Tropical ideals do not realise all Bergman fans

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Tropicalisation

Setting: K an infinite field and $v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ a non-Archimedean valuation:

- $v^{-1}(\infty) = \{0\}$
- $v(ab) = v(a) + v(b)$
- $v(a+b) \geq \min\{v(a), v(b)\}$

$\overline{\mathbb{R}}$ is a *semifield* with respect to $\oplus = \min$ and $\odot = +$

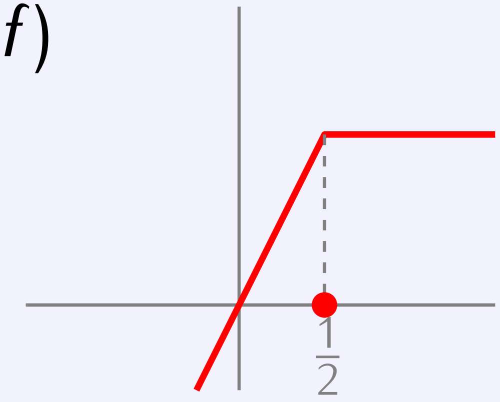
Tropicalising polynomials

$\mathrm{Trop} : K[x_1, \ldots, x_n] \to \overline{\mathbb{R}}[x_1, \ldots, x_n]$

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \mapsto \bigoplus_\alpha v(c_\alpha) \odot x^{\odot\alpha} = \mathrm{Trop}(f)$$

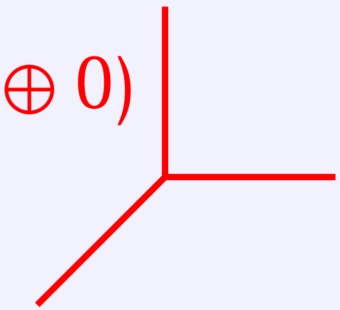
Example: $K = \mathbb{Q}$, $v = 2$-adic, $f = x^2 - 2 \rightsquigarrow$

$\mathrm{Trop}(f) = (0 \odot x^{\odot 2}) \oplus 1 : x \mapsto \min\{2x, 1\}$



Tropicalisation

$V_{\overline{\mathbb{R}}}(x_1 \oplus x_2 \oplus 0)$



Tropical hypersurface

$h \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$

$\rightsquigarrow V_{\mathbb{R}}(h) := \{p \in \mathbb{R}^n \mid$ the min in $h(p)$ is attained $\geq$ twice$\}$

Tropicalising an ideal

$I \subseteq K[x_1, \ldots, x_n] \rightsquigarrow \mathrm{Trop}(I) := \{\mathrm{Trop}(f) \mid f \in I\} \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$

[Bieri-Groves, Einsiedler-Kapranov-Lind, Speyer-Sturmfels, Payne, D, …]

Tropicalising a variety, fundamental theorem

K algebraically closed, v nontrivial, $I \subseteq K[x_1, \ldots, x_n]$, then

$V_{\mathbb{R}}(\mathrm{Trop}(I)) := \bigcap_{h \in \mathrm{Trop}(I)} V_{\mathbb{R}}(h) = \overline{\{v(x) \mid x \in V_{K^*}(I)\}}$.

Structure theorem: If $V_{K^*}(I)$ is irreducible of dim d, then $V_{\mathbb{R}}(\mathrm{Trop}(I))$ is the support of a finite, balanced, weighted polyhedral complex of dimension d.

Tropical ideals [Maclagan-Rincón]

Proposal to axiomatise the *algebra* side of tropical geometry.

Notation: $f \in K[x_1, \ldots, x_n]$ or $f \in \overline{\mathbb{R}}[x_1, \ldots, x_n] \rightsquigarrow$ write $[f]_{x^\alpha}$ for the coefficient of x^α in f.

Observation: if $f, g \in I \subseteq K[x_1, \ldots, x_n]$ with $[f]_{x^\alpha} = [g]_{x^\alpha}$, then $h := f - g \in I$ has the following properties:
- $[h]_{x^\alpha} = 0$ and
- for all $\beta \neq \alpha$, $v([h]_{x^\beta}) \geq \min\{v([f]_{x^\beta}), v([g]_{x^\beta})\}$ with equality if the two valuations are different.

Definition: an $\overline{\mathbb{R}}$-subsemimodule $J \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is a *tropical ideal* if $x_i \odot J \subseteq J$ and for all $f, g \in J$ and x^α with $[f]_{x^\alpha} = [g]_{x^\alpha}$ there exists an $h \in J$ such that $[h]_{x^\alpha} = \infty$ and for $\beta \neq \alpha$: $[h]_{x^\beta} \geq \min\{[f]_{x^\beta}, [g]_{x^\beta}\}$ with equality if distinct.

Tropical ideals [Maclagan-Rincón]

Equivalently: $J_{\leq d} := \overline{\mathbb{R}}[x_1, \ldots, x_n]_{\leq d}$ is a *tropical linear space* related to a *valuated matroid* [Dress-Wenzel], for each d; and $x_i \odot J_{\leq d} \subseteq J_{\leq d+1}$. Note that $\mathrm{Trop}(I)$ is a tropical ideal.

Key results *[Maclagan-Rincón]*

• tropical ideals satisfy the ascending chain condition (but are not finitely generated) and

• a tropical ideal J defines a finite polyhedral complex $\bigcap_{h \in J} V_{\mathbb{R}}(h)$ equipped with weights, called its *tropical variety*.

Motivating question:

Which weighted polyhedral complexes arise in this manner?

It is not known if balancedness is necessary, nor do we have a notion of prime tropical ideal.

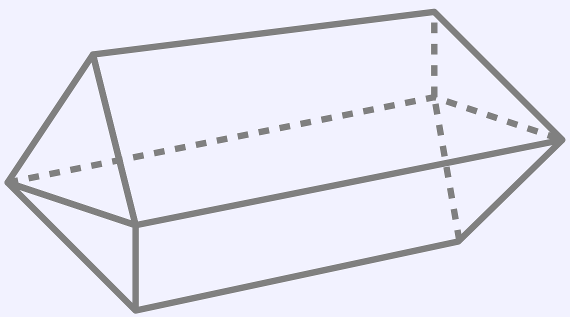
Main result

Theorem *[Draisma-Rincón]*

The Bergman fan of $U_{2,3} \oplus V_8$, with weight 1 on the top-dimensional fans, is *not* the tropical variety of any tropical ideal.

- U_{23} is the uniform matroid of rank 2 on 3 elements, with Bergman fan: $(\mathbb{R}_{\geq 0}e_1 \cup \mathbb{R}_{\geq 0}e_2 \cup \mathbb{R}_{\geq 0}e_3) + \mathbb{R}(1,1,1) \subseteq \mathbb{R}^3$.

- V_8 is the *Vámos matroid* of rank 4 on 8 elements, so its Bergman fan is a 4-dimensional fan in $\mathbb{R}^8$.



The direct sum of these fans is a 6-dimensional fan in $\mathbb{R}^{11}$, and is not the tropical variety of any tropical ideal in 11 variables.

Tensor products of matroids? [Las Vergnas]

Given K-vector spaces V, W and nonzero vectors $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_n \in W$, get vectors $v_i \otimes w_j \in V \otimes W$ and three matroids: M on $[m]$, N on $[n]$ and P on $[m] \times [n]$.

P has the following properties:

- for each $i \in [m]$, $j \mapsto (i, j)$ is an iso from M to $P|_{\{i\}\times[n]}$;
- for each $j \in [n]$, $i \mapsto (i, j)$ is an iso from N to $P|_{[m]\times\{j\}}$; and
- $\mathrm{rk}\,(P) \geq \mathrm{rk}\,(M) \times \mathrm{rk}\,(N)$.

Question: for *general* matroids M and N on $[m], [n]$, does a *tensor product* P with these properties exist?

Theorem *[Las Vergnas]*

No, e.g., not for $M = U_{2,3}$ and $N = V_8$.

Proof sketch of our theorem

Theorem (D-Rincón)
The Bergman fan of $U_{2,3} \oplus V_8$, with weight 1 on the top-dimensional fans, is *not* the tropical variety of any tropical ideal J.

• Call the first three variables x_1, x_2, x_3 and the last eight variables $y_1, \ldots, y_8$. Reduce to the case where J is homogeneous and saturated with respect to $\langle x_1, x_2, x_3, y_1, \ldots, y_8 \rangle$.

• Show that $M(J_1)|_{\{x_1,x_2,x_3\}} = U_{2,3}$ and $M(J_1)|_{\{y_1,\ldots,y_8\}} = V_8$.

• Show that $M(J_2), M(J_2)|_{\{x_i x_j\}}, M(J_2)|_{\{y_i y_j\}}$ have ranks:

$$\binom{2+4+1}{2} = 21, \ \leq \binom{2+1}{2} = 3, \ \leq \binom{4+1}{2} = 10,$$

so $M(J_2)|_{\{x_i y_j\}}$ has rank $\geq 21 - 13 = 8 = 2 \cdot 4$. Find: $M(J_2)_{\{x_i y_j\}}$ is a tensor product of $U_{2,3}$ and V_8, a contradiction. $\square$

Summary and outlook

• Not all tropical linear spaces are the tropical varieties of tropical ideals. (How about Bergman(V_8)?)

• Challenge: describe properties of varieties of tropical ideals, e.g. balancedness?

• Challenge: are there notions of prime tropical ideal, irreducible tropical variety, and tropical-algebraic matroids?

Thank you!