A new formula for Stanley's chromatic symmetric function for unit interval graphs and e-positivity for triangular ladder graphs

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Overview

1. Chromatic symmetric functions.
2. The $(3+1)$-free conjecture.
3. Triangular ladders, TL_n.
4. Chromatic symmetric functions in non-commuting variables.
5. Deletion-contraction.
6. Semi-symmetrized e-positivity.
7. Signed formula for unit interval graphs.
8. Sign-reversing involution for TL_n.
9. Further work.

Graphs colorings

Given G with vertex set V a *proper coloring* κ of G is

$$\kappa : V \to \{1, 2, 3, \ldots\}$$

so if $u, v \in V$ are joined by an edge then

$$\kappa(u) \neq \kappa(v).$$

Chromatic symmetric function: Stanley 1995

Given a proper coloring κ of vertices $v_1, \ldots, v_N$ we associate a monomial in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}.$$

$x_1^2 x_2$ $\qquad$ $x_1 x_2 x_3$

The *chromatic symmetric function* is

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

summed over all proper colorings κ.

$$X_{P_3} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \cdots$$
$$+ 6 x_1 x_2 x_3 + \cdots$$

Symmetric functions

The *algebra of symmetric functions*, Λ, contains all formal power series f in commuting variables $x_1, x_2, \ldots$ such that for all permutations π

$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

$$f(x) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \cdots$$

Fact: Any basis of Λ is indexed by integer partitions.

An *integer partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n, $\lambda \vdash n$, is a list of positive integers whose *parts* λ_i weakly decrease and sum to n.

$$(3, 1, 1) = (3, 1^2) \vdash 5$$

Classic Bases: elementary

The *i-th elementary symmetric function*, e_i, is

$$e_i = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} \ldots x_{j_i}$$

and

$$e_\lambda = e_{\lambda_1} \ldots e_{\lambda_\ell}.$$

$$e_{(2,1)} = e_2 e_1 = (x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$

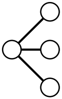
For the complete graph K_n on n vertices

$$X_{K_n} = n! e_n.$$

e-POSITIVITY

Call a graph G *e-positive* if X_G is a non-negative sum of elementary symmetric functions.

$G =$  has $X_G = e_{(2,1)} + 3e_{(3)}$. ✔

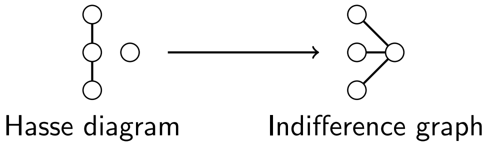
$K_{31} =$  has $X_{K_{31}} = e_{(2,1,1)} - 2e_{(2,2)} + 5e_{(3,1)} + 4e_{(4)}$. ✘

The *claw,* K_{31}, is the smallest graph that is not e-positive.

$(3+1)$-free conjecture

Conjecture (Stanley-Stembridge 1993)

If G is an indifference graph of a $(3+1)$-free poset then X_G is e-positive.



Theorem (Guay-Paquet 2013)

It is sufficient to prove the Stanley-Stembridge conjecture for all $(2+2)$ and $(3+1)$-free posets.

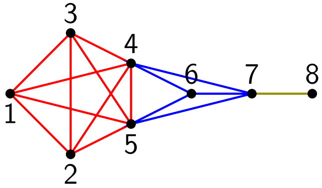
Interval graphs

The indifference graphs for $(2+2)$ and $(3+1)$-free posets are *unit interval graphs*. Construct an unit interval graphs from a collection of integer intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_l, b_l].$$

On each interval we place a complete graph.

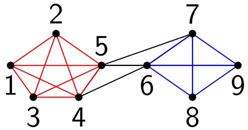
The interval graph for the intervals $[1,5]$, $[4,7]$ and $[7,8]$ is



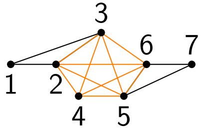
KNOWN e-POSITIVE UNIT INTERVAL GRAPHS

- The paths $[1,2],[2,3],\ldots,[n-1,n]$ (Stanley 1995).

- Any list containing $[1,j]$ and $[j+1,n]$ (Stanley 1995).



- Any list containing $[2,n-1]$ (Cho and Huh 2017).



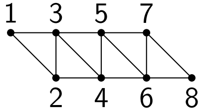
- Any list $[1,j_1],[j_1,j_2],\ldots,[j_k,n]$ (Gebhard and Sagan 2001).

Triangular Ladders

The graph $P_{n,2}$ comes from intervals $[1,3],[2,4],\ldots,[n-2,n]$, which we will call *triangular ladders*, TL_n.

The graph TL_8 comes from

$$[1,3],[2,4],[3,5],[4,6],[5,7],[6,8].$$



In 1995 Stanley wrote

"It remains open whether $P_{d,2}$ is e-positive."

Chromatic symmetric functions in non-commuting variables

A generalization by Gebhard and Sagan (2001).

Fix an ordering on the vertices vertices $v_1, \ldots, v_N$. The *chromatic symmetric function in non-commuting variables* is

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

summed over all proper colorings κ.

$G =$ 

$$Y_G = x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + x_3x_1x_3 + \cdots$$
$$+ x_1x_2x_3 + x_1x_3x_2 + x_2x_1x_3 + x_2x_3x_1 + \cdots$$

Fact: The vertex labeling matters.

Symmetric functions in non-commuting variables

The *algebra of symmetric functions in non-commuting variables*, NCSym, contains all formal power series f in non-commuting variables $x_1, x_2, \ldots$ such that for all permutations π

$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

$$f(x) = x_1x_1x_2 + x_2x_2x_1 + x_1x_1x_3 + x_3x_3x_1 + x_2x_2x_3 + x_3x_3x_2 + \cdots$$

Fact: Any basis of NCSym is indexed by set partitions.

An *set partition* $\pi = B_1/B_2/\cdots/B_k$ of $[n] = \{1, 2, \ldots n\}$, $\pi \vdash [n]$ is a collection of nonempty disjoint subsets B_i called *blocks* that union to $[n]$.

$$\{1,4\}/\{2,5\}/\{3\} = 14/25/3 \vdash [5]$$

Classic Bases: elementary

For $\pi \vdash [n]$ the *elementary symmetric function in non-commuting variables*, e_π, is

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

where $i_j \neq i_k$ if j and k are in the same block of π.

$$e_{12/3} = x_1 x_2 x_2 + x_1 x_2 x_1 + \cdots + x_1 x_2 x_3 + \cdots$$

For the complete graph K_n on n vertices

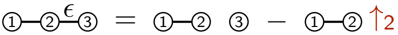
$$Y_{K_n} = e_{12\cdots n}.$$

Deletion-contraction for Y_G

To *delete* an edge ϵ, $G - \epsilon$, remove ϵ.

To *contract* an edge ϵ between u and v, G/ϵ, merge u and v and any multiedges created.

Deletion-Contraction for Y_G



$$Y_{P_3} = (x_1x_2x_1 + x_1x_2x_2 + x_1x_2x_3 + \cdots) - (x_1x_2x_2 + \cdots)$$

Given a monomial of degree $n-1$ define the *induced monomial* for $j < n$ to be

$$x_{i_1}x_{i_2}\cdots x_{i_j}\cdots x_{i_{n-1}} \uparrow_j = x_{i_1}x_{i_2}\cdots x_{i_j}\cdots x_{i_{n-1}}x_{i_j}.$$

Theorem (Gebhard and Sagan 2001)

For G with vertices $V = [n]$ and an edge ϵ between vertices j and n we have

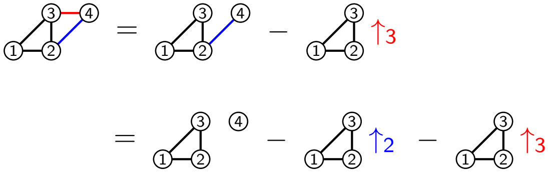
$$Y_G = Y_{G-\epsilon} - Y_{G/\epsilon}\uparrow_j .$$

Induction on monomials

Theorem (D 2018)

G is a unit interval graph with intervals $[a_1, 1], [a_2, 2], \ldots, [a_n, n]$ and G' is G after removing vertex n. Then,

$$Y_G = Y_{G'} Y_{K_1} - \sum_{i=a_n}^{n-1} Y_{G'} \uparrow_i .$$



Semi-symmetrizing

$$e_{12}\uparrow_1 = \frac{1}{2}\left(e_{12/3} + e_{1/23} - e_{13/2} - e_{123}\right) \equiv \frac{1}{2}\left(e_{12/3} - e_{123}\right)$$

For $\pi \vdash [n]$ let $\lambda(\pi) \vdash n$ be formed by all the block sizes.

$$\lambda(1/23) = \lambda(13/2) = (2,1) \quad \text{and} \quad 1/23 \sim 13/2$$

Say two set partitions $\pi \vdash [n]$ and $\sigma \vdash [n]$ are *related*, $\pi \sim \sigma$, if

1. $\lambda(\pi) = \lambda(\sigma)$ and
2. the sizes of the blocks containing n are the same.

If $\pi \sim \sigma$ we say e_π and e_σ are *equivalent*,

$$e_\pi \equiv e_\sigma.$$

Extend this definition linearly.

Semi-symmetrizing

For $\pi \vdash [n-1]$ define

$$\pi \oplus_j n \vdash [n]$$

to be the integer partition where we place n in the same block as j.

Theorem (Gebhard and Sagan 2001)

For $\pi \vdash [n-1]$, $j < n$ and b the size of the block in π containing $n-1$ we have

$$e_\pi \uparrow_j \equiv \frac{1}{b}\left(e_{\pi/n} - e_{\pi \oplus_j n}\right).$$

Call G *semi-symmetrized e-positive* if $Y_G \equiv f$ for some $f \in \text{NCSym}$ that is a sum of nonnegative e_π.

Fact: If G is semi-symmetrized e-positive then G is e-positive.

New formula for unit interval graphs

Theorem (D 2018)

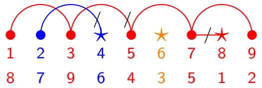
For a unit interval graph G on n vertices,

$$Y_G \equiv \frac{1}{n!} \sum_{D \in \mathcal{A}'_L(G)} (-1)^{t(D)} e_{\pi(D)}.$$

Arc diagrams $D \in \mathcal{A}'_L(G)$ are defined by:

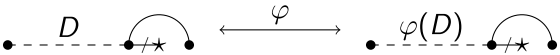
- all vertices have at most one left arc,
- each arc possibly has a tic mark,
- a permutation labeling increasing on all pieces and
- one vertex in each right-most piece is marked with a star.

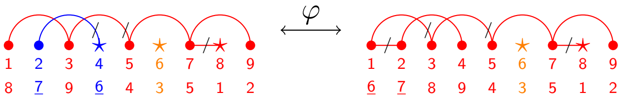
$D \in \mathcal{A}'_L(TL_9)$ with $t(D) = 3$ and $\pi(D) = 13/2/4/57/6/89$.



The sign-reversing involution for TL_n

The general inductive idea:





$\pi(D) = 13/2/4/57/6/89$ $\quad$ $\pi(\varphi(D)) = 1/2/34/57/6/89$

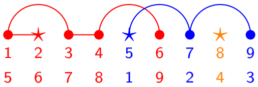
$t(D) = 3$ $\quad$ $t(\varphi(D)) = 4$

The sign-reversing involution:

- changes $t(D)$ by one and
- has $\pi(D) \sim \pi(\varphi(D))$.
- There are 18 cases where D is a fixed point.

The sign-reversing involution

$D \in \mathcal{A}_L(TL_9)$ with $\pi(D) = 12346/579/8$ is a fixed point.



Fixed points:

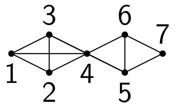
- have no tic marks,
- have a star on each connected component and
- satisfy 5 other more detailed conditions.

New family of e-positive graphs

Theorem (D 2018)

The triangular ladder TL_n, $n \geq 1$, is semi-symmetrized e-positive and so e-positive.

Given a graph G on $[n]$ and H on $[m]$ their *concatenation* is the graph $G \cdot H$ on $[n+m-1]$ where G is on the first n vertices and H is on the last m.

$K_4 \cdot TL_4 =$ 

Theorem (Gebhard and Sagan 2001)

If a graph G is semi-symmetrized e-positive then so is the concatenation $G \cdot K_m$ and $G \cdot TL_4$.

New family of e-positive graphs

Proposition (D 2018)

Any graph G such that

$$G = G_1 \cdot G_2 \cdots G_l,$$

where $G_i = TL_{n_i}$ or $G_i = K_{n_i}$, is a semi-symmetrized e-positive graph, so also e-positive.

Further work:

- Investigate the relationship between the positive terms of TL_n and acyclic orientations.
- By computer calculation all unit interval graphs up to $n = 7$ vertices are semi-symmetrized e-positive. Investigate if this is true for all unit interval graphs.

Thank you very much!