SET-PARTITION TABLEAUX

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Set-Partition Tableaux

Integer Partition:

		$8, \underline{13}$	$\underline{14}$
$2, \underline{3}$	$1, 6, \underline{10}$	$\underline{16}$	
$\underline{4}$	$\underline{12}$		
$5, 7, \underline{9}$	$15, \underline{17}$		
$\underline{11}$			

Set Partition:

$$\left\{ \begin{array}{l} \{2, \underline{3}\}, \{\underline{4}\}, \{5, 7, \underline{9}\}, \\ \{1, 6, \underline{10}\}, \{\underline{11}\}, \{\underline{12}\}, \\ \{8, \underline{13}\}, \{\underline{14}\}, \{\underline{16}\}, \\ \{15, \underline{17}\} \end{array} \right\}$$

Organization:

I. Origins: Representation Theory of the Symmetric Group

II. Schur-Weyl Duality: The Partition Algebra and Other Diagram Algebras

III. Insertion Bijections

May 2016:

[Benkart-H-Harmon] *Dimensions of irreducible modules ...*

[Orellana-Zabrocki] *Symmetric group characters as symmetric functions*

I. Symmetric Group Tensor Power Representations

Origins: The Symmetric Group $\mathbf{S}_n$

- $\mathbf{M}_n$ = n-dimensional permutation module
  $\cong \mathbf{S}_n^{\ydiagram{6}} \oplus \mathbf{S}_n^{\ydiagram{5,1}}$
  - Basis: $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with group action: $\sigma(\mathbf{v}_i) = \mathbf{v}_{\sigma(i)}$
- $\mathbf{S}_n^{\lambda}$ = irreducible $\mathbb{C}\mathbf{S}_n$-module, $\lambda \vdash n$
- $\mathbf{M}_n^{\otimes k}$ = k-fold tensor product module
  - Diagonal action on basis of simple tensors:
    $\sigma(\mathbf{v}_{i_1} \otimes \mathbf{v}_{i_2} \otimes \cdots \otimes \mathbf{v}_{i_k}) = \mathbf{v}_{\sigma(i_1)} \otimes \mathbf{v}_{\sigma(i_2)} \otimes \cdots \otimes \mathbf{v}_{\sigma(i_k)}$

Question

Determine the multiplicity $\mathbf{m}_{k,n}^{\lambda}$ in the decomposition:

$$\mathbf{M}_n^{\otimes k} = \bigoplus_{\lambda \vdash n} \mathbf{m}_{k,n}^{\lambda} \, \mathbf{S}_n^{\lambda}$$

Method 1: Restriction-Induction

Tensor Identity: tensoring with the permutation module is the same as restriction and induction

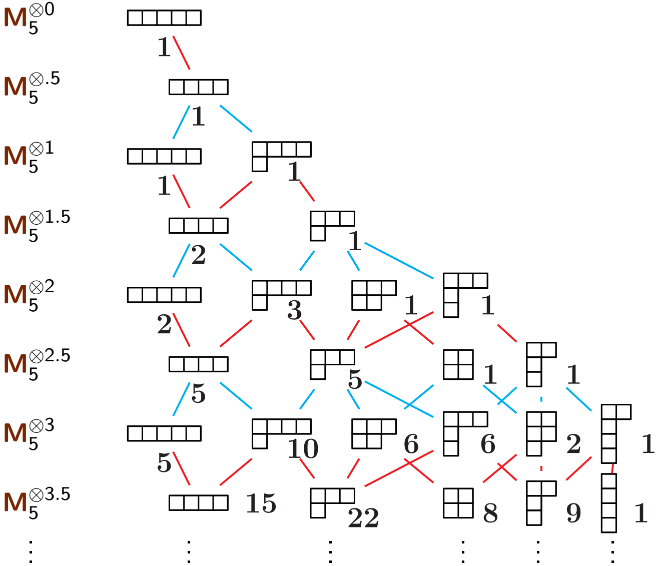
$$\mathbf{S}_n^\lambda \otimes \mathbf{M}_n \cong \mathbf{Ind}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n} \mathbf{Res}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n} (\mathbf{S}_n^\lambda)$$

$$\cong \mathbf{Ind}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n} \bigoplus_{\nu=\lambda-\square} \mathbf{S}_{n-1}^\nu$$

$$\cong \bigoplus_{\mu=\nu+\square} \bigoplus_{\nu=\lambda-\square} \mathbf{S}_n^\mu$$

$$\mathbf{S}_n^\lambda \otimes \mathbf{M}_n \cong \bigoplus_{\mu=(\lambda-\square)+\square} \mathbf{S}_n^\mu$$

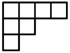
Bratteli Diagram: $\mathcal{B}(\mathbf{S}_5, \mathbf{S}_4)$ for $\mathbf{M}_5^{\otimes k}$

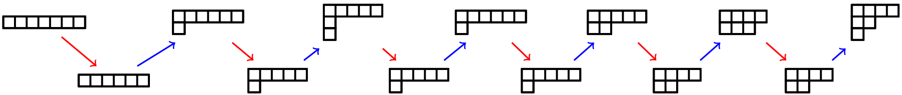


Vacillating Tableaux

A vacillating tableaux of shape $\lambda \vdash n$ is a sequence of partitions for which

$$\lambda_{i+\frac{1}{2}} = \lambda_i - \square \quad \text{and} \quad \lambda_{i+1} = \lambda_{i+\frac{1}{2}} + \square.$$

Example: A vacillating tableaux of length 6 and shape :



The multiplicity of $\mathbf{S}_n^{\lambda}$ in $\mathbf{M}_n^{\otimes k}$ is given by

$$\mathbf{m}_{k,n}^{\lambda} = \#\ \text{vacillating tableaux of length } k \text{ and shape } \lambda.$$

Method 2: Decompose $\mathbf{M}_n^{\otimes k}$ into Permutation Modules

- Diagonal Action:

$$\sigma(\mathbf{v}_a \otimes \mathbf{v}_a \otimes \mathbf{v}_b \otimes \mathbf{v}_a \otimes \mathbf{v}_b \otimes \mathbf{v}_c \otimes \mathbf{v}_d \otimes \mathbf{v}_c)$$
$$= \mathbf{v}_{\sigma(a)} \otimes \mathbf{v}_{\sigma(a)} \otimes \mathbf{v}_{\sigma(b)} \otimes \mathbf{v}_{\sigma(a)} \otimes \mathbf{v}_{\sigma(b)} \otimes \mathbf{v}_{\sigma(c)} \otimes \mathbf{v}_{\sigma(d)} \otimes \mathbf{v}_{\sigma(c)}$$

- Partition tensor positions: $P = \{1, 2, 4 \mid 3, 5 \mid 6, 8 \mid 7\}$:
  $\mathbf{v}_{i_j} = \mathbf{v}_{i_\ell}$ iff $j \sim \ell$ in P

- As $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ vary over distinct elements of $\{1, \ldots, n\}$, these simple tensors span a *submodule* isomorphic to the permutation module

$$\mathcal{M}^{(n-4,1,1,1,1)} = \mathbf{Ind}_{\mathbf{S}_{n-4} \times \mathsf{S}_1 \times \mathsf{S}_1 \times \mathsf{S}_1 \times \mathsf{S}_1}^{\mathbf{S}_n}(\mathbf{1}).$$

Decompose $\mathbf{M}_n^{\otimes k}$ into Permutation Modules

$$\mathbf{M}_n^{\otimes k} \cong \bigoplus_{t=0}^{n} \left\{ {k \atop t} \right\} \mathcal{M}^{(n-t,1^t)} \cong \bigoplus_{t=0}^{n} \left\{ {k \atop t} \right\} \bigoplus_{\lambda \vdash n} \mathbf{K}_{\lambda,(n-t,1^t)} \mathbf{S}_n^{\lambda}$$

$$\cong \bigoplus_{t=0}^{n} \bigoplus_{\lambda \vdash n} \left\{ {k \atop t} \right\} \mathbf{f}^{\lambda/(n-t)} \mathbf{S}_n^{\lambda}$$

- $\left\{ {k \atop t} \right\} = \#$ set partitions of $\{1, \ldots, k\}$ into t subsets (Stirling 2^{nd})
- $\mathbf{K}_{\lambda,(n-t,1^t)}$ = Kostka number = #semistandard tableaux of shape λ filled with $\underbrace{0, \ldots, 0}_{n-t}, 1, 2, \ldots, t$.
- Example:

0	0	0	3	7
1	4	6	8	
2	9			
5				

has $\lambda = (5, 4, 2, 1)$, $n = 12$, $t = 9$.

- $\mathbf{K}_{\lambda,(n-t,1^t)} = \mathbf{f}^{\lambda/(n-t)} = \#$ standard tableaux shape $\lambda/(n-t)$

Decompose $\mathbf{M}_n^{\otimes k}$ into Permutation Modules

$$\mathbf{M}_n^{\otimes k} \cong \bigoplus_{\lambda \vdash n} \bigoplus_{t=0}^{n} \left\{ {k \atop t} \right\} \mathbf{f}^{\lambda/(n-t)} \mathbf{S}_n^{\lambda}$$

$$\mathbf{m}_{k,n}^{\lambda} = \sum_{t=0}^{n} \left\{ {k \atop t} \right\} \mathbf{f}^{\lambda/(n-t)} = \#\left\{ (P,T) \,\middle|\, \begin{array}{l} P = \text{partition of } \{1,\ldots,k\} \text{ into } t \text{ parts} \\ T = \text{standard tableau of shape } \lambda/(n-t) \end{array} \right\}$$

Example. A standard set-partition tableau of shape $\lambda = (5,4,2,1)$

$P = \{1,\underline{6}|4,7,9,\underline{10}|2,11,\underline{12}|8,\underline{14}|15,\underline{16}|5,13,\underline{18}|3,17,\underline{19}|\underline{20}\}$ $\quad t = 8.$

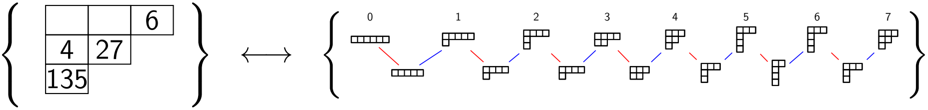
$T =$

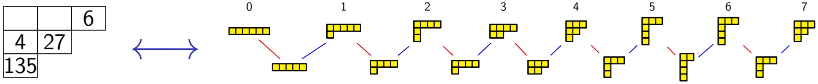
				$3,17,\underline{19}$
$1,\underline{6}$	$4,7,9,\underline{10}$	$5,13,\underline{18}$	$\underline{20}$	
$2,11,\underline{12}$	$8,\underline{14}$			
$15,\underline{16}$				

$n = 12$

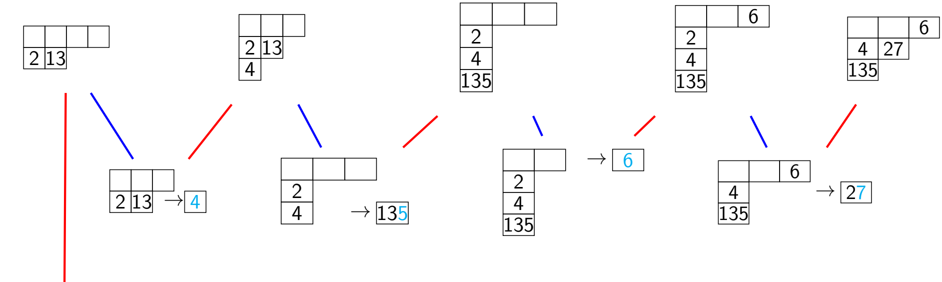
Set-Partition Tableaux $\longleftrightarrow$ Vacillating Tableaux

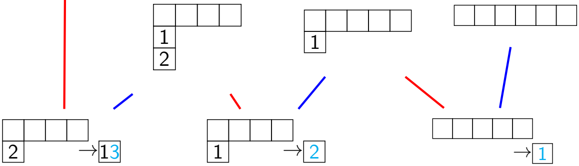
$$\mathbf{m}^{\lambda}_{k,n} = \# \left\{ \begin{array}{l} \text{Standard set-partition} \\ \text{tableaux of shape} \\ \lambda/(n-k) \end{array} \right\} = \# \left\{ \begin{array}{l} \text{Vacillating tableaux} \\ \text{of length } k \\ \text{and shape } \lambda \end{array} \right\}$$





[BH'19] H-Benkart, [COSSZ'19] Colmenarejo, Orellana, Saliola, Schilling, Zabrocki





1. Remove box containing $m = max(T)$
2. Delete m from the box
3. Schensted insert box in $T_{>1}$

II. Schur-Weyl Duality and the Partition Algebra

Centralizer Algebra of $\mathbf{S}_n$ on $\mathbf{M}_n^{\otimes k}$

Centralizer Algebra:

$$\mathbf{Z}_{k,n} := \mathbf{End}_{\mathbf{S}_n}(\mathbf{M}_n^{\otimes k}) = \left\{ \phi \in \mathbf{End}(\mathbf{M}_n^{\otimes k}) \mid \phi\sigma(x) = \sigma\phi(x), \sigma \in \mathbf{S}_n \right\}$$

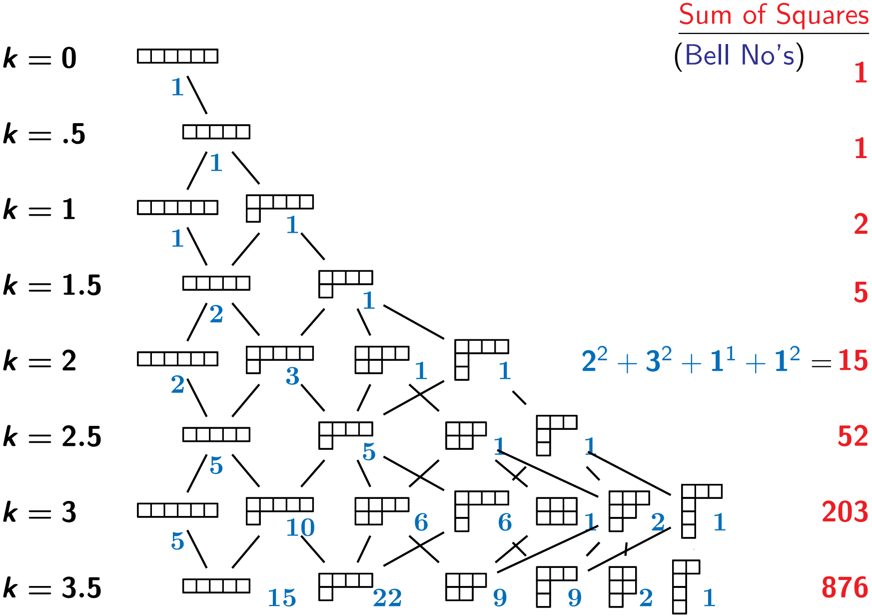
Schur-Weyl Duality:

$$\mathbf{M}_n^{\otimes k} \cong \underbrace{\bigoplus_{\lambda \vdash n} \mathbf{m}_{k,n}^{\lambda} \mathbf{S}_n^{\lambda}}_{\text{as an } \mathbf{S}_n\text{-module}} \cong \underbrace{\bigoplus_{\lambda \vdash n} \mathbf{f}^{\lambda} \mathbf{Z}_{k,n}^{\lambda}}_{\text{as a } \mathbf{Z}_{k,n}\text{-module}}$$

- $\mathbf{m}_{k,n}^{\lambda} = \mathrm{mult}_k(\mathbf{S}_n^{\lambda}) = \dim(\mathbf{Z}_{k,n}^{\lambda}) =$ #(Standard Set-Partition Tableaux)
- $\mathbf{f}^{\lambda} = \dim(\mathbf{S}_n^{\lambda}) = \mathrm{mult}(\mathbf{Z}_{k,n}^{\lambda}) =$ # (Standard Tableaux)

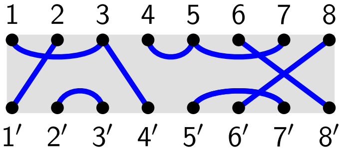
Artin-Wedderburn theory: $\dim(\mathbf{Z}_{k,n}) = \sum_{\lambda \vdash n} (\mathbf{m}_{k,n}^{\lambda})^2$

Bratteli Diagram: $\mathcal{B}(\mathbf{S}_6, \mathbf{S}_5) = \mathcal{B}(\mathbf{Z}_{k,6})$



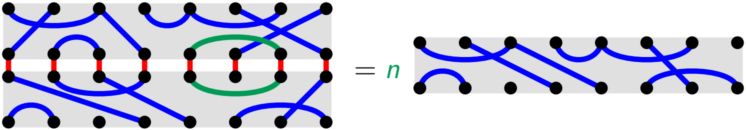
Partition Algebra $\mathbf{P}_k(n)$ [P.P. Martin, V.F.R. Jones, ≈1993]

Basis of set partitions of $\{1, \ldots, k, 1', \ldots, k'\}$.

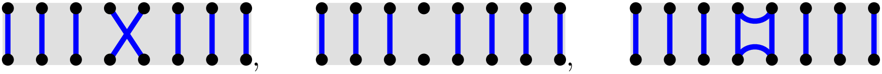


$$= \left\{ \begin{array}{l} \{1,3,4'\}, \{2,1'\}, \{4,5,7\}, \\ \{6,8'\}, \{8,6'\}, \{2',3'\}, \{5',7'\} \end{array} \right\}$$

Multiplication given by diagram concatenation:

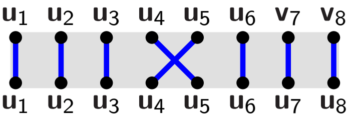


Generated by 3 types of diagrams:

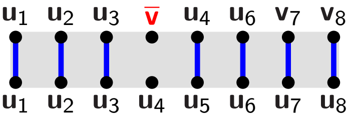


Action of $\mathbf{P}_k(n)$ on Tensor Space $\mathbf{M}_n^{\otimes k}$

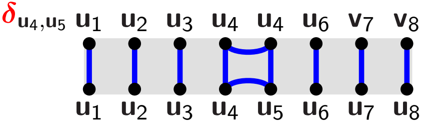
Transposition:



- Commutes with $\mathbf{S}_n$: $\mathbf{P}_k(n) \to \mathrm{End}_{\mathbf{S}_n}(\mathbf{M}_n^{\otimes k})$
  - is surjective
  - it is injective if $n \geq 2k$ the stable case
  - kernel? [Benkart-H'19]



$\overline{\mathbf{v}} = \sum_{i=1}^{n} \mathbf{v}_i$ projection onto trivial module



Irreducible Modules for the Partition Algebra

Schur-Weyl Duality: $\mathbf{M}_n^{\otimes k} \cong \underbrace{\bigoplus_{\lambda \vdash n} \mathbf{m}_{k,n}^{\lambda} \mathbf{S}_n^{\lambda}}_{\text{as an } \mathbf{S}_n\text{-module}} \cong \underbrace{\bigoplus_{\lambda \vdash n} \mathbf{f}^{\lambda} \mathbf{P}_k^{\lambda}}_{\text{as a } \mathbf{P}_k(n)\text{-module}}$

- $\mathbf{m}_{k,n}^{\lambda} = \dim(\mathbf{P}_k^{\lambda}) = \#$(Standard Set-Partition Tableaux)
- $\mathbf{f}^{\lambda} = \dim(\mathbf{S}_n^{\lambda}) = \#$ (Standard Tableaux)

The irreducible partition algebra module:

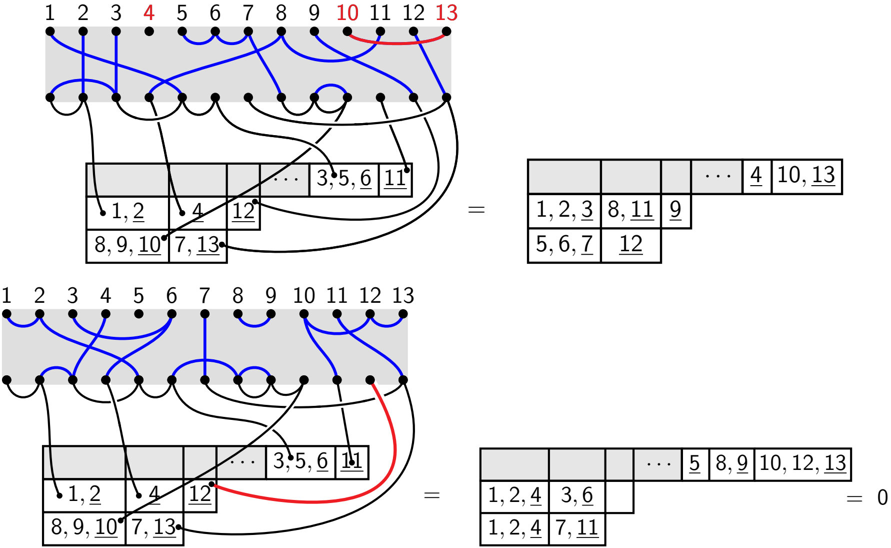
$$\mathbf{P}_k^{\lambda} = \mathbb{C}\text{-span}\left\{ \mathbf{v}_T \;\middle|\; T \in \left\{ \begin{array}{l} \text{standard set-partition} \\ \text{tableaux of shape } \lambda \end{array} \right\} \right\}$$

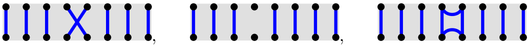
Question

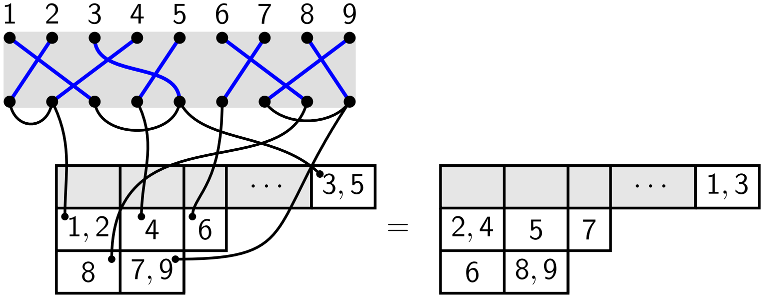
Is there a combinatorial action analogous to Young's representations of $\mathbf{S}_n$ on standard tableaux?

Action on Basis Indexed by Set-Partition Tableaux

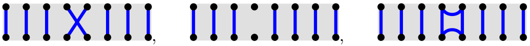
[H-Jacobson, 2018]

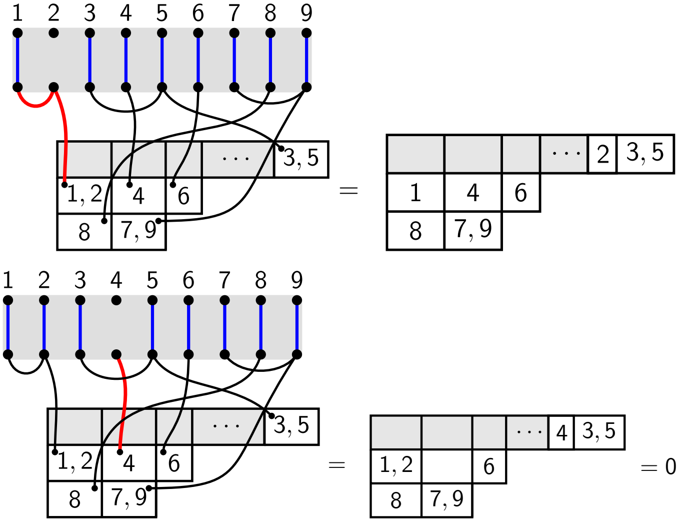


Action of Generators 

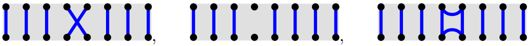


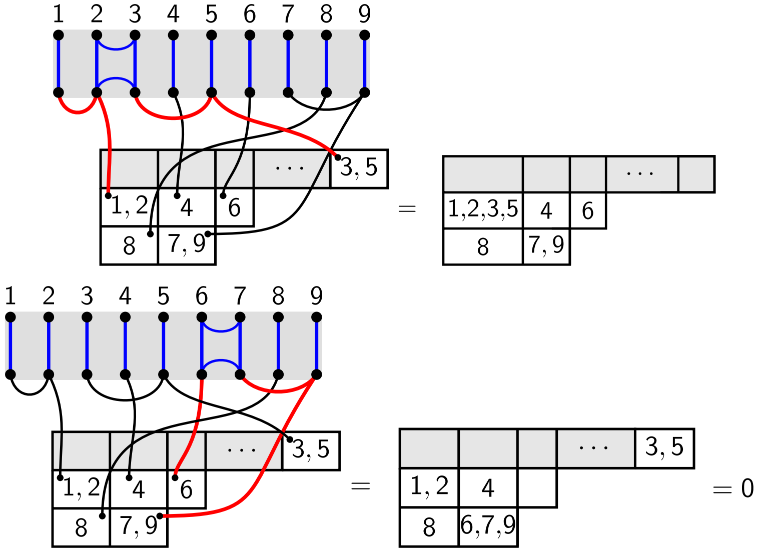
Action of Generators





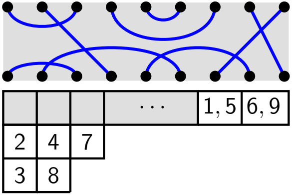
Action of Generators



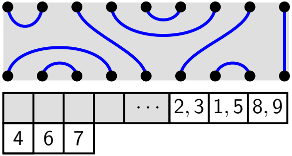


Restricts Naturally to Subalgebras of the Partition Algebra

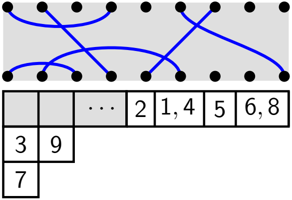
Brauer Algebra
(matchings)



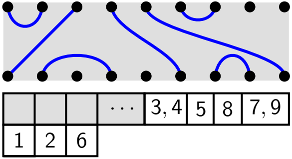
Temperley-Lieb
(planar matchings)



Rook Brauer Algebra
(partial matchings)



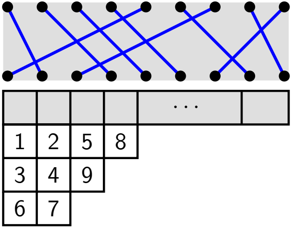
Motzkin Algebra
(planar partial matchings)



Restricts Naturally to Subalgebras of the Partition Algebra

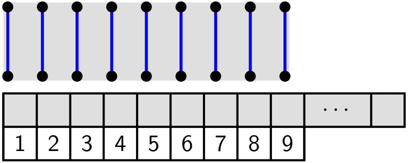
Symmetric Group

(permutations)



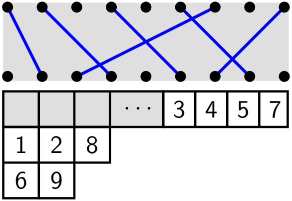
Identity

(planar permutations)



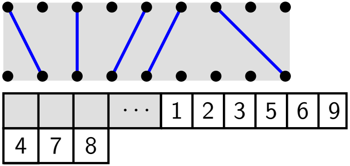
Rook Monoid

(partial permutations)



Planar Rook Monoid

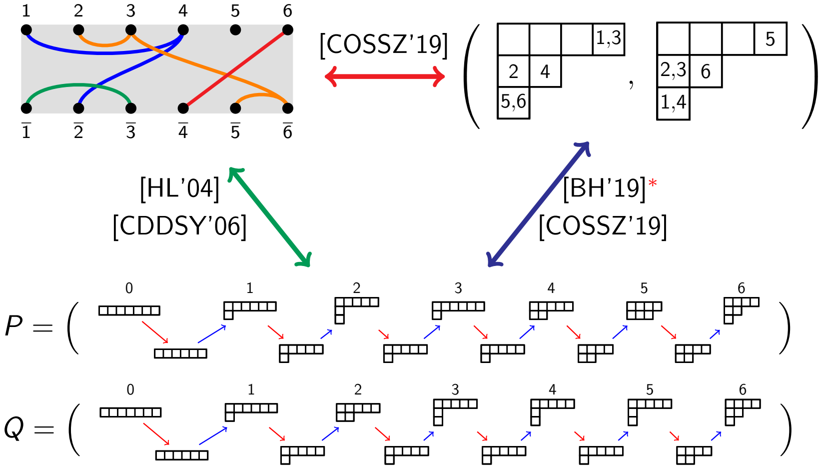
(planar partial permutations)

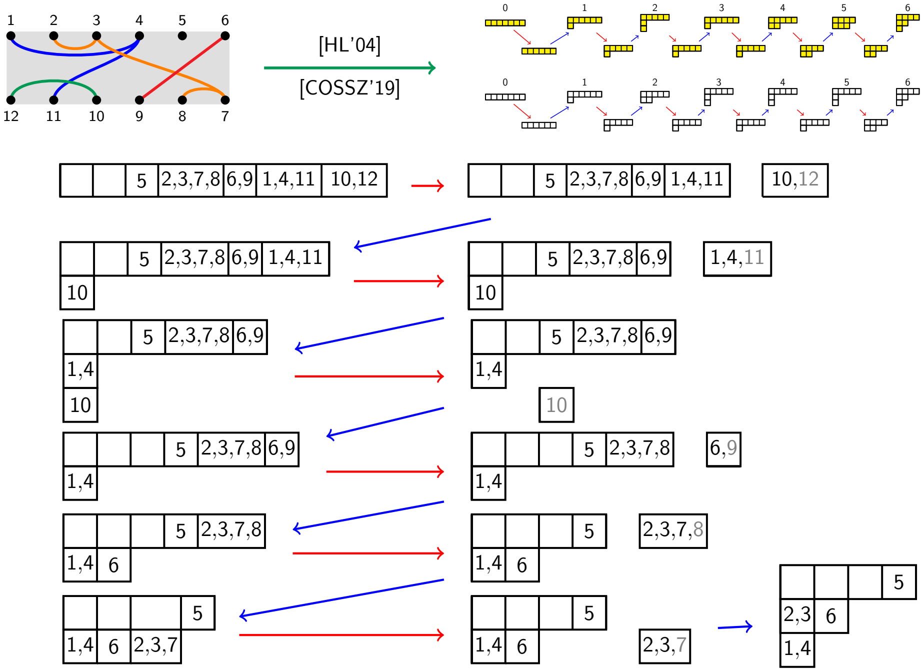


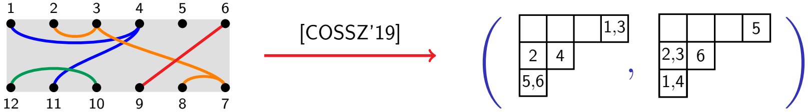
III. Insertion Bijections

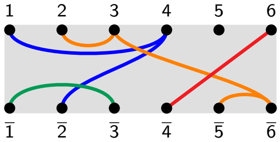
III. Insertion Bijections:

$$\dim(\mathbf{P}_k(n)) = \sum_{\lambda \vdash n} (\mathbf{m}_{k,n}^{\lambda})^2$$




[HL'04]
[COSSZ'19]
1 2 3 4 5 6
12 11 10 9 8 7
5 | 2,3,7,8 | 6,9 | 1,4,11 | 10,12
10,12
1,4,11
10
1,4
6,9
2,3,7,8
6
2,3,7
2,3 | 6
5



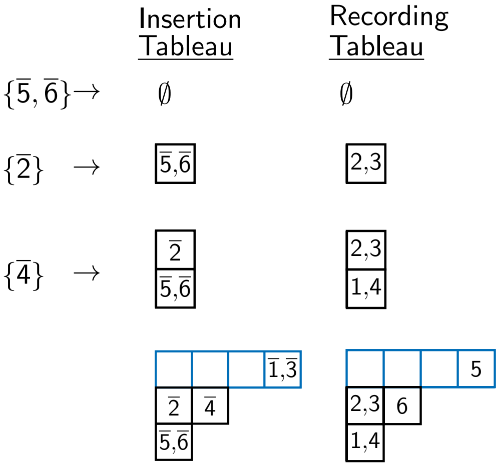


Generalized permutation of propagating blocks:

$$\longrightarrow \begin{pmatrix} \{2,3\} & \{1,4\} & \{6\} \\ \{\bar{5},\bar{6}\} & \{\bar{2}\} & \{\bar{4}\} \end{pmatrix}$$

non-propagating blocks:

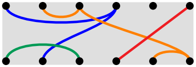
$\{5\}, \{\bar{1},\bar{3}\}$

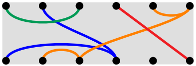


Many Nice Properties

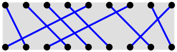
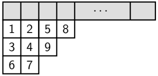
here are a few

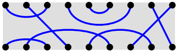
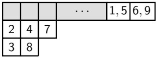
1. Reflect the diagram over its horizontal axis:

$d =$  $\longrightarrow \quad (P, Q)$

$\text{flip}(d) =$  $\longrightarrow \quad (Q, P)$

2. Consequence: $|\{\text{symmetric diagrams}\}| = \sum_\lambda \dim(\mathbf{P}_k^\lambda)$.
   Implies: a "model" representation on symmetric diagrams ([H-Reeks'15]).
3. Respects subalgebras:

 $\leftrightarrow$ 

 $\leftrightarrow$ 

A Few References

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