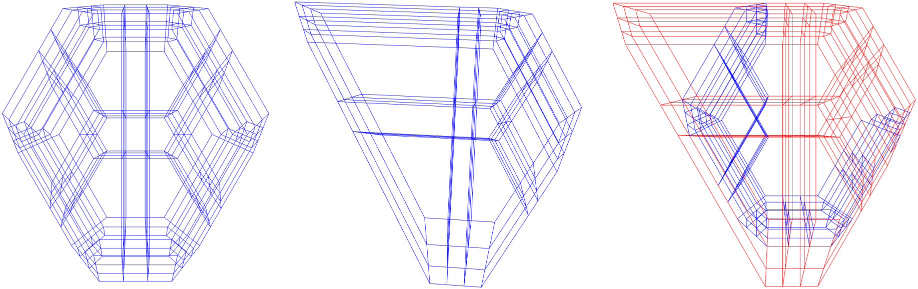
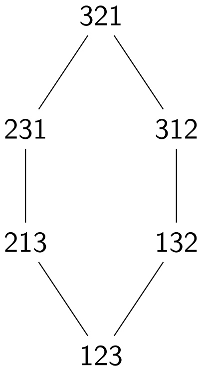
s-weak order and s-permutahedra

Cesar Ceballos – Viviane Pons
Univ. of Vienna – LRI, Univ. Paris-Sud



Weak Order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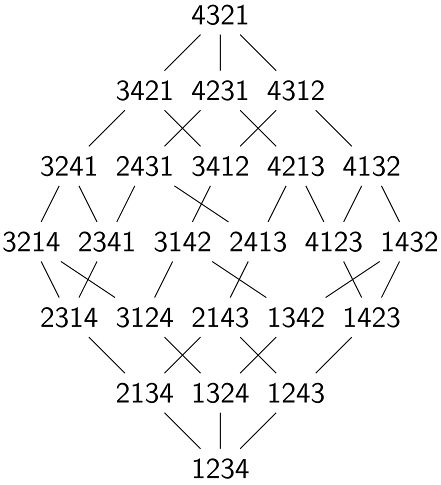
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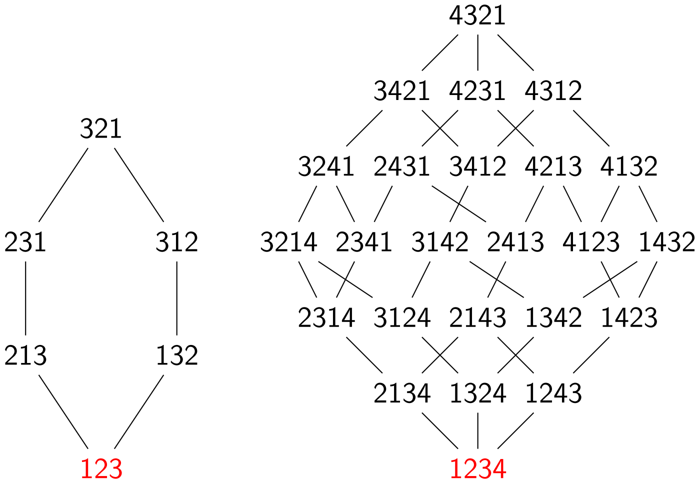
Weak Order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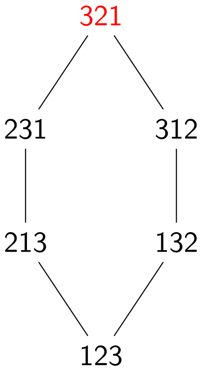
Weak Order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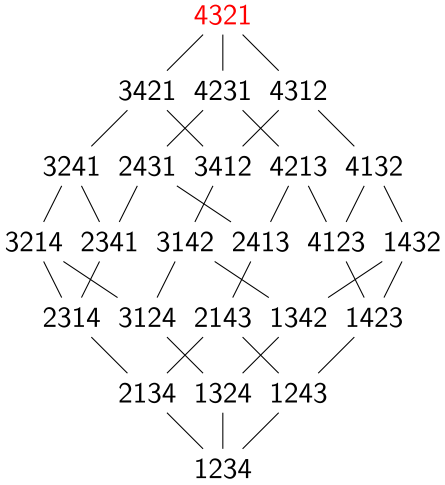
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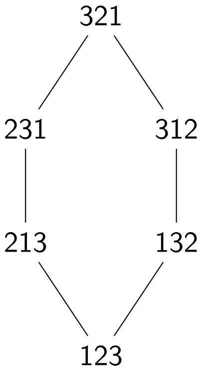
Weak Order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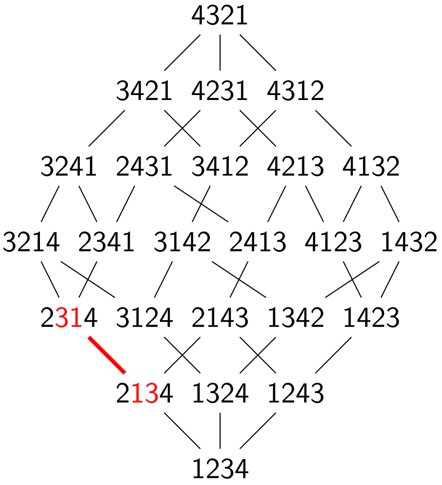
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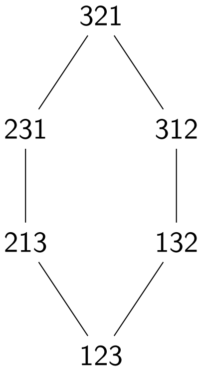
Weak Order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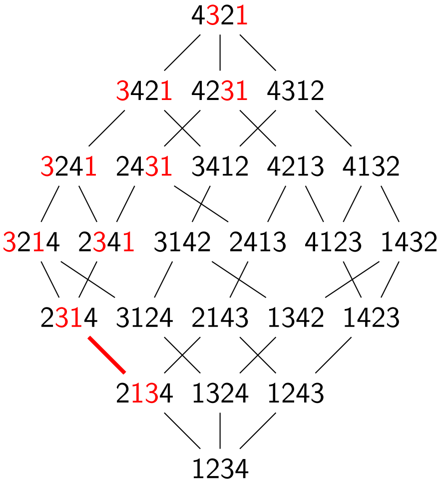
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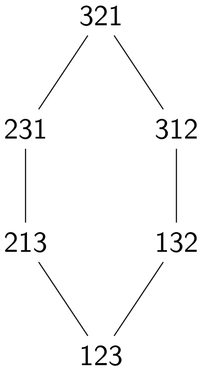
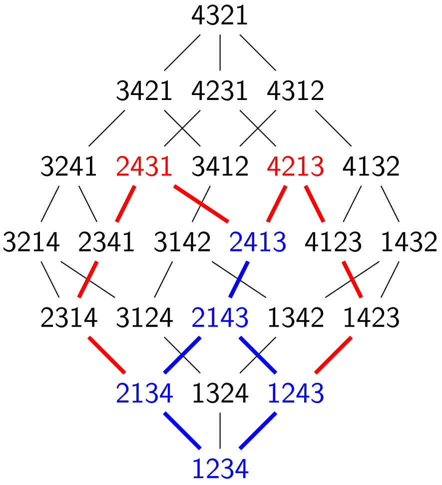
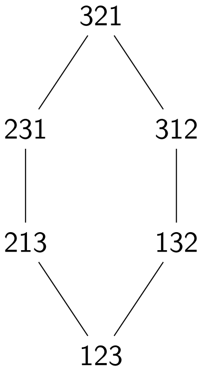
Weak Order

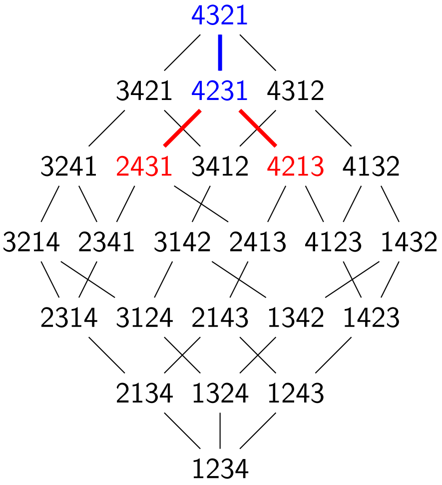




$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

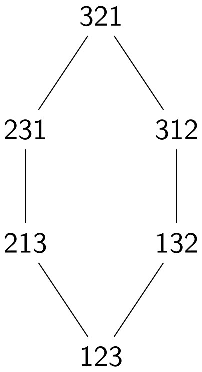
Weak Order

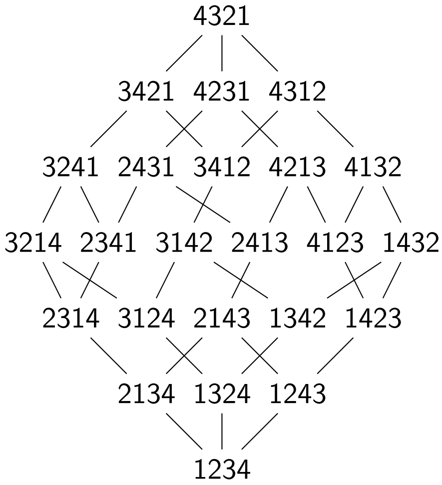




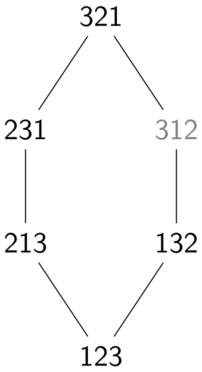
$2413 \wedge 4213 = 2413$ $\qquad$ $2413 \vee 4213 = 4231$

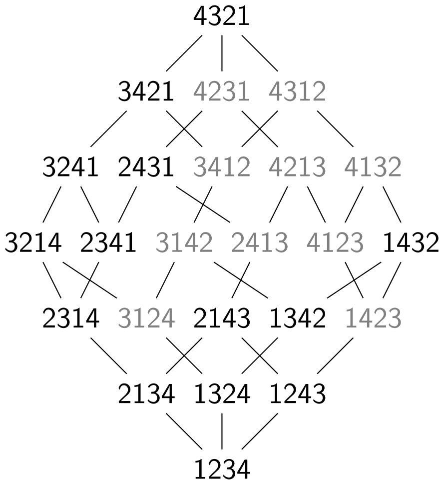
From the Weak Order to the Tamari lattice



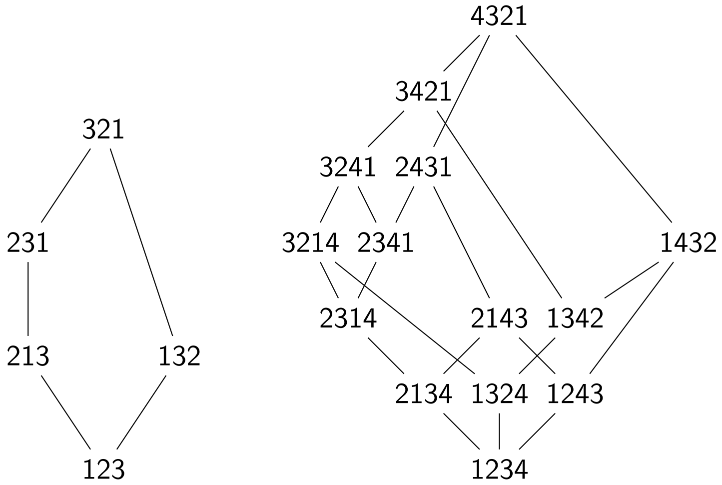


From the Weak Order to the Tamari lattice

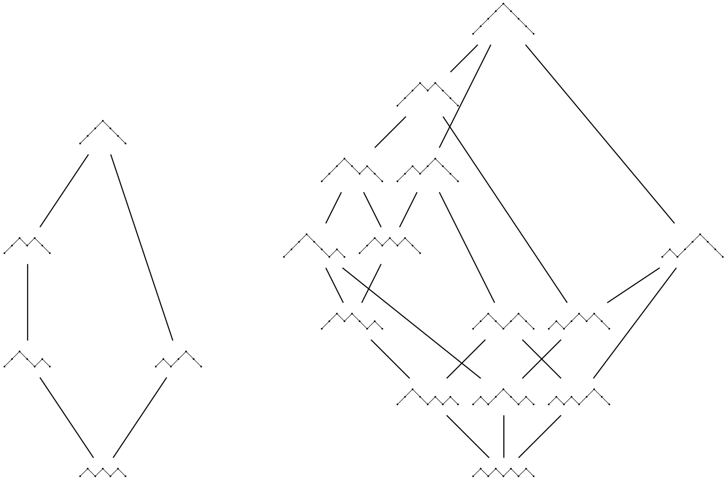


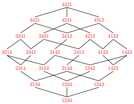
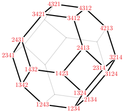
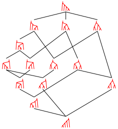
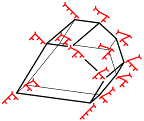
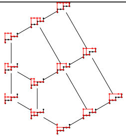
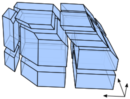


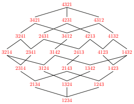
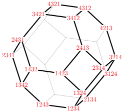
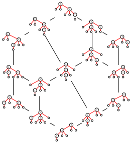
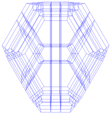
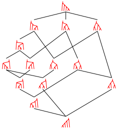
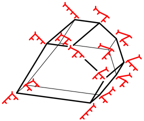
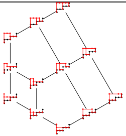
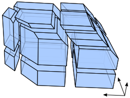
From the Weak Order to the Tamari lattice



From the Weak Order to the Tamari lattice



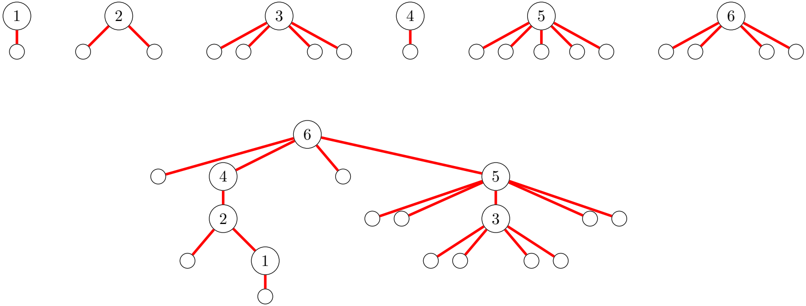
Weak order	Permutahedron	s-Weak order?	s-Permutahedron?
			
 Tamari lattice	 Associahedron	 ν-Tamari Préville-Ratelle, Viennot	 ν-Associahedron Ceballos, Padrol, Sarmiento

Weak order	Permutahedron	s-Weak order?	s-Permutahedron?
			
			
Tamari lattice	Associahedron	ν-Tamari	ν-Associahedron
		Préville-Ratelle, Viennot	Ceballos, Padrol, Sarmiento

s-decreasing trees

Let s be a sequence of n non-negative integers. An s-decreasing tree is a planar tree labeled with $1 \dots n$ such that each node i has $s(i) + 1$ children and labels are decreasing from root to leaves.

$$s = (0, 1, 3, 0, 4, 3)$$



How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

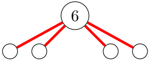
Number of s-decreasing trees:

How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)$

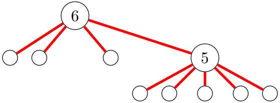


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)$

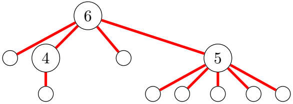


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$(1+3)\times(1+3+4)\times(1+3+4+0)$

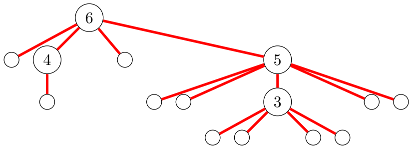


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)$$

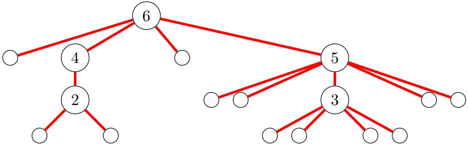


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)\times(1+3+4+0+3+1)$$

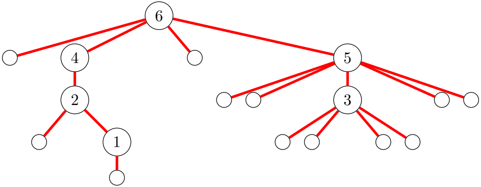


How many trees?

$$s = (0, 1, 3, 0, 4, 3)$$

Number of s-decreasing trees:

$$(1+3)\times(1+3+4)\times(1+3+4+0)\times(1+3+4+0+3)\times(1+3+4+0+3+1)$$

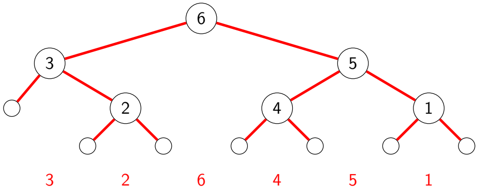


Permutations

$$s = (1, 1, 1, 1, 1, 1)$$

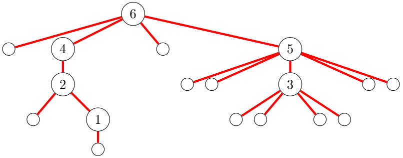
Number of s-decreasing trees: 6!

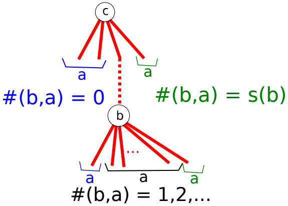
$$(1+1)\times(1+1+1)\times(1+1+1+1)\times(1+1+1+1+1)\times(1+1+1+1+1+1)$$



Tree-inversions

For all $b > a$, we define $0 \leq \#(b,a) \leq s(b)$.





$$
\begin{array}{lllll}
\#(6,5)=3 & \#(6,4)=1 & \#(6,3)=3 & \#(6,2)=1 & \#(6,1)=1 \\
 & \#(5,4)=0 & \#(5,3)=2 & \#(5,2)=0 & \#(5,1)=0 \\
 & & \#(4,3)=0 & \#(4,2)=0 & \#(4,1)=0 \\
 & & & \#(3,2)=0 & \#(3,1)=0 \\
 & & & & \#(2,1)=1
\end{array}
$$

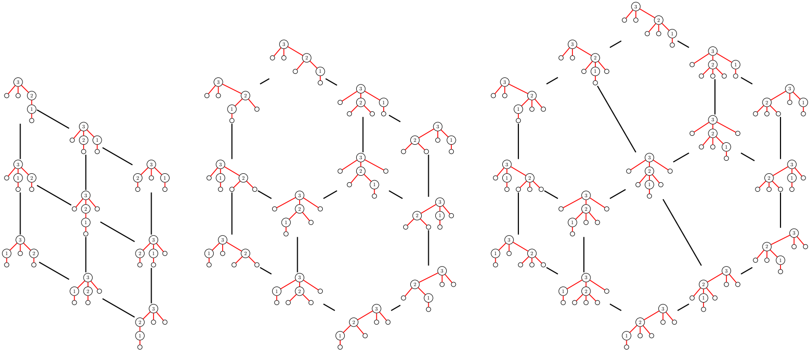
The s-weak order

R, T, s-decreasing trees:

$$R \preccurlyeq T \Leftrightarrow \forall b > a, \#_R(b,a) \leq \#_T(b,a)$$

Theorem

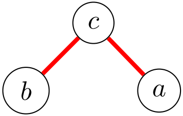
The s-weak order is always a lattice.



$(0, 0, 2)$ $(0, 1, 2)$ $(0, 2, 2)$

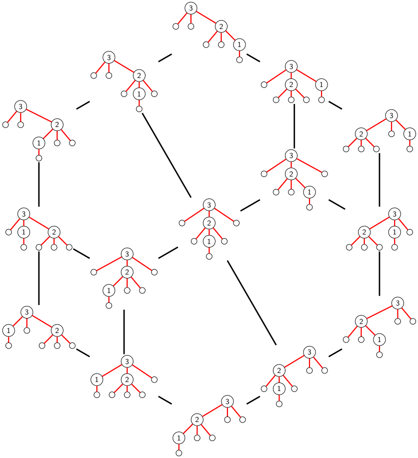
s-Tamari lattice

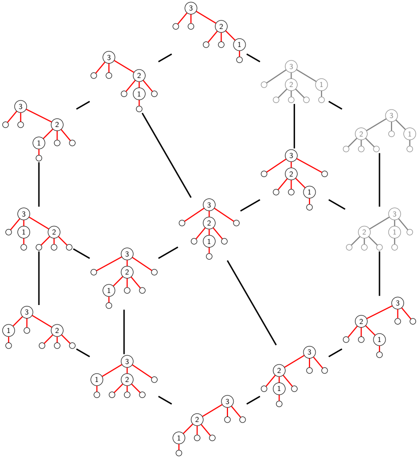
Select trees which avoid "pattern 231": $a < b < c$

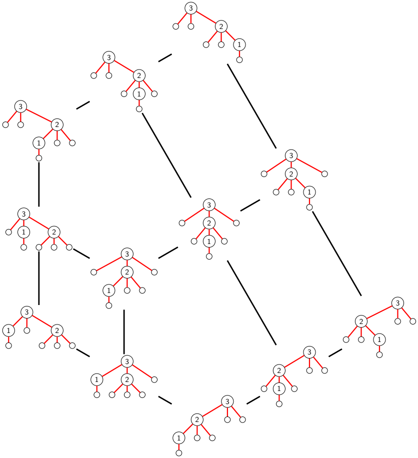


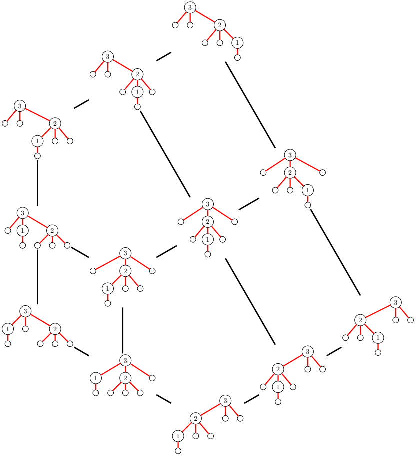
Theorem

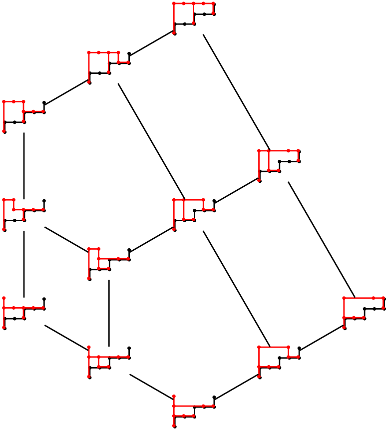
The set of 231-avoiding s-decreasing trees form a sublattice, the s-Tamari lattice, isomorphic to the ν-Tamari lattice.



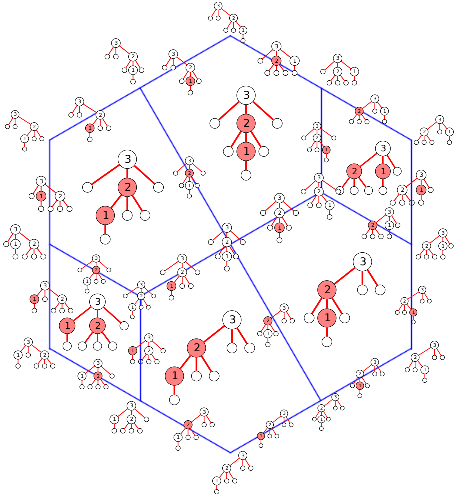


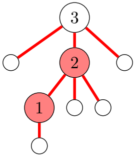






Geometry: the s-Permutahedron

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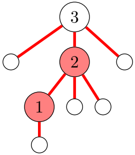
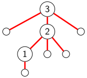
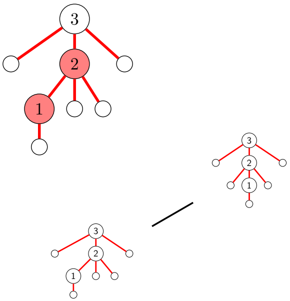
2
1



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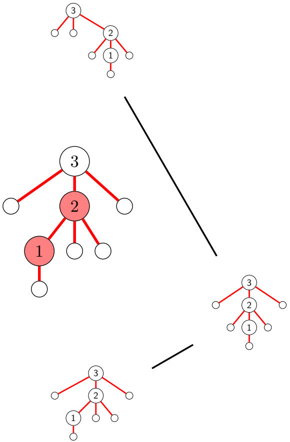
1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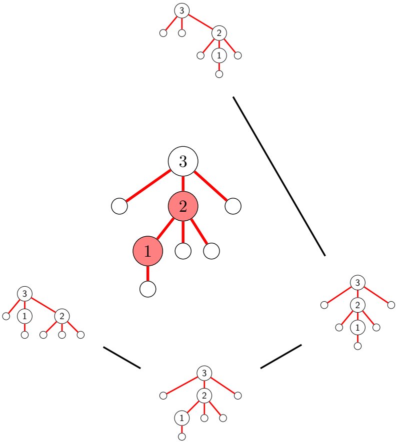
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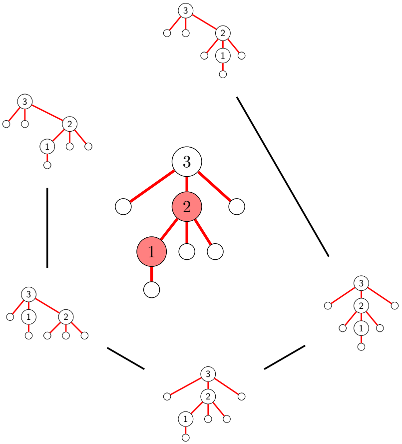
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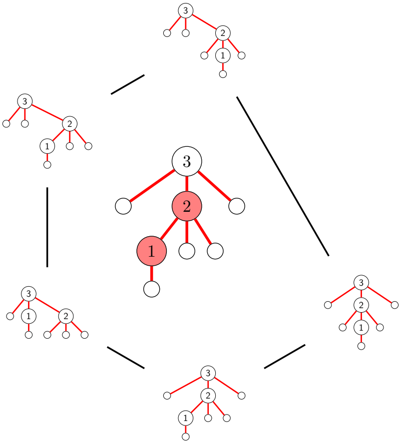
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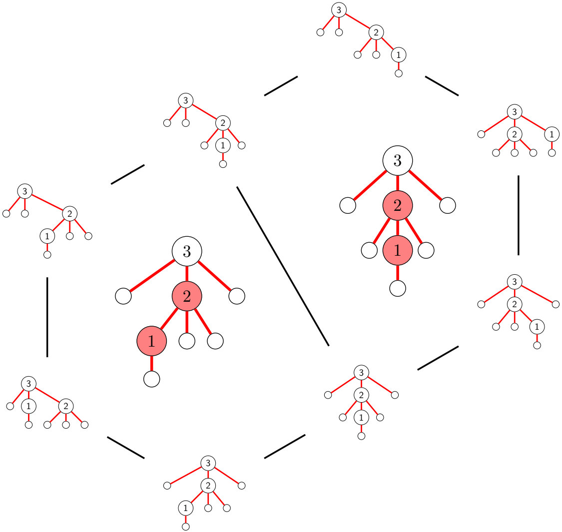
2
1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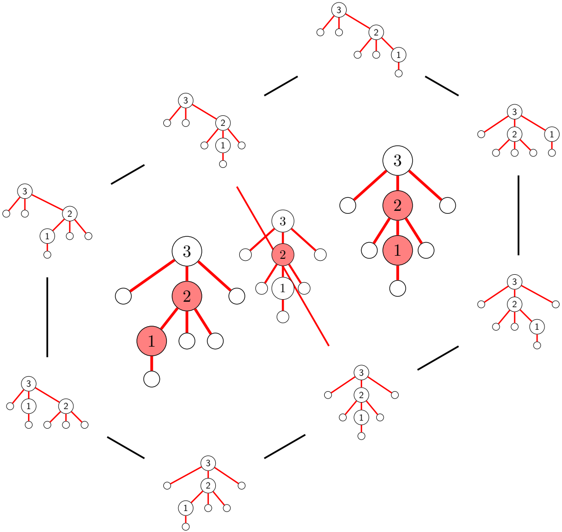
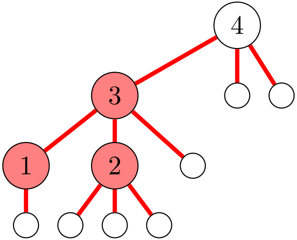
1
3
2
1

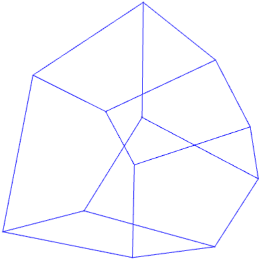

3
2
1

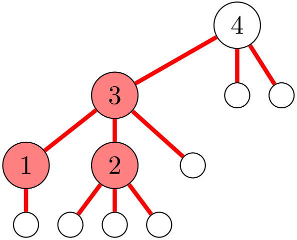
Polytopal complex? Ascentopes

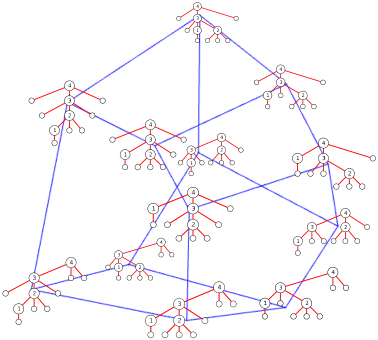
Can each face be realized as a polytope?

To each facet, we associate a **generalized permutahedron**: we define a injection between maximal faces included in a given facet and the facets of the permutahedron.

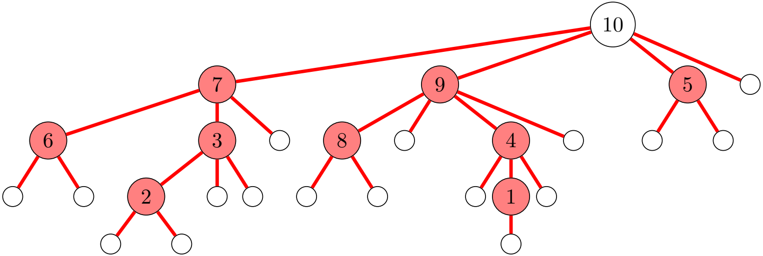

4
3
1
2





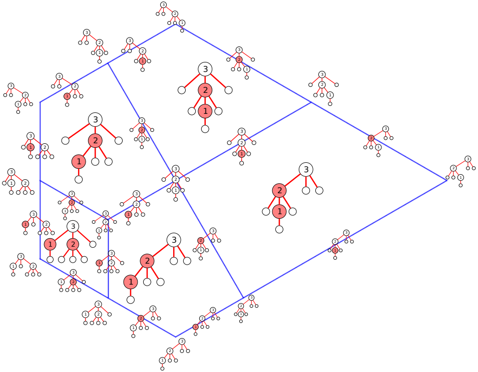


- 12 vertices
- 18 edges
- 8 facets



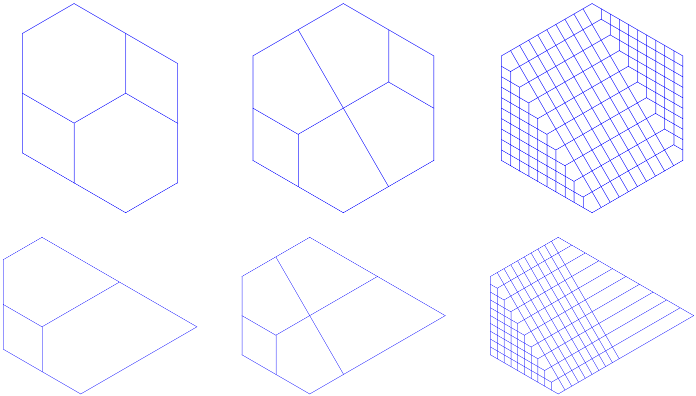
f vector $=$
$(1, 2178, 9801, 19008, 20790, 14082, 6099, 1680, 282, 26, 1)$

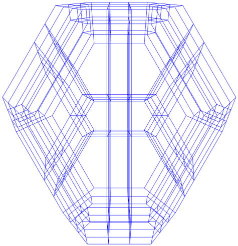
The s-Associahedron

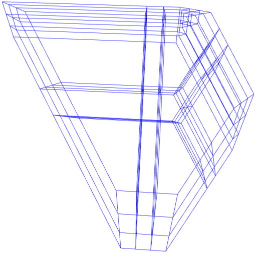


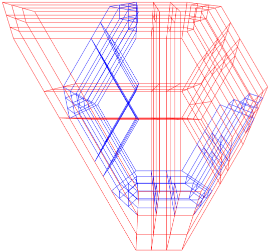
Theorem: The s-associahedron is isomorphic to the ν-associahedron for $\nu = NE^{s(n)} \dots NE^{s(1)}$.

Polytopal subdivision









Conjecture 1

The s-permutohedron can be realized as a polytopal subdivision of the permutohedron.

Conjecture 2

One can obtain a realization of the s-associahedron by removing some facets of the s-permutohedron realization.