Alcove random walks and k-Schur functions

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0. General considerations

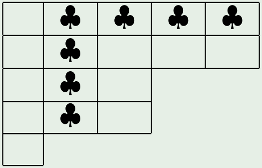
1. There exist natural generalizations of the Young lattice.
2. Their extremal harmonic functions make appear interesting families of polynomials.
3. These harmonic functions also control the behavior of certain random walks.

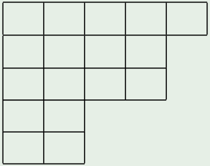
I. Combinatorics

A partition of rank l is a nonincreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m) \in \mathbb{Z}_{\geq 0}$ s.t. $\lambda_1 + \cdots + \lambda_m = l$.

The partition λ is encoded by its Young diagram.
Each cell c of λ has a hook length $h(c)$.
Let $\mathrm{hook}(\lambda) = \lambda_1 + d - 1$ where $d = \max(i \mid \lambda_i > 0\}$.

Example

$\lambda = (5,5,3,3,1) \leftrightarrows$  with $h(1,2) = 7$, $\mathrm{hook}(\lambda) = 9$ and

transposed partition $tr(\lambda) = (5,4,4,2,2) \leftrightarrows$ 

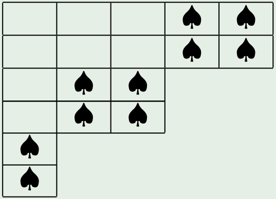
Fix $k \geq 1$ an integer

A $(k+1)$-core is a partition λ with no hook length equal to $k+1$.

Write $|\lambda|_k$ for the number of cells with hook length less or equal to k.

Example

The partition

$\lambda =$ 

is a 4-core with $|\kappa|_3 = 10$ (but not a 3-core).

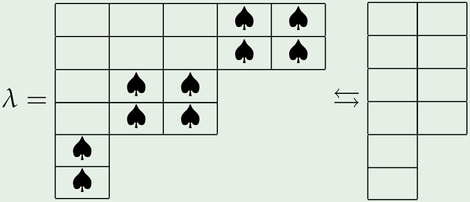
Observe λ is a $(k+1)$-core i.f.f $tr(\lambda)$ is.

A partition is k-bounded when its parts are less or equal to k.
For k fixed, there is a bijection

$$\{\lambda \mid k+1\text{-core with } |\lambda|_k = l\} \underset{\mathfrak{c}^{-1}}{\overset{\mathfrak{c}}{\leftrightarrows}} \{\mu \mid k\text{-bounded of rank } l\}$$

obtained by deleting the cells with hook lengths greater than $k+1$ and next left align.

Example



The map $\iota = \mathfrak{c}^{-1} \circ tr \circ \mathfrak{c}$ is an involution on the k-bounded partitions.

Let $\mathcal{Y}^k$ be the graph with vertices the k-bounded partitions and arrows $\lambda \to \mu$ when

- μ is obtained by adding one cell to λ

Observe that $\lim_{k \to +\infty} \mathcal{Y}^k = \mathcal{Y}$ is the Young lattice of ordinary partitions.

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- $\iota(\mu)$ is obtained by adding one cell to $\iota(\lambda)$.

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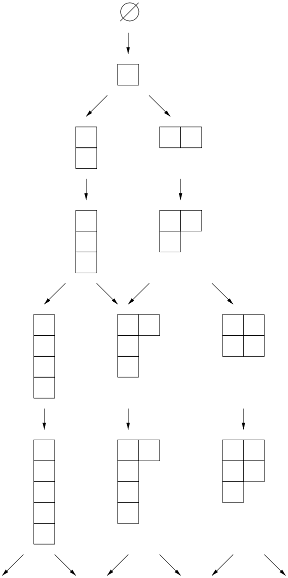

The graph $\mathcal{Y}^2$

II. Harmonic functions

A function $f : \mathcal{Y}^k \to \mathbb{R}_{\geq 0}$ is harmonic when $f(\emptyset) = 1$ and for any $\lambda \in \mathcal{Y}^k$

$$f(\lambda) = \sum_{\lambda \to \mu} f(\mu).$$

Positive harmonic functions parametrize central Markov chains on $\mathcal{Y}^k$: the transition matrix associated to f is

$$\Pi(\lambda, \mu) = \frac{f(\mu)}{f(\lambda)} 1_{\lambda \to \mu}$$

and

$$\Pi(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)}) = \frac{f(\lambda^{(l)})}{f(\lambda^{(1)})}$$

only depends on the ends of the trajectory $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)}$.

Problem (Minimal boundary of $\mathcal{Y}^k$)

What are the extremal nonnegative harmonic functions on $\mathcal{Y}^k$?

The graph $\mathcal{Y}^k$ is multiplicative : there exists a $\mathbb{R}$-algebra $\mathcal{A}$ with a distinguished basis $\mathbb{B} = \{s_\lambda^{(k)} \mid \lambda \in \mathcal{Y}^k\}$ s.t.

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- $s_\lambda^{(k)} s_1^{(k)} = \sum_{\lambda \to \mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} s_\mu^{(k)}$ decomposes on $\mathbb{B}$ with nonnegative coefficients (only a geometric proof by Lam 2008).

Theorem (Kerov-Vershik 1989)

The extremal harmonic functions on $\mathcal{Y}^k$ *correspond to the morphisms* $\theta : \mathcal{A} \to \mathbb{R}$ *s.t.* $\theta(s_1^{(k)}) = 1$ *and* $\theta(s_\lambda^{(k)}) \geq 0$ *for any* $\lambda \in \mathcal{Y}^k$ *by setting*

$$f(\lambda) = \theta(s_\lambda^{(k)})$$

III. k-Schur functions

Let $\Lambda = Sym_{\mathbb{R}}(x_1, \dots, x_n, \dots)$ the algebra of symmetric functions.

Recall that the

$$h_a = \sum_{1 \le i_1 \le \cdots \le i_a} x_{i_1} \cdots x_{i_a}$$

with $a \geq 1$ algebraically generate Λ

Set $\mathcal{A} = \langle h_1, \dots, h_k \rangle$ and the $s_\lambda^{(k)}, \lambda \in \mathcal{Y}^{(k)}$ are the k-Schur functions of Lapointe, Lascoux and Morse (2003).
They are defined from the multiplication $s_\lambda^{(k)} \times h_a$ (k-Pieri rule) which is encoded in $\mathcal{Y}^{(k)}$

We have $\lim_{k \to +\infty} s_\lambda^{(k)} = s_\lambda$ the Schur function associated to λ.

When $\mathrm{hook}(\lambda) \leq k$, we have $s_\lambda^{(k)} = s_\lambda$.

$\lambda \in \mathcal{Y}^k$ is k-irreducible when λ does not contain any rectangle $R_a = (k - a + 1) \times a$ for $a = 1, \ldots, k$.

Theorem (Lapointe, Morse (2007))

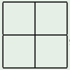
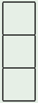
For any $\lambda \in \mathcal{P}^{(k)}$, there is a unique factorization

$$s_\lambda^{(k)} = s_{R_1}^{p_1} \cdots s_{R_k}^{p_k} s_\kappa^{(k)}$$

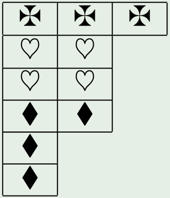
with $\kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}$ (the set of k-irreducible partitions).

Example

For $k = 3$ we have

$R_1 =$ , $R_2 =$ , $R_3 =$ 

and for

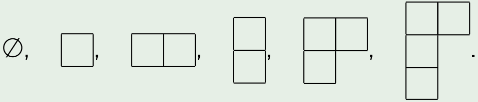
$\lambda = (3, 2, 2, 2, 1, 1) =$ 

$$s_\lambda^{(3)} = s_{(3)} s_{(2,2)} s_{(2,1,1)}^{(3)}.$$

We have $\mathrm{card}(\mathcal{P}_{\mathrm{irr}}^{(k)}) = k!$

Example

For $k = 3$, there are 6 irreducible 3-restricted partitions



Thus, the relevant morphisms θ are those s.t.

$$\begin{cases} \theta(s_1) = 1 \\ \theta(s_{R_a}) \geq 0 \text{ for any } a = 1, \ldots, k \\ \theta(s_\kappa^{(k)}) \geq 0 \text{ for any } \kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}. \end{cases}$$

From the rectangle factorization, one can write for any $\kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}$

$$s_\kappa^{(k)} \cdot s_{(1)} = \sum_{\kappa \to \mu} s_\mu^{(k)} = \sum_{\kappa' \in \mathcal{P}_{\mathrm{irr}}^{(k)}} m_{\kappa,\kappa'}(s_{R_1}, \dots, s_{R_k}) s_{\kappa'}^{(k)}$$

where $m_{\kappa,\kappa'}(s_{R_1}, \dots, s_{R_k}) \in \mathbb{Z}_{\geq 0}[s_{R_1}, \dots, s_{R_k}]$ define a $k! \times k!$ matrix $M_k(s_{R_1}, \dots, s_{R_k})$.

Theorem (L, Tarrago (2018))

By specializing $s_{R_a} = r_a \in \mathbb{R}_{\geq 0}$, $M_k(r_1, \dots, r_k)$ is irreducible i.f.f $r_a + r_{a+1} > 0$ for any $a = 1, \dots, k-1$.

Example

For $k = 3$, on gets

$$M_3 = \begin{pmatrix} 0 & 0 & s_{R_1} & s_{R_3} & s_{R_2} & 0 \\ 1 & 0 & 0 & 0 & 0 & s_{R_2} \\ 0 & 1 & 0 & 0 & 0 & s_{R_3} \\ 0 & 1 & 0 & 0 & 0 & s_{R_1} \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

IV. Primitive element theorem

Remind $\mathcal{R} = \mathbb{R}[s_{R_1}, \dots, s_{R_k}]$ is a subalgebra of $\mathcal{A}$.

Let $\mathbb{L}$ and $\mathbb{K}$ be the fraction fields of $\mathcal{A}$ and $\mathcal{R}$, respectively.

Theorem (L, Tarrago (2018))

$\mathbb{L}$ is a separable and algebraic extension of $\mathbb{K}$ of degree $k!$. Moreover $\mathbb{L} = \mathbb{K}[s_1]$ i.e. s_1 is a primitive element in $\mathbb{L}$.

Corollary

There exists a polynomial $\Delta \in \mathcal{R}$ s.t. each polynomial $s_\kappa^{(k)}$ with $\kappa \in \mathcal{P}_{\mathrm{irr}}^{(k)}$ can be written on the form

$$s_\kappa^{(k)} = \frac{1}{\Delta} P_\kappa(s_1)$$

with $P_\kappa \in \mathcal{R}[X]$.

V. Reduced alcove walks of type A

For $i = 1, \ldots, k-1$ set $\alpha_i = e_i - e_{i+1}$ in $\mathbb{R}^k$ and put $\alpha_0 = -(\alpha_1 + \cdots + \alpha_k)$.

There is a tesselation of $\mathbb{R}^k$ by alcoves supported by the hyperplanes

$$H_{i,m} = \{v \in \mathbb{R}^k \mid (v, \alpha_i) = m\}$$

with $i = 0, \ldots, k-1$ and $m \in \mathbb{Z}$.

The dominant alcoves are those in the cone delimited by the hyperplanes $H_{\alpha_i^\vee} = H_{i,0}$.

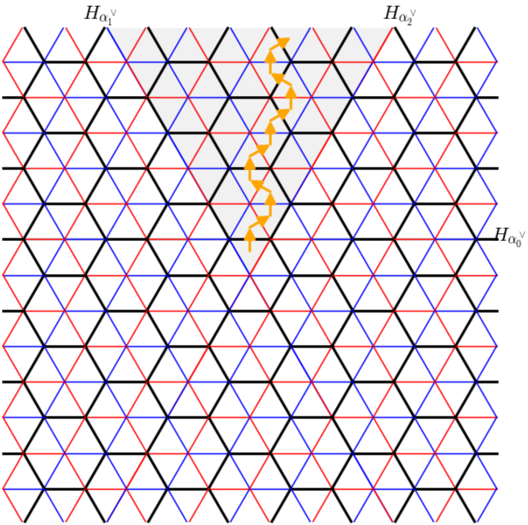


Figure: A reduced walk on dominant alcoves for $k = 2$

The dominant alcoves are in bijection with the k-bounded partitions.

A random central Markov chain on $\mathcal{Y}^k$ gives a random central alcove walk : it can only cross each hyperplane once.

Different random alcove walks have been considered by Lam (2015) which are central only when

$$k = 2, \quad \theta(s_{R_1}) = \theta(h_2) = \frac{1}{2} \text{ and } \quad \theta(s_{R_2}) = \theta(e_2) = \frac{1}{2}.$$

VI. Main results

Consider the simplex $\mathcal{S}_k = \{(v_1, \ldots, v_k) \in \mathbb{R}^k_{\geq 0} \mid v_1 + \cdots + v_k = 1\}$.

Theorem (L, Tarrago (2018))

1. *The morphisms $\theta : \mathcal{A} \to \mathbb{R}$ nonnegative on the k-Schur functions are uniquely determined by the $\theta(s_{R_a}) = r_a \geq 0$, $a = 1, \ldots, k$.*

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2. *The $\theta(s_\kappa), \kappa \in \mathcal{P}_{\mathrm{irr}}$ can be essentially computed from the $\theta(s_{R_a})$, $a = 1, \ldots, k$ by applying Perron Frobenius theorem to the matrix M_k and using continuity arguments.*

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3. *The minimal boundary of $\mathcal{Y}_k$ is homeomorphic to $\mathcal{S}_k$ (recall the additional condition $\theta(h_1) = 1$).*
4. *This permits to recover Rietsch's parametrization (2002) of totally nonnegative unitriangular Toeplitz matrices*