Eulerian orientations and the six-vertex model on planar maps

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Planar maps

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A CHRONOLOGY OF PLANAR MAPS

- **Recursive approach**: Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...
- **Matrix integrals**: Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Kostov, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections**: Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, Bousquet-Mélou, Chapuy...
- **Geometric properties of random maps**: Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne...

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Maps equipped with an additional structure

- **How many maps equipped with...**
  - a spanning tree [Mullin 67, Bernardi]
  - a spanning forest? [Bouttier et al., Sportiello et al., Bousquet-Mélou & Courtiel]
  - a self-avoiding walk? [Duplantier & Kostov; Gwynne & Miller]
  - a proper $q$-colouring? [Tutte 74-83, Bouttier et al.]
  - a bipolar orientation? [Kenyon, Miller, Sheffield, Wilson, Fusy, Bousquet-Mélou...]

- **What is the expected partition function of...**
  - the Ising model? [Boulatov, Kazakov, Bousquet-Mélou, Schaeffer, Chen, Turunen, Bouttier et al., Albenque, Ménard...]
  - the Potts model? [Eynard-Bonnet, Baxter, Bousquet-Mélou & Bernardi, Guionnet et al., Borot et al., ...]
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Eulerian orientations generating functions

The 4-valent case: the ice model

\[ G(t) = 2Q(t, 0) \]

The 6-vertex model

\[ Q(t, \gamma) \]

- Non-alternating
  - (weight \( t \))
- Alternating
  - (weight \( t\gamma \))

Each vertex has equally many incoming as outgoing edges.
Part 1: Counting Eulerian orientations
**Aim:** Determine the number $g_n$ of (rooted planar) Eulerian orientations with $n$ edges

The generating function $G(t) = \sum_{n=1}^{\infty} g_n t^n = t + 5t^2 + \ldots$
Problem posed by Bonichon, Bousquet-Mélou, Dorbec and Pennarun in 2016.

In 2017, E.P. and Guttmann:
- Computed the number $g_n$ of Eulerian orientations for $n < 100$.
- Predicted that
  \[ g_n \sim \kappa g \frac{(4\pi)^n}{n^2(\log n)^2}. \]
- This led us to conjecture the exact solution.
Let $R_0(t)$ be the unique power series with constant term 0 satisfying
\[ t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} R_0(t)^{n+1}. \]

The generating function $G(t) = \sum_{n=0}^{\infty} g_n t^n$ of rooted planar Eulerian orientations counted by edges is given by
\[ G(t) = \frac{1}{4t^2} (t - 2t^2 - R_0(t)). \]
EULERIAN ORIENTATIONS OUTLINE

- Bijections
- Functional equations
- Guess and check solution
Step 1: Bijection to labelled maps

BIJECTION TO LABELLED MAPS

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Labelled maps are rooted planar maps with labelled vertices such that:

- The root edge is labelled from 0 to 1.
- Adjacent labels differ by 1.

By the bijection, $G(t)$ counts labelled maps by edges.
By our bijection, $Q(t, \gamma)$ counts labelled quadrangulations by faces ($t$) and alternating faces ($\gamma$).

Non-alternating
(weight $t$)

Alternating
(weight $t\gamma$)

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By our bijection, \( Q(t, \gamma) \) counts labelled \textit{quadrangulations} by faces \((t)\) and \textit{alternating} faces \((\gamma)\).

\( Q(t, 0) \) counts labelled quadrangulations with no alternating faces.
Step 2: Bijection between labelled quadrangulations with no alternating faces and labelled maps

(Miermont (2009)/Ambjørn and Budd (2013)).
Highlight edges according to the rule. The red edges (sometimes) form a labelled map. The bijection implies that $Q(t, 0) = 2G(t)$. 
Exact solution using labelled quadrangulations at $\gamma = 0$

(Bousquet-Mélou and E.P.)
Decomposition of labelled quadrangulations

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The series $2G(t) = Q(t, 0)$ is given by

$$Q(t, 0) = [y^1]P(t, y) - 1,$$

where the series $P(t, y)$, $C(t, x, y)$ and $D(t, x, y)$ are characterised by the equations

$$P(t, 0) = 1,$$

$$P(t, y) = \frac{1}{y}[x^1]C(t, x, y),$$

$$D(t, x, y) = \frac{1}{1 - C(t, \frac{1}{1-x}, y)},$$

$$C(t, x, y) = xy[x \geq 0] \left( P(t, tx)D(t, \frac{1}{x}, y) \right).$$

We solve these using a guess and check method.
Let $R_0(t)$ be the unique power series with constant term 0 satisfying
\[ t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 R_0(t)^{n+1}. \]

Then the series $P(t, y)$, $C(t, x, y)$ and $D(t, x, y)$ are given by:
\[ tP(t, ty) = \sum_{n \geq 0} \sum_{j=0}^{n} \frac{1}{n+1} \binom{2n}{n} \binom{2n-j}{n} y^j R_0^{n+1}, \]
\[ C(t, x, ty) = 1 - \exp \left( - \sum_{n \geq 0} \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{1}{n+1} \binom{2n-j}{n} \binom{2n-i}{n} x^i y^j R_0^{n+1} \right), \]
\[ D(t, x, ty) = \exp \left( \sum_{n \geq 0} \sum_{j=0}^{n} \sum_{i \geq 0} \frac{1}{n+1} \binom{2n-j}{n} \binom{2n+i+1}{n} x^i y^j R_0^{n+1} \right). \]
Let $R_0(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n + 1} \binom{2n}{n}^2 R_0(t)^{n+1},$$

Then the generating function of rooted planar Eulerian orientations counted by edges is

$$G(t) = \frac{1}{2} Q(t, 0) = \frac{1}{4t^2} (t - 2t^2 - R_0(t)).$$

Asymptotically, the coefficients behave as

$$g_n \sim \kappa \frac{\mu^{n+2}}{n^2 (\log n)^2},$$

where $\kappa = 1/16$ and $\mu = 4\pi$. 
Let $R_1(t)$ be the unique power series with constant term 0 satisfying
\[ t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} R_1(t)^{n+1}, \]

Then the generating function of quartic rooted planar Eulerian orientations counted by edges is
\[ Q(t, 1) = \frac{1}{3t^2}(t - 3t^2 - R_1(t)). \]

Asymptotically, the coefficients behave as
\[ q_n \sim \kappa \frac{\mu^{n+2}}{n^2 (\log n)^2}, \]

where $\kappa = 1/18$ and $\mu = 4\sqrt{3\pi}$. 

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Part 2:
Solution for general $\gamma$

(Kostov/ E.P. and Zinn-Justin)

Non-alternating
(weight $t$)

Alternating
(weight $t\gamma$)
Solved at criticality by Zinn-Justin in 2000.
Exactly solved by Kostov later in 2000 (to the satisfaction of physicists).
Solution was not completely rigorous.
We corrected a mistake and simplified the form of the solutions.
RECALL: SOLUTIONS AT $\gamma = 0, 1$

The generating function $Q(t, 0)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 R_0(t)^{n+1},$$

$$Q(t, 0) = \frac{1}{2t^2}(t - 2t^2 - R_0(t)).$$

The generating function $Q(t, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} R_1(t)^{n+1},$$

$$Q(t, 1) = \frac{1}{3t^2}(t - 3t^2 - R_1(t)).$$
Define
\[ \vartheta(z, q) = \sum_{n=0}^{\infty} (-1)^n (e^{(2n+1)iz} - e^{-(2n+1)iz}) q^{(2n+1)^2/8}. \]

Let \( q = q(t, \alpha) \) be the unique series satisfying
\[ t = \frac{\cos \alpha}{64 \sin^3 \alpha} \left( - \frac{\vartheta(\alpha, q) \vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)^2} + \frac{\vartheta''(\alpha, q)}{\vartheta'(\alpha, q)} \right). \]

Define \( R(t, \gamma) \) by
\[ R(t, -2 \cos(2\alpha)) = \frac{\cos^2 \alpha}{96 \sin^4 \alpha \vartheta'(\alpha, q)^2} \left( - \frac{\vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)} + \frac{\vartheta'''(0, q)}{\vartheta'(0, q)} \right). \]

Then
\[ Q(t, \gamma) = \frac{1}{(\gamma + 2)t^2} \left( t - (\gamma + 2)t^2 - R(t, \gamma) \right). \]
Outline for general $\gamma$

- Bijection
- Functional equations
- Solution using analytic methods
Q(t, γ) is characterised by equations relating it to generating functions W(x) ≡ W(t, ω, x) and H(x, y) ≡ H(t, ω, x, y).

\[ Q(t, \omega^2 + \omega^{-2}) = H(t, \omega, 0, 0) \equiv H(0, 0) \]

\[ W(x) = x^2 t W(x)^2 + \omega x t H(0, x) + \omega^{-1} x t H(x, 0) + 1 \]

\[ H(x, y) = W(x) W(y) + \frac{\omega}{y} (H(x, y) - H(x, 0)) + \frac{\omega^{-1}}{x} (H(x, y) - H(0, y)) \]
SOLVING FOR $C(t, \omega)$

Think of $W(x)$ and $H(x, y)$ as analytic functions then consider

$$U(x) = \frac{x(\omega^2 + 1)}{1 + ix(\omega^2 + 1)} W \left( \frac{\omega + \omega^{-1}}{1 + ix(\omega^2 + 1)} \right)$$

$$+ \frac{x(\omega^{-2} + 1)}{1 - ix(\omega^{-2} + 1)} W \left( \frac{\omega + \omega^{-1}}{1 - ix(\omega^{-2} + 1)} \right)$$

$$+ \frac{ix^2}{t(\omega^2 - \omega^{-2})} - \frac{x}{t(\omega + \omega^{-1})^2}$$

- $U(x)$ is analytic except on two cuts $i\omega[x_1, x_2]$ and $-i\omega^{-1}[x_1, x_2]$
- $U(x)$ satisfies

$$U(i\omega(x \pm i0)) = U(-i\omega^{-1}(x \mp i0)).$$

These + initial conditions characterise $U(x)$. 
Define

\[ \vartheta(z, q) = \sum_{n=0}^{\infty} (-1)^n (e^{(2n+1)iz} - e^{-(2n+1)iz}) q^{(2n+1)^2/8} \]

and let \( \omega = ie^{-i\alpha} \). Then \( U(x) \) is determined by

\[ U \left( x_0 \frac{\vartheta(z + \alpha, q)}{\vartheta(z, q)} \right) = A + B \wp(z), \]

where \( \wp(z) \) is the Weierstrass function and \( x_0, A, B \) and \( q \) are explicit “constants” (they depend on \( t \) and \( \alpha \) but not \( z \)).
**Solution for** $Q(t, \gamma)$

Define

$$\vartheta(z, q) = \sum_{n=0}^{\infty} (-1)^n (e^{(2n+1)iz} - e^{-(2n+1)iz})q^{(2n+1)^2/8}.$$

Let $q = q(t, \alpha)$ be the unique series satisfying

$$t = \frac{\cos \alpha}{64 \sin^3 \alpha} \left( - \frac{\vartheta(\alpha, q)\vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)^2} + \frac{\vartheta''(\alpha, q)}{\vartheta'(\alpha, q)} \right).$$

Define $R(t, \gamma)$ by

$$R(t, -2 \cos(2\alpha)) = \frac{\cos^2 \alpha}{96 \sin^4 \alpha} \vartheta(\alpha, q)^2 \left( - \frac{\vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)^2} + \frac{\vartheta'''(0, q)}{\vartheta'(0, q)} \right).$$

Then

$$Q(t, \gamma) = \frac{1}{(\gamma + 2)t^2} \left( t - (\gamma + 2)t^2 - R(t, \gamma) \right).$$
Part 3:
Comparison between solutions
We now have a general expression for $Q(t, \gamma)$ and a simpler expression at $\gamma = 0, 1$.

**Question:** Are these actually the same expression?

**Answer:** Yes.

Ideas of proof (for $\gamma = 0$):

- We just need to prove that

\[ t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 R(t, 0)^{n+1} = R(t, 0)_{2F1} \left( \begin{array}{c|c} \frac{1}{2}, \frac{1}{2} ; 2 & 16R(t, 0) \end{array} \right). \]

- Our proof involves relations of Ramanujan between theta functions and $2F1$ and some “well known” theta function identities.
Comparison between solutions

Question: Which solution method is more powerful?
FURTHER QUESTIONS

- Find bijective proofs of the formulas for $Q(t, 0)$ and $Q(t, 1)$. These each count a class of labelled trees.
- What do large random Eulerian orientations look like?

A random quadrangulation

A random bipolar triangulation

Images from Jérémie Bettinelli’s home page
Thank you!