbf{P}} \{ v(\mathbf{P}) \ : \ \text{self-dual puzzles } \mathbf{P} \text{ with boundary } \lambda, \nu \} \right) S_\nu,$$

where $((i,j) - \text{rhombus}) \overset{v}{\mapsto} z_i - z_j$, $z_i = y_i$, or $-y_{2n+1-i}$ if $i \leq n$, or $n+1 \leq i \leq 2n$.

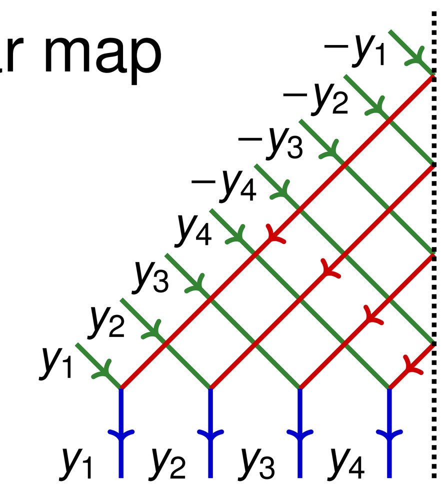
Example: $\iota^*(S_{110101}) = (y_2 - y_3)S_{10,1,0} + S_{10,1,1} + S_{1,10,0}$



Idea of proof

Tensor calculus Consider the puzzle labels $\{0, 10, 1\}$ as indexing bases for three spaces $\mathbb{C}_G^3, \mathbb{C}_R^3, \mathbb{C}_B^3$ (Green, Red, Blue).

We get a *scattering diagram* by taking the graph dual of an **unlabeled** size $2n$ half-puzzle triangle $2n$ with assigned "spectral parameters" $y_1,\ldots,y_n,-y_n,\ldots,-y_1$ on the Northwest side. Associate

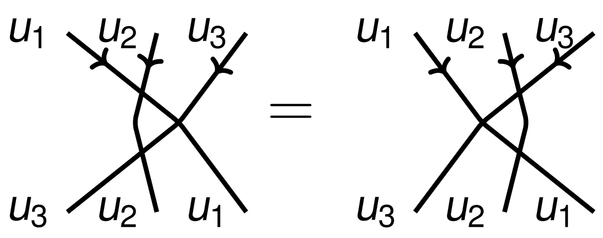


- to a crossing with parameters a and b, a linear map $R_{CD}(a-b) = \overset{C\ \ D}{\times} : \mathbb{C}_C^3 \otimes \mathbb{C}_D^3 \longrightarrow \mathbb{C}_D^3 \otimes \mathbb{C}_C^3$;
- to a wall-bounce of a strand with parameter a, $K_C(a) = \overset{C}{\underset{D}{\rangle}} : \mathbb{C}_C^3 \to \mathbb{C}_D^3$, (and $a \mapsto -a$);
- to a trivalent vertex with both parameters a, $U(a) = \curlyvee : \mathbb{C}_G^3 \otimes \mathbb{C}_R^3 \longrightarrow \mathbb{C}_B^3$.

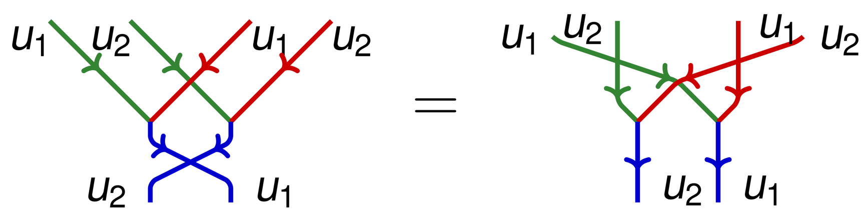
Gluing strands corresponds to composing the maps, so the whole $2n\triangle$ corresponds to a linear map $\Phi : (\mathbb{C}^3_G)^{\otimes 2n} \longrightarrow (\mathbb{C}^3_B)^{\otimes n}$.

We ask that these maps satisfy the following identities:

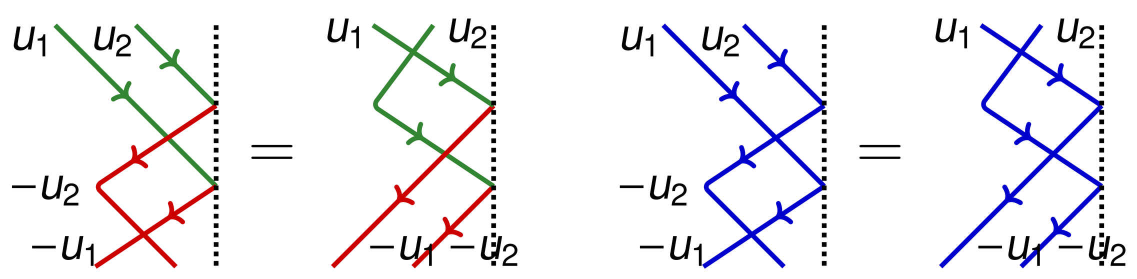
1. The Yang-Baxter equation. (With colors GRR/GGR/GGG/BBB)



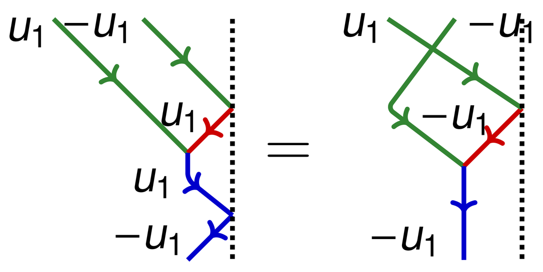
2. Swapping of two trivalent vertices.



3. The reflection equation.



4. K-fusion. (Is this a known equation?)



$$K_B(u_1) \circ U_{GR}(u_1) \circ (\mathrm{Id} \otimes K_G(-u_1)) = U_{GR}(-u_1) \circ (\mathrm{Id} \otimes K_G(u_1)) \circ R_{GG}(2u_1)$$

Puzzle values: Let $\mathbf{P}$ range among all (self-dual) half-puzzles with labels $\lambda\!\!\!\diagup_{\nu}$, where $\lambda \in 0^k 1^{2n-k}$ and $\nu \in (10)^{n-k}\{0,1\}^k$. Then,

$$\boxed{\text{the } (\nu,\lambda) \text{ matrix entry of } \Phi = \textstyle\sum_{\mathbf{P}} v(\mathbf{P}) \quad \left(\overset{\text{Main}}{\underset{\text{Theorem}}{=}} c_\nu^\lambda\right)}.$$

The AJS/Billey formula: Computing restriction to fixed points

Proposition (AJS/Billey using scattering diagrams)

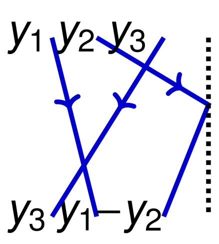
Let $(\lambda, \mu =$ strings in $0, 10, 1 \leftrightarrow$ cosets $W_P\backslash W)$, W of type A/C, P maximal. To compute $S_\lambda|_\mu$: make a scattering diagram by

- *taking a reduced word for the shortest lift $\widetilde{\mu}^{-1}$*
- *for each crossing, compose with $R_{BB}(= R_{RR})$*
- *(in type C) for each wall-reflection, compose with K_B.*

Then $S_\lambda|_\mu$ is the $(\omega_{G/P}, \lambda)$ matrix entry of the resulting map, where

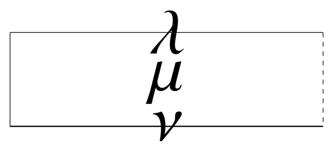
$$\omega_{Gr} = 0\ldots0\,1\ldots1 \in 0^k 1^{2n-k} \quad \text{for } G/P = Gr(k, 2n),$$
$$\omega_{SpGr} = 0\ldots0\,10\ldots10 \in 0^k (10)^{n-k} \quad \text{for } G/P = SpGr(k, 2n).$$

Example For $G = Sp_6$ and $Q = s_2 s_3 s_1$, the scattering diagram is



$$(R_{BB}(y_1 - y_3) \otimes \mathrm{Id}) \circ (Id^{\otimes 2} \otimes K_B(y_2)) \circ (\mathrm{Id} \otimes R_{BB}(y_2 - y_3)) : (\mathbb{C}^3_B)^{\otimes 3} \to (\mathbb{C}^3_B)^{\otimes 3}$$

For $\lambda, \mu, \nu \in W_P\backslash W$ as above, we denote

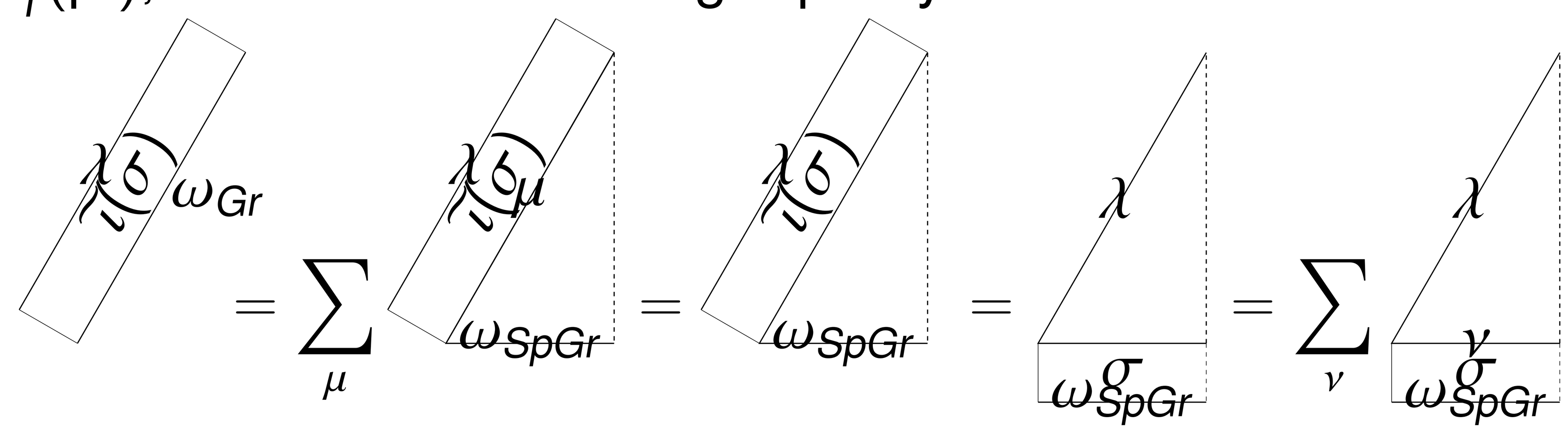


$:=$ the (ν, λ) matrix entry for the scattering diagram map coming from a reduced word for $\widetilde{\mu}^{-1}$.

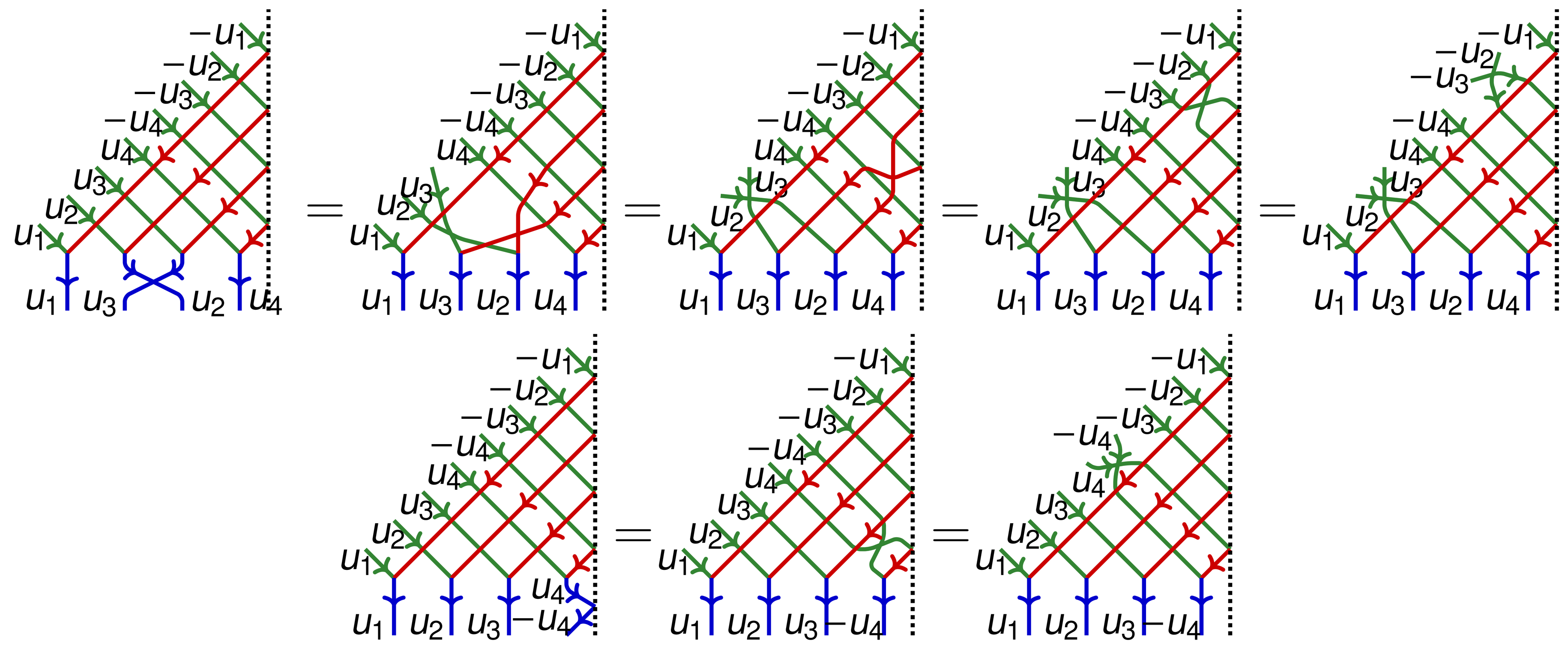
By the proposition, when $\nu = \omega_{G/P}$ this gives $S_\lambda|_\mu$.

Proof of the main theorem.

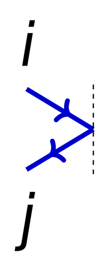
In $H_T^*(\mathrm{pt})$, we have the following equality



In the second and fourth equality, the strings μ and ν have content $0^k 1^{2n-k}$ and $(10)^{n-k}\{0,1\}^k$ respectively, and all other terms of the sum vanish.



Appendix: Example of a K-matrix.



$$K_B(a) : \;\; \overset{i}{\underset{j}{\;}} = \begin{cases} 1 & \text{if } i = j, \\ 2a & \text{if } (i,j) = (1,0) \end{cases}$$

Thank You!

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