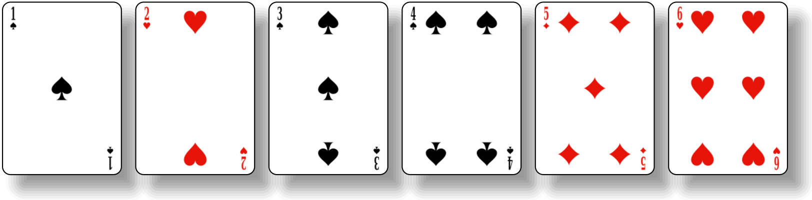
Eigenvalues of symmetrized shuffling operators

Nadia Lafrenière

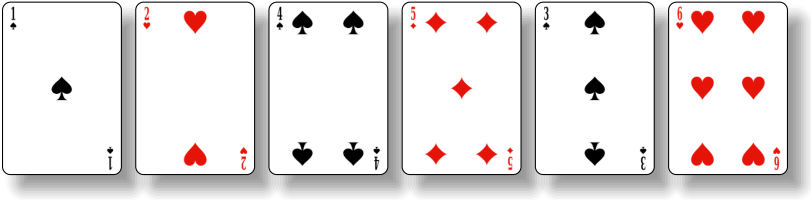
Université du Québec à Montréal

FPSAC 2019

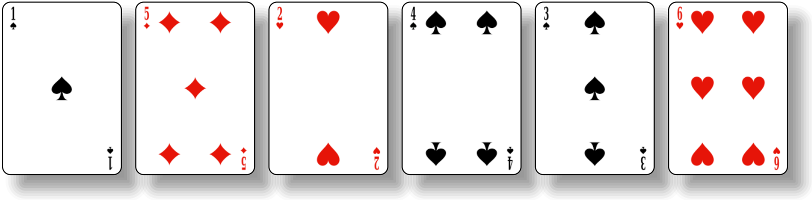
Pick a card—any card!



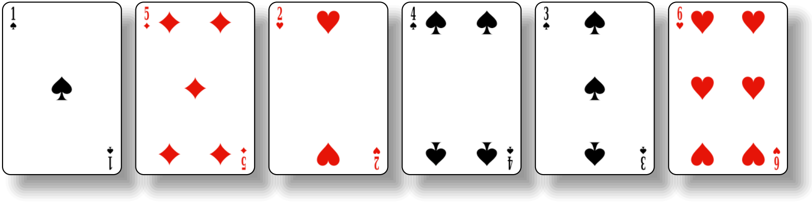
Pick a card—any card!



Pick a card—any card!



Pick a card—any card!



The random-to-random shuffle

The *random-to-random* shuffle on a deck of cards is defined as the action of taking any card randomly, with uniform probability, and placing it back anywhere, with uniform probability.

The random-to-random shuffle

What if we move more cards?

2 possible ways.

We move k cards and reinsert them in *the same order.*

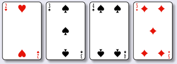
We move k cards, and reinsert them in the deck *after shuffling them.*

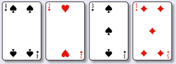
The operator ν_k removes k card from the deck and reinsert them one after the other, without necessarily preserving their order.

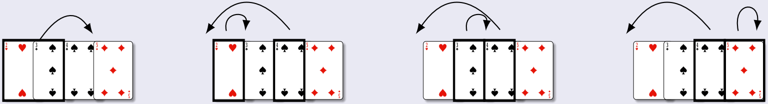
Symmetrized shuffling operators

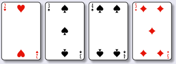
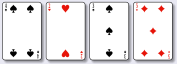
The operator ν_k removes k cards from the deck of cards and reinsert them one after the other, without necessarily preserving their order.

Example

Consider the sequence  *. The operator* ν_2 *acts on it by moving two cards, and the result is the linear combination of the possible results.*

For example, one can obtain  *in four different ways:*



Thus, $\nu_2($$) = \ldots + 4 \cdot$  $+ \ldots$

Symmetrized shuffling operators

Example

One can express the linear operators $\{\nu_k\}_{k\in\mathbb{N}}$ *as matrices:*

$$\nu_1 = \begin{array}{c} \\ \text{ABC} \\ \text{ACB} \\ \text{BAC} \\ \text{BCA} \\ \text{CAB} \\ \text{CBA} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{ABC} & \text{ACB} & \text{BAC} & \text{BCA} & \text{CAB} & \text{CBA} \end{array} \\ \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 & 0 & 1 \\ 1 & 0 & 2 & 3 & 1 & 2 \\ 1 & 2 & 0 & 1 & 3 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 \end{pmatrix} \end{array}.$$

Random walk properties and the transition matrix

Shuffling operators are random walks, and their properties can be translated in terms of Markov chains.

Questions about random walk	**Algebraic properties of transition matrix**
probability after m steps?	entries of T^m
long-term behaviour? (limiting distribution)	eigenvectors $\vec{v}$ s.t. $\vec{v}\,\mathsf{T} = \vec{v}$
rate of convergence to limiting distribution?	controlled by the eigenvalues of T

Slide courtesy of Franco Saliola

A memory of FPSAC 2009

In FPSAC '09, Volkmar Welker presented the following conjecture:

Conjecture (Reiner, Saliola, Welker)

The eigenvalues of the symmetrized shuffling operators are real, nonnegative and integers.

Problem

The number of states is very large. We cannot compute with the usual algorithms the eigenvalues of the operators.

The solution: use the representation theory of the symmetric group.

How to compute the eigenvalues?

Our operators act on the algebra of the symmetric group, $\mathbb{C}S_n$.

We divide $\mathbb{C}S_n$ into subspaces that are stable for the action of any shuffling operators. They are called *submodules.*

The simple submodules of $\mathbb{C}S_n$ are the *Specht modules* :

$$\begin{aligned}\mathbb{C}S_n &\cong \bigoplus_{\lambda \vdash n} f^\lambda S^\lambda \\ &\cong \bigoplus_{\substack{t \text{ a standard} \\ \text{tableau}}} S^{\mathrm{shape}(t)}\end{aligned}$$

where f^λ is the number of standard Young tableaux of shape λ.

Standard Young Tableaux $\leftrightarrow$ Copies of simple modules

How to compute the eigenvalues?

Schur's lemma: A homomorphism from a simple module to itself is a multiple of the identity.

To each copy of simple modules, one can associate exactly one eigenvalue.

How to compute the eigenvalues?

Standard Young Tableaux ↔ Copies of simple modules

$+$

To each copy of simple modules, one can associate exactly one eigenvalue.

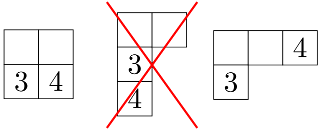
$=$

To each standard Young tableau, one can associate exactly one eigenvalue.

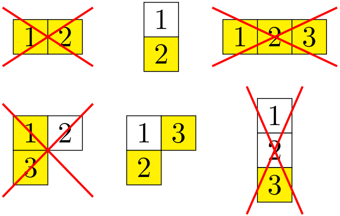
When the eigenvalues are 0

Some standard Young tableaux are always associated with the eigenvalue 0. Those are the *desarrangement tableaux*, due to Désarménien and Wachs.

An *ascent* in a standard Young tableau is either the largest entry or an entry i such that $i+1$ is located to the North-East of i.



A desarrangement tableau is a tableau with its first ascent even.



Theorem (Reiner, Saliola, Welker, 2014)

The eigenvalue of any symmetrized shuffling operator associated with a desarrangement tableau is 0.

Eigenvalues for other tableaux

Theorem (Branching Rule)

If $\lambda \vdash n$*, then*

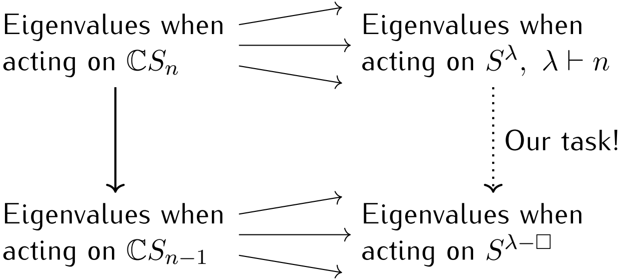
$$S^{\lambda} \downarrow_{S_{n-1}} \cong \bigoplus_{\lambda^-} S^{\lambda^-},$$

where λ^- *is the set of all diagrams obtained from* λ *by removing a cell.*

Example

$$S^{\ydiagram{2,1,1}} \downarrow_{S_3} \cong S^{\ydiagram{2,1}} \oplus S^{\ydiagram{1,1,1}}.$$

The eigenvalues for the Specht modules



Eigenvalues when acting on $\mathbb{C}S_n$

Eigenvalues when acting on $S^{\lambda},\ \lambda \vdash n$

Our task!

Eigenvalues when acting on $\mathbb{C}S_{n-1}$

Eigenvalues when acting on $S^{\lambda-\square}$

Jeu-de-taquin: Δ operator

1	2	5
3	4	7
6		

- Take a standard Young tableau

Jeu-de-taquin: Δ operator

•	2	5
3	4	7
6		

- Take a standard Young tableau
- Remove the 1 entry
- Execute jeu-de-taquin moves to slide the empty box to the border of the diagram

Jeu-de-taquin: Δ operator

2	•	5
3	4	7
6		

- Take a standard Young tableau
- Remove the 1 entry
- Execute jeu-de-taquin moves to slide the empty box to the border of the diagram

Jeu-de-taquin: Δ operator

2	4	5
3	•	7
6		

- Take a standard Young tableau
- Remove the 1 entry
- Execute jeu-de-taquin moves to slide the empty box to the border of the diagram

Jeu-de-taquin: Δ operator

2	4	5
3	7	•
6		

- Take a standard Young tableau
- Remove the 1 entry
- Execute jeu-de-taquin moves to slide the empty box to the border of the diagram

Jeu-de-taquin: Δ operator

2	4	5
3	7	
6		

- Take a standard Young tableau
- Remove the 1 entry
- Execute jeu-de-taquin moves to slide the empty box to the border of the diagram
- Replace the values $2, \dots, n$ by $1, \dots, n-1$

Jeu-de-taquin: Δ operator

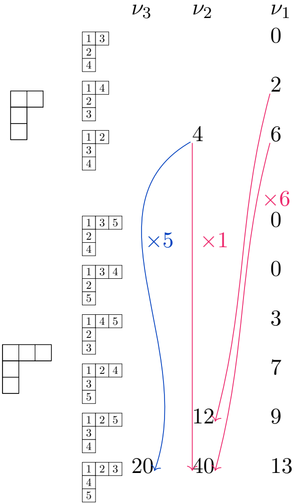
1	3	4
2	6	
5		

- Take a standard Young tableau
- Remove the 1 entry
- Execute jeu-de-taquin moves to slide the empty box to the border of the diagram
- Replace the values $2, \ldots, n$ by $1, \ldots, n-1$

The Δ operator associates a standard Young tableau of size n to a standard Young tableau of size $n-1$.

Eigenvalues, in general

Using SageMath, we found the following recursions:



Observation

If t is not a desarrangement tableau, the eigenvalue of ν_k associated with the tableau t comes from the eigenvalues of ν_k and ν_{k-1} associated with $\Delta(t)$.

$$40 = 4 + (5 + 1 - 2 + 3 - 1) \cdot 6$$
$$20 = 0 + (5 + 1 - 3 + 3 - 1) \cdot 4$$

Eigenvalues, in general

Theorem (Lafrenière, 2019)

The eigenvalues of ν_k *are indexed by the standard Young tableaux of size* n*. For a given tableau* t*, the eigenvalue* $v_k(t)$ *is*

- 0*, if there exists* $i < k$ *such that* $\Delta^i(t)$ *is a desarrangement tableau,*
- $$v_k(\Delta(t)) + (n + 1 - k + \lambda_a - a) \cdot v_{k-1}(\Delta(t))$$
  otherwise, where a *is the line in which lies the only cell of* $t/\Delta(t)$*.*

The multiplicity of the eigenvalue for the tableau t *is the number of standard Young tableaux that have the same shape.*

Corollary

All the eigenvalues of ν_k *are integers.*

Computation example

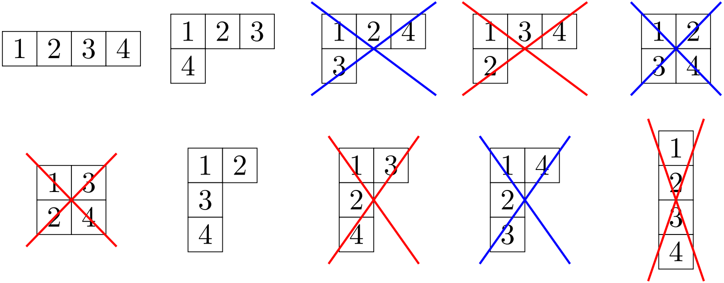
$$\nu_2 = \begin{pmatrix}
6 & 5 & 5 & 4 & 4 & 3 & 5 & 4 & 4 & 3 & 3 & 2 & 4 & 3 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 1 & 1 & 0 \\
5 & 6 & 4 & 3 & 5 & 4 & 4 & 5 & 3 & 2 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 4 & 3 & 3 & 2 & 2 & 1 \\
5 & 4 & 6 & 5 & 3 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 3 & 1 & 0 & 2 & 1 \\
4 & 3 & 5 & 6 & 4 & 5 & 3 & 2 & 2 & 1 & 1 & 0 & 4 & 5 & 3 & 2 & 4 & 3 & 3 & 4 & 2 & 1 & 3 & 2 \\
4 & 5 & 3 & 4 & 6 & 5 & 3 & 4 & 2 & 1 & 3 & 2 & 2 & 3 & 1 & 0 & 2 & 1 & 5 & 4 & 4 & 3 & 3 & 2 \\
3 & 4 & 4 & 5 & 5 & 6 & 2 & 3 & 1 & 0 & 2 & 1 & 3 & 4 & 2 & 1 & 3 & 2 & 4 & 5 & 3 & 2 & 4 & 3 \\
5 & 4 & 4 & 3 & 3 & 2 & 6 & 5 & 5 & 4 & 4 & 3 & 3 & 2 & 4 & 3 & 1 & 2 & 2 & 1 & 3 & 2 & 0 & 1 \\
4 & 5 & 3 & 2 & 4 & 3 & 5 & 6 & 4 & 3 & 5 & 4 & 2 & 1 & 3 & 2 & 0 & 1 & 3 & 2 & 4 & 3 & 1 & 2 \\
4 & 3 & 3 & 2 & 2 & 1 & 5 & 4 & 6 & 5 & 3 & 4 & 4 & 3 & 5 & 4 & 2 & 3 & 1 & 0 & 2 & 3 & 1 & 2 \\
3 & 2 & 2 & 1 & 1 & 0 & 4 & 3 & 5 & 6 & 4 & 5 & 3 & 2 & 4 & 5 & 3 & 4 & 2 & 1 & 3 & 4 & 2 & 3 \\
3 & 4 & 2 & 1 & 3 & 2 & 4 & 5 & 3 & 4 & 6 & 5 & 1 & 0 & 2 & 3 & 1 & 2 & 4 & 3 & 5 & 4 & 2 & 3 \\
2 & 3 & 1 & 0 & 2 & 1 & 3 & 4 & 4 & 5 & 5 & 6 & 2 & 1 & 3 & 4 & 2 & 3 & 3 & 2 & 4 & 5 & 3 & 4 \\
4 & 3 & 5 & 4 & 2 & 3 & 3 & 2 & 4 & 3 & 1 & 2 & 6 & 5 & 5 & 4 & 4 & 3 & 1 & 2 & 0 & 1 & 3 & 2 \\
3 & 2 & 4 & 5 & 3 & 4 & 2 & 1 & 3 & 2 & 0 & 1 & 5 & 6 & 4 & 3 & 5 & 4 & 2 & 3 & 1 & 2 & 4 & 3 \\
3 & 2 & 4 & 3 & 1 & 2 & 4 & 3 & 5 & 4 & 2 & 3 & 5 & 4 & 6 & 5 & 3 & 4 & 0 & 1 & 1 & 2 & 2 & 3 \\
2 & 1 & 3 & 2 & 0 & 1 & 3 & 2 & 4 & 5 & 3 & 4 & 4 & 3 & 5 & 6 & 4 & 5 & 1 & 2 & 2 & 3 & 3 & 4 \\
2 & 1 & 3 & 4 & 2 & 3 & 1 & 0 & 2 & 3 & 1 & 2 & 4 & 5 & 3 & 4 & 6 & 5 & 3 & 4 & 2 & 3 & 5 & 4 \\
1 & 0 & 2 & 3 & 1 & 2 & 2 & 1 & 3 & 4 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 2 & 3 & 3 & 4 & 4 & 5 \\
3 & 4 & 2 & 3 & 5 & 4 & 2 & 3 & 1 & 2 & 4 & 3 & 1 & 2 & 0 & 1 & 3 & 2 & 6 & 5 & 5 & 4 & 4 & 3 \\
2 & 3 & 3 & 4 & 4 & 5 & 1 & 2 & 0 & 1 & 3 & 2 & 2 & 3 & 1 & 2 & 4 & 3 & 5 & 6 & 4 & 3 & 5 & 4 \\
2 & 3 & 1 & 2 & 4 & 3 & 3 & 4 & 2 & 3 & 5 & 4 & 0 & 1 & 1 & 2 & 2 & 3 & 5 & 4 & 6 & 5 & 3 & 4 \\
1 & 2 & 0 & 1 & 3 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 3 & 5 & 6 & 4 & 5 \\
1 & 2 & 2 & 3 & 3 & 4 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 2 & 3 & 5 & 4 & 4 & 5 & 3 & 4 & 6 & 5 \\
0 & 1 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 3 & 4 & 2 & 3 & 3 & 4 & 4 & 5 & 3 & 4 & 4 & 5 & 5 & 6
\end{pmatrix}$$

Computation example

With SageMath, one can compute the eigenvalues of ν_2:

Eigenvalues	Multiplicity
0	17
4	3
20	3
72	1

One can also find the eigenvalues using the standard Young tableaux.



Computation example

<table>
<tr><td></td><td>$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$</td><td>$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$</td><td>$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$</td></tr>
<tr><td>Eigenvalue</td><td>72</td><td>20</td><td>4</td></tr>
</table>

Example

$$\begin{tikzcd}
\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \arrow[d, mapsto, "\Delta"'] \\
\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}
\end{tikzcd}$$

$$\begin{aligned}
v_2\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\right) &= v_2\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) + (4+1-2+2-1)\cdot v_1\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)\\
&= 0 + 4\cdot v_1\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)\\
&= 4\cdot 1 = 4.
\end{aligned}$$

Hvala!