Eulerian orientations and the six-vertex model on planar maps

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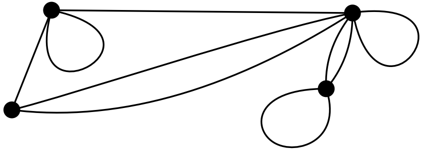
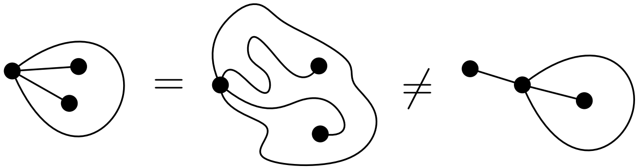
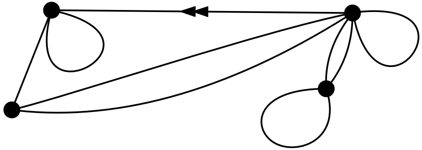
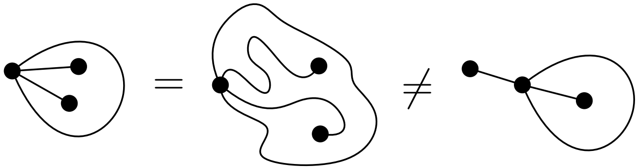
Planar maps



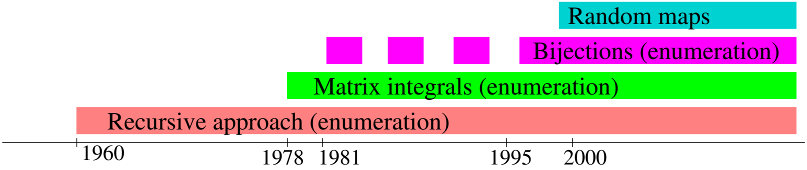


Rooted planar maps





A chronology of planar maps



- **Recursive approach:** Tutte, Brown, Bender, Canfield, Richmond, Goulden, Jackson, Wormald, Walsh, Lehman, Gao, Wanless...
- **Matrix integrals:** Brézin, Itzykson, Parisi, Zuber, Bessis, Ginsparg, Kostov, Zinn-Justin, Boulatov, Kazakov, Mehta, Bouttier, Di Francesco, Guitter, Eynard...
- **Bijections:** Cori & Vauquelin, Schaeffer, Bouttier, Di Francesco & Guitter (BDG), Bernardi, Fusy, Poulalhon, Bousquet-Mélou, Chapuy...
- **Geometric properties of random maps:** Chassaing & Schaeffer, BDG, Marckert & Mokkadem, Jean-François Le Gall, Miermont, Curien, Albenque, Bettinelli, Ménard, Angel, Sheffield, Miller, Gwynne...

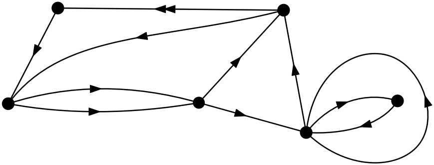
Maps equipped with an additional structure

- **How many maps equipped with...**
  - a spanning tree [Mullin 67, Bernardi]
  - a spanning forest? [Bouttier et al., Sportiello et al., Bousquet-Mélou & Courtiel]
  - a self-avoiding walk? [Duplantier & Kostov; Gwynne & Miller]
  - a proper q-colouring? [Tutte 74-83, Bouttier et al.]
  - a bipolar orientation? [Kenyon, Miller, Sheffield, Wilson, Fusy, Bousquet-Mélou...]
- **What is the expected partition function of...**
  - the Ising model? [Boulatov, Kazakov, Bousquet-Mélou, Schaeffer, Chen, Turunen, Bouttier et al., Albenque, Ménard...]
  - the hard-particle model? [Bousquet-Mélou, Schaeffer, Jehanne, Bouttier et al.]
  - the Potts model? [Eynard-Bonnet, Baxter, Bousquet-Mélou & Bernardi, Guionnet et al., Borot et al., ...]

EULERIAN ORIENTATIONS GENERATING FUNCTIONS

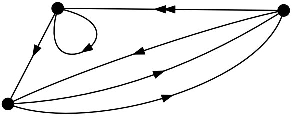
The 6-vertex model

$Q(t, \gamma)$

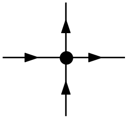


$G(t) = 2Q(t, 0)$

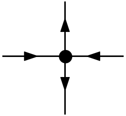
The 4-valent case: the ice model



$Q(t, 1)$



Non-alternating
(weight t)



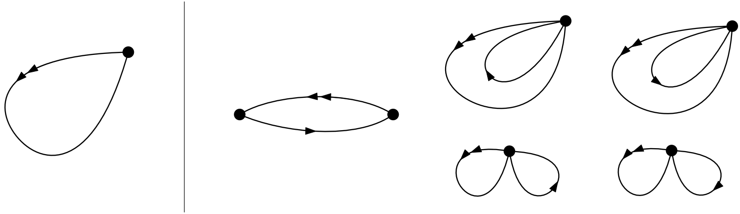
Alternating
(weight $t\gamma$)

Each vertex has equally many incoming as outgoing edges.

Part 1:
Counting Eulerian orientations

EULERIAN ORIENTATIONS

Aim: Determine the number g_n of (rooted planar) Eulerian orientations with n edges



The generating function $\mathsf{G}(t) = \sum_{t=1}^{\infty} g_n t^n = t + 5t^2 + \ldots$

Enumerating Eulerian orientations

- Problem posed by Bonichon, Bousquet-Mélou, Dorbec and Pennarun in 2016.
- In 2017, E.P. and Guttmann:
  - Computed the number g_n of Eulerian orientations for $n < 100$.
  - Predicted that
$$g_n \sim \kappa_g \frac{(4\pi)^n}{n^2(\log n)^2}.$$
- This led us to conjecture the exact solution.

Preview: Exact solution

Let $\mathsf{R}_0(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1}.$$

The generating function $\mathsf{G}(t) = \sum_{n=0}^{\infty} g_n t^n$ of rooted planar Eulerian orientations counted by edges is given by

$$\mathsf{G}(t) = \frac{1}{4t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

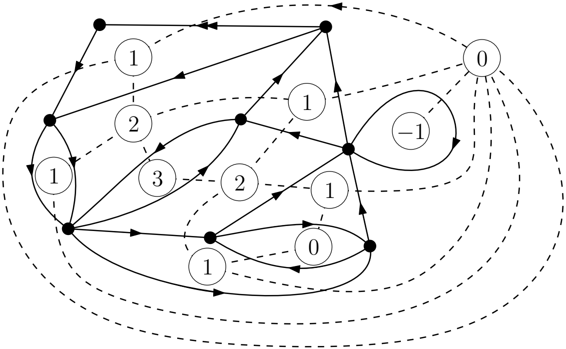
EULERIAN ORIENTATIONS OUTLINE

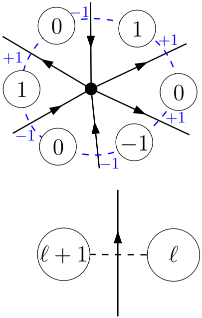
- Bijections
- Functional equations
- Guess and check solution

Step 1: Bijection to labelled maps

(EP and Guttmann, 2017)

BIJECTION TO LABELLED MAPS



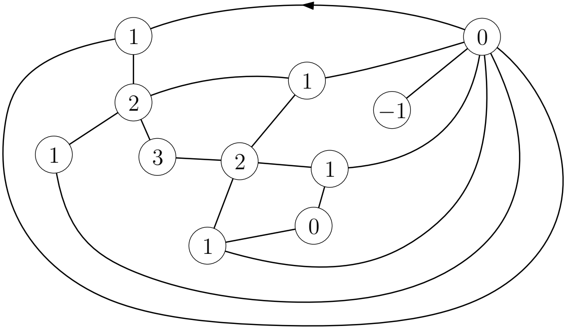


Labelled maps

Labelled maps are rooted planar maps with labelled vertices such that:

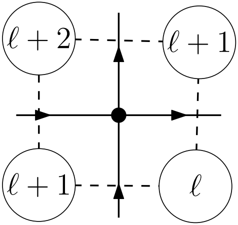
- The root edge is labelled from 0 to 1.
- Adjacent labels differ by 1.

By the bijection, $\mathsf{G}(t)$ counts labelled maps by edges.

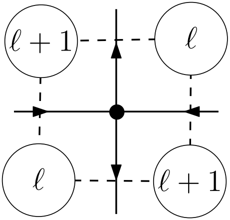


Labelled quadrangulations

By our bijection, $\mathsf{Q}(t,\gamma)$ counts labelled *quadrangulations* by faces (t) and *alternating* faces (γ).



Non-alternating
(weight t)

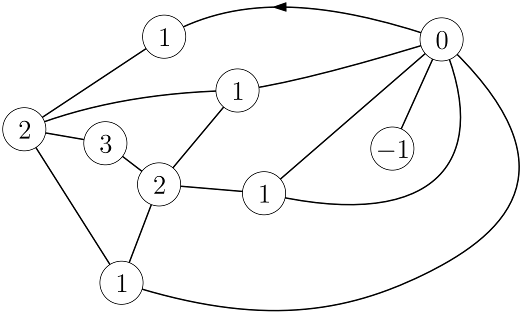


Alternating
(weight $t\gamma$)

Labelled quadrangulations

By our bijection, $\mathsf{Q}(t, \gamma)$ counts labelled *quadrangulations* by faces (t) and *alternating* faces (γ).

$\mathsf{Q}(t, 0)$ counts labelled quadrangulations with no alternating faces.

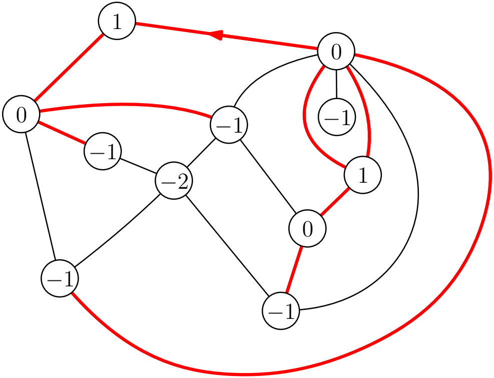


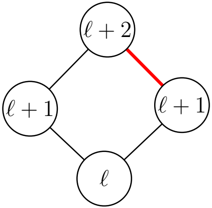
Step 2: Bijection between labelled quadrangulations with no alternating faces and labelled maps

(Miermont (2009)/Ambjørn and Budd (2013)).

Labelled quadrangulations to labelled maps

Highlight edges according to the rule. The red edges (sometimes) form a labelled map. The bijection implies that $\mathsf{Q}(t,0) = 2\mathsf{G}(t)$.

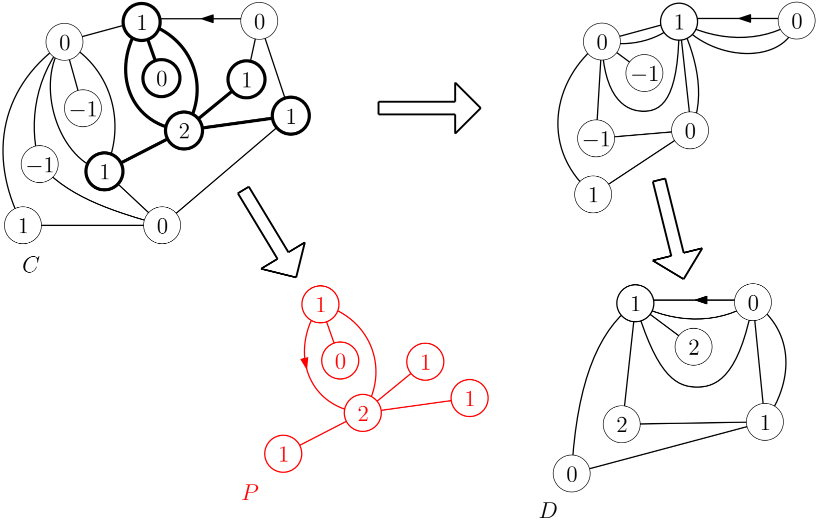




Exact solution using labelled quadrangulations at $\gamma = 0$

(Bousquet-Mélou and E.P.)

DECOMPOSITION OF LABELLED QUADRANGULATIONS



Equations for planar Eulerian orientations

The series $2\mathsf{G}(t) = \mathsf{Q}(t,0)$ is given by

$$\mathsf{Q}(t,0) = [y^1]\mathsf{P}(t,y) - 1,$$

where the series $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$ are characterised by the equations

$$\begin{aligned}
\mathsf{P}(t,0) &= 1 \\
\mathsf{P}(t,y) &= \frac{1}{y}[x^1]\mathsf{C}(t,x,y), \\
\mathsf{D}(t,x,y) &= \frac{1}{1-\mathsf{C}\left(t,\frac{1}{1-x},y\right)}, \\
\mathsf{C}(t,x,y) &= xy[x^{\geq 0}]\left(\mathsf{P}(t,tx)\mathsf{D}\left(t,\frac{1}{x},y\right)\right),
\end{aligned}$$

We solve these using a guess and check method.

SOLUTION FOR PLANAR EULERIAN ORIENTATIONS

Let $\mathsf{R}_0(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1}.$$

Then the series $\mathsf{P}(t,y)$, $\mathsf{C}(t,x,y)$ and $\mathsf{D}(t,x,y)$ are given by:

$$t\mathsf{P}(t,ty) = \sum_{n\geq 0}\sum_{j=0}^{n} \frac{1}{n+1}\binom{2n}{n}\binom{2n-j}{n} y^j \mathsf{R}_0^{n+1},$$

$$\mathsf{C}(t,x,ty) = 1-\exp\left(-\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{n} \frac{1}{n+1}\binom{2n-j}{n}\binom{2n-i}{n} x^{i+1}y^{j+1}\mathsf{R}_0^{n+1}\right),$$

$$\mathsf{D}(t,x,ty) = \exp\left(\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i\geq 0} \frac{1}{n+1}\binom{2n-j}{n}\binom{2n+i+1}{n} x^{i}y^{j+1}\mathsf{R}_0^{n+1}\right).$$

Solution for planar Eulerian orientations

Let $\mathsf{R}_0(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

Then the generating function of rooted planar Eulerian orientations counted by edges is

$$\mathsf{G}(t) = \frac{1}{2}\mathsf{Q}(t,0) = \frac{1}{4t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

Asymptotically, the coefficients behave as

$$g_n \sim \kappa \frac{\mu^{n+2}}{n^2(\log n)^2},$$

where $\kappa = 1/16$ and $\mu = 4\pi$.

Solution for quartic Eulerian orientations

Let $\mathsf{R}_1(t)$ be the unique power series with constant term 0 satisfying

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

Then the generating function of quartic rooted planar Eulerian orientations counted by edges is

$$\mathsf{Q}(t,1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

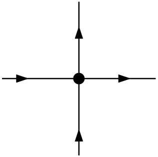
Asymptotically, the coefficients behave as

$$q_n \sim \kappa \frac{\mu^{n+2}}{n^2 (\log n)^2},$$

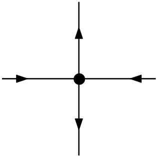
where $\kappa = 1/18$ and $\mu = 4\sqrt{3}\pi$.

Part 2:
Solution for general γ

(Kostov/ E.P. and Zinn-Justin)



Non-alternating
(weight t)



Alternating
(weight $t\gamma$)

BACKGROUND (FROM PHYSICS)

- Solved at criticality by Zinn-Justin in 2000.
- Exactly solved by Kostov later in 2000 (to the satisfaction of physicists).
- Solution was not completely rigorous.
- We corrected a mistake and simplified the form of the solutions.

Recall: Solutions at $\gamma = 0, 1$

The generating function $\mathsf{Q}(t, 0)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{R}_0(t)^{n+1},$$

$$\mathsf{Q}(t, 0) = \frac{1}{2t^2}(t - 2t^2 - \mathsf{R}_0(t)).$$

The generating function $\mathsf{Q}(t, 1)$ is given by

$$t = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}_1(t)^{n+1},$$

$$\mathsf{Q}(t, 1) = \frac{1}{3t^2}(t - 3t^2 - \mathsf{R}_1(t)).$$

PREVIEW: SOLUTION FOR $\mathsf{Q}(t, \gamma)$

Define

$$\vartheta(z, q) = \sum_{n=0}^{\infty} (-1)^n \left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right) q^{(2n+1)^2/8}.$$

Let $q = q(t, \alpha)$ be the unique series satisfying

$$t = \frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha, q)\vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)^2} + \frac{\vartheta''(\alpha, q)}{\vartheta'(\alpha, q)}\right).$$

Define $\mathsf{R}(t, \gamma)$ by

$$\mathsf{R}(t, -2\cos(2\alpha)) = \frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha, q)^2}{\vartheta'(\alpha, q)^2}\left(-\frac{\vartheta'''(\alpha, q)}{\vartheta'(\alpha, q)} + \frac{\vartheta'''(0, q)}{\vartheta'(0, q)}\right).$$

Then

$$\mathsf{Q}(t, \gamma) = \frac{1}{(\gamma+2)t^2}\left(t - (\gamma+2)t^2 - \mathsf{R}(t, \gamma)\right).$$

Outline for general γ

- Bijection
- Functional equations
- Solution using analytic methods Solution using analytic methods

Functional equations for the six vertex model

$\mathsf{Q}(t,\gamma)$ is characterised by equations relating it to generating functions $\mathsf{W}(x) \equiv \mathsf{W}(t,\omega,x)$ and $\mathsf{H}(x,y) \equiv \mathsf{H}(t,\omega,x,y)$.

$$\mathsf{Q}(t,\omega^2+\omega^{-2}) = \mathsf{H}(t,\omega,0,0) \equiv \mathsf{H}(0,0)$$

$$\mathsf{W}(x) = x^2 t\mathsf{W}(x)^2 + \omega x t\mathsf{H}(0,x) + \omega^{-1} x t\mathsf{H}(x,0) + 1$$

$$\mathsf{H}(x,y) = \mathsf{W}(x)\mathsf{W}(y) + \frac{\omega}{y}\left(\mathsf{H}(x,y) - \mathsf{H}(x,0)\right) + \frac{\omega^{-1}}{x}\left(\mathsf{H}(x,y) - \mathsf{H}(0,y)\right).$$

Solving for $\mathsf{C}(t,\omega)$

Think of $\mathsf{W}(x)$ and $\mathsf{H}(x,y)$ as analytic functions then consider

$$U(x) = \frac{x(\omega^2+1)}{1+ix(\omega^2+1)}\mathsf{W}\left(\frac{\omega+\omega^{-1}}{1+ix(\omega^2+1)}\right) + \frac{x(\omega^{-2}+1)}{1-ix(\omega^{-2}+1)}\mathsf{W}\left(\frac{\omega+\omega^{-1}}{1-ix(\omega^{-2}+1)}\right) + \frac{ix^2}{t(\omega^2-\omega^{-2})} - \frac{x}{t(\omega+\omega^{-1})^2}$$

- $U(x)$ is analytic except on two cuts $i\omega[x_1,x_2]$ and $-i\omega^{-1}[x_1,x_2]$
- $U(x)$ satisfies

$$U(i\omega(x\pm i0)) = U(-i\omega^{-1}(x\mp i0)).$$

These + initial conditions characterise $U(x)$.

Solution for $U(x)$

Define

$$\vartheta(z,q)=\sum_{n=0}^{\infty}(-1)^n(e^{(2n+1)iz}-e^{-(2n+1)iz})q^{(2n+1)^2/8}$$

and let $\omega=ie^{-i\alpha}$. Then $U(x)$ is determined by

$$U\left(x_0\frac{\vartheta(z+\alpha,q)}{\vartheta(z,q)}\right)=A+B\wp(z),$$

where $\wp(z)$ is the Weierstrass function and x_0, A, B and q are explicit "constants" (they depend on t and α but not z).

Solution for $\mathsf{Q}(t,\gamma)$

Define

$$\vartheta(z,q)=\sum_{n=0}^{\infty}(-1)^n\left(e^{(2n+1)iz}-e^{-(2n+1)iz}\right)q^{(2n+1)^2/8}.$$

Let $q=q(t,\alpha)$ be the unique series satisfying

$$t=\frac{\cos\alpha}{64\sin^3\alpha}\left(-\frac{\vartheta(\alpha,q)\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)^2}+\frac{\vartheta''(\alpha,q)}{\vartheta'(\alpha,q)}\right).$$

Define $\mathsf{R}(t,\gamma)$ by

$$\mathsf{R}(t,-2\cos(2\alpha))=\frac{\cos^2\alpha}{96\sin^4\alpha}\frac{\vartheta(\alpha,q)^2}{\vartheta'(\alpha,q)^2}\left(-\frac{\vartheta'''(\alpha,q)}{\vartheta'(\alpha,q)}+\frac{\vartheta'''(0,q)}{\vartheta'(0,q)}\right).$$

Then

$$\mathsf{Q}(t,\gamma)=\frac{1}{(\gamma+2)t^2}\left(t-(\gamma+2)t^2-\mathsf{R}(t,\gamma)\right).$$

Part 3:
Comparison between solutions

Comparison between solutions

We now have a general expression for $\mathsf{Q}(t,\gamma)$ and a simpler expression at $\gamma = 0, 1$.

Question: Are these actually the same expression?

Answer: Yes.

Ideas of proof (for $\gamma = 0$):

- We just need to prove that

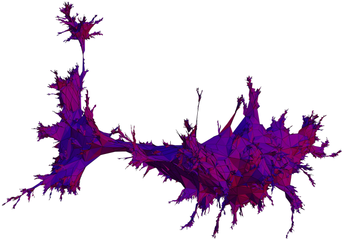
$$t = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}^2 \mathsf{R}(t,0)^{n+1} = \mathsf{R}(t,0)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2\,\middle|\,16\mathsf{R}(t,0)\right).$$

- Our proof involves relations of Ramanujan between theta functions and ${}_2F_1$ and some “well known” theta function identities.

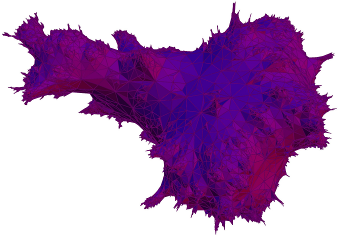
Question: Which solution method is more powerful??

FURTHER QUESTIONS

- Find bijective proofs of the formulas for $\mathsf{Q}(t, 0)$ and $\mathsf{Q}(t, 1)$. These each count a class of labelled trees.
- What do large random Eulerian orientations look like?



A random quadrangulation



A random bipolar triangulation

Images from Jérémie Bettinelli's home page

Thank you!