Counting factorizations in complex reflection groups

Joel Brewster Lewis (George Washington University)
joint work with Alejandro Morales (UMass Amherst)

arXiv:1906.11961

FPSAC Ljubljana, July 5, 2019

Intro: counting factorizations in the symmetric group

Question

How many ways are there to write the n-cycle $\varsigma = (1 \cdots n)$ *in* $\mathfrak{S}_n$ *as a product*

$$\varsigma = \pi_1 \cdots \pi_k$$

of permutations $\pi_1, \ldots, \pi_k$*?*

Answer:

$$|\mathfrak{S}_n|^{k-1} = (n!)^{k-1}$$

(any group, nothing special about permutations)

Intro: counting factorizations in the symmetric group

Question

How many ways are there to write the n-cycle $\varsigma = (1 \cdots n)$ in $\mathfrak{S}_n$ as a product

$$\varsigma = \pi_1 \cdots \pi_k$$

of permutations $\pi_1, \ldots, \pi_k$ so that π_i has exactly r_i cycles?

Since # cycles is conjugacy invariant, the following general tool works:

Lemma (Frobenius, 1898)

Let G be a finite group, $g \in G$, and $A_1, \ldots, A_k$ subsets of G that are closed under conjugacy by G. Then the number of factorizations of g as a product $g = t_1 \cdots t_k$ such that $t_i \in A_i$ for each i is

$$\frac{1}{|G|} \sum_{\lambda \in \mathrm{Irr}(G)} \dim(\lambda)^{1-k} \chi_\lambda(g^{-1}) \chi_\lambda(z_1) \cdots \chi_\lambda(z_k),$$

where $\mathrm{Irr}(G)$ is the set of irreducible representations of G, $\dim(\lambda)$ is the dimension of the representation λ, χ_λ is the character associated to λ, and z_i is the formal sum in the group algebra of elements in A_i.

Jackson's theorem

Theorem (Jackson (1988) as formulated by Schaeffer–Vassilieva (2008))

Let ς be a fixed n-cycle in $\mathfrak{S}_n$, and let $a_{r_1,\ldots,r_k}$ be the number of tuples $(\pi_1,\ldots,\pi_k)$ of elements in $\mathfrak{S}_n$ such that π_i has r_i cycles for all i and $\pi_1\cdots\pi_k=\varsigma$. Then

$$\sum_{r_1,\ldots,r_k\geq 1} a_{r_1,\ldots,r_k} x_1^{r_1}\cdots x_k^{r_k} = (n!)^{k-1}\cdot \sum_{p_1,\ldots,p_k\geq 1} M^{n-1}_{p_1-1,\ldots,p_k-1}\frac{(x_1)_{p_1}}{p_1!}\cdots\frac{(x_k)_{p_k}}{p_k!}$$

where $(x)_p$ denotes the falling factorial $(x)_p := x(x-1)\cdots(x-p+1)$ and

$$M^n_{p_1,\ldots,p_k} := [x_1^{p_1}\cdots x_k^{p_k}]\Big((1+x_1)\cdots(1+x_k)-x_1\cdots x_k\Big)^n.$$

Observation: $\frac{(x)_p}{p!}=\binom{x}{p}$. Interpretation: $(n!)^{k-1}M^{n-1}_{p_1-1,\ldots,p_k-1}$ counts *cycle-colored factorizations*.

Recent combinatorial proof: Bernardi–Morales (2016), via *maps on surfaces* & sign-reversing involutions

What is this talk about?

- Often, interesting questions about $\mathfrak{S}_n$ have nice generalizations or analogues in other contexts
- E.g.: Jackson (1988) also enumerated factorizations of an n-cycle as a product of transpositions
- This was extended to *well generated complex reflection groups* by Chapuy–Stump (2014)
- I'm going to describe some initial attempts to do the same thing for the more general result
- First step: translate $\mathfrak{S}_n$

$\mathfrak{S}_n$ is a reflection group

- V a $\mathbb{C}$-vector space; a transformation $T : V \to V$ is a *reflection* if it fixes a hyperplane pointwise
- a *complex reflection group* is a finite subgroup of $\mathrm{GL}(V)$ that is generated by its subset of reflections
- E.g.: $\mathfrak{S}_n$ is a CRG
  - $\mathfrak{S}_n$: $n \times n$ permutation matrices, act on $\mathbb{C}^n$
  - Transpositions are the ref'ns: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ y \\ w \end{bmatrix}$ fixes $w = z$
  - $\mathfrak{S}_n$ generated by transpositions
  - In general, # cycles = dimension of fixed space:

$$(153)(26)(4) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

fixes vectors like (a, b, a, c, a, b)

$\mathfrak{S}_n$ is a reflection group

- V a $\mathbb{C}$-vector space; a transformation $T : V \to V$ is a *reflection* if it fixes a hyperplane pointwise
- a *complex reflection group* is a finite subgroup of $\mathrm{GL}(V)$ that is generated by its subset of reflections
- E.g.: $\mathfrak{S}_n$ is a CRG
- E.g.: group of $n \times n$ signed permutation matrices is a CRG
- E.g.: every finite Coxeter group is a CRG
- E.g.: wreath product $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$ is a CRG (definition on next slide!)

Generalized permutations

Think of $\mathbb{Z}/m\mathbb{Z}$ as m-th complex roots of unity; wreath product is

$$(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n \cong \Big\{ n \times n \text{ monomial matrices} \\ \text{whose nonzero entries are } m\text{-th roots of } 1 \Big\}$$

- $m = 1$: $\mathfrak{S}_n$, Coxeter type A_{n-1}
- $m = 2$: signed permutations, Coxeter type B_n
- $m \geq 3$: things like $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \exp(4\pi i/5) \end{bmatrix}$, not Coxeter groups

The **weight** of an element is the $\mathbb{Z}/m\mathbb{Z}$ value corresponding to the product of the nonzero entries: matrix above is $m = 10$, weight $= 9$ because $-1 \cdot 1 \cdot \exp(4\pi i/5) = \exp(2\pi i \cdot \frac{9}{10})$.

Another example: the subgroup $G(m, m, n)$ of $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$ consisting of elements of weight 0 (when $m = 2$, Coxeter type D_n)

n-cycles are Coxeter elements

- In $\mathfrak{S}_n$, we factor n-cycle
- Analogue in a well generated CRG is a *Coxeter element*
- In $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, with $\omega = \exp(2\pi i/m)$, these are things like

$$\begin{bmatrix} & & & \omega \\ 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 & \end{bmatrix}$$

- In $G(m, m, n)$, things like

$$\begin{bmatrix} & & & \omega & \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & & \omega^{-1} \end{bmatrix}$$

Our question

Jackson counted factorizations of the n-cycle in the symmetric group $\mathfrak{S}_n$ as an arbitrary product, keeping track of the number of cycles of each factor, using a change of basis to binomial coefficients.

We count factorizations of a Coxeter element in $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$ (or other CRG) as an arbitrary product, keeping track of the fixed space dimension of each factor, using a change of basis to ???????.

Reminder for comparison: Jackson

Theorem (Jackson (1988))

Let ς be a fixed n-cycle in $\mathfrak{S}_n$, and let $a_{r_1,\ldots,r_k}$ be the number of tuples $(\pi_1,\ldots,\pi_k)$ of elements in $\mathfrak{S}_n$ such that π_i has r_i cycles for all i and $\pi_1\cdots\pi_k=\varsigma$. Then

$$\sum_{r_1,\ldots,r_k\geq 1} a_{r_1,\ldots,r_k} x_1^{r_1}\cdots x_k^{r_k} = (n!)^{k-1}\cdot \sum_{p_1,\ldots,p_k\geq 1} M^{n-1}_{p_1-1,\ldots,p_k-1}\frac{(x_1)_{p_1}}{p_1!}\cdots\frac{(x_k)_{p_k}}{p_k!}$$

where $(x)_p$ denotes the falling factorial $(x)_p := x(x-1)\cdots(x-p+1)$ and

$$M^n_{p_1,\ldots,p_k} := [x_1^{p_1}\cdots x_k^{p_k}]\Big((1+x_1)\cdots(1+x_k)-x_1\cdots x_k\Big)^n.$$

Our answer (wreath product)

Theorem (L–Morales (2019))

For $m > 1$, let c be a fixed Coxeter element in $G = (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, and let $a_{r_1,\ldots,r_k}$ be the number of factorizations of c as a product of k elements of G with fixed space dimensions $r_1, \ldots, r_k$, respectively. Then

$$\sum_{\substack{r_1,\ldots,r_k \\ r_i \geq 0}} a_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = |G|^{k-1} \sum_{\substack{p_1,\ldots,p_k \\ p_i \geq 0}} M^n_{p_1,\ldots,p_k} \frac{(x_1 - 1)^{(m)}_{p_1}}{m^{p_1} p_1!} \cdots \frac{(x_k - 1)^{(m)}_{p_k}}{m^{p_k} p_k!},$$

$(x-1)^{(m)}_p := (x-1)(x-m-1)\cdots(x-m(p-1)-1)$ *and* $M^n_{p_1,\ldots,p_k}$ *is exactly the same coefficient as before.*

Observation: $|(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_p| = m^p p!$

Proof idea

- Proof with character theory is technical, straightforward, not illuminating
- We think it is more interesting to give a combinatorial proof
- Fixed space dimension counts cycles of weight 0
- Rewrite the desired result as

$$\sum_{\substack{r_1,\ldots,r_k\\ r_i\geq 0}} a_{r_1,\ldots,r_k}(mx_1+1)^{r_1}\cdots(mx_k+1)^{r_k} = |G|^{k-1}\sum_{\substack{p_1,\ldots,p_k\\ p_i\geq 0}} M^n_{p_1,\ldots,p_k}\binom{x_1}{p_1}\cdots\binom{x_k}{p_k}$$

- Interpret left side as an elaborate cycle-coloring scheme
- Colored factorizations in $(\mathbb{Z}/m\mathbb{Z})\wr\mathfrak{S}_n$ project to colored factorizations of an n-cycle in $\mathfrak{S}_n$
- Carefully count pre-images to get result
- Recover Chapuy–Stump result (reflection factorizations) for this group as corollary

What about other CRGs?

$$G(m,m,n) = \Big\{ n \times n \text{ monomial matrices whose nonzero entries are } m\text{-th roots of 1 and have product } 1 \Big\}$$

Coxeter elements look like

$$\begin{bmatrix} & & & \omega & \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & & \omega^{-1} \end{bmatrix}$$

Under projection, gives an $(n-1)$-cycle in $\mathfrak{S}_n$.

Aside: factoring an $(n-1)$-cycle

Two very different ways to factor an $(n-1)$-cycle:

$(13)(24)(5)\cdot(1432)(5) = (1234)(5)$ vs. $(15)(24)(3)\cdot(152)(34) = (1234)(5)$

- Say $(\pi_1, \ldots, \pi_k)$ is a **transitive** factorization of its product if group $\langle \pi_1, \ldots, \pi_k \rangle$ acts transitively on $\{1, \ldots, n\}$
- *Every* factorization of an n-cycle in $\mathfrak{S}_n$ is transitive
- A factorization of an $(n-1)$-cycle in $\mathfrak{S}_n$ is nontransitive if and only if all factors share a fixed point.

Aside: factoring an $(n-1)$-cycle

Two very different ways to factor an $(n-1)$-cycle:

$(13)(24)(5)\cdot(1432)(5) = (1234)(5)$ vs. $(15)(24)(3)\cdot(152)(34) = (1234)(5)$

Theorem (L–Morales (2019) (??!!??))

Let ς_{n-1} be a fixed $(n-1)$-cycle in $\mathfrak{S}_n$. For integers $r_1, \ldots, r_k$ let $b_{r_1,\ldots,r_k}$ be the number of k-tuples $(\pi_1, \ldots, \pi_k)$ of elements in $\mathfrak{S}_n$ such that π_i has r_i cycles for $i = 1, \ldots, k$, $\pi_1 \cdots \pi_k = \varsigma_{n-1}$, and these tuples are a transitive factorization. Then

$$\sum_{r_1,\ldots,r_k \geq 1} b_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \frac{n!^{k-1}}{n^k} \sum_{p_1,\ldots,p_k \geq 1} M^n_{p_1,\ldots,p_k} \frac{(x_1)_{p_1}}{(p_1-1)!} \cdots \frac{(x_k)_{p_k}}{(p_k-1)!},$$

where $M^n_{p_1,\ldots,p_k}$ still the same as ever.

Proof. Character theory. (Is there a combinatorial proof?) $\square$

Main theorem in $G(m, m, n)$

- Say $(u_1, \ldots, u_k) \in G(m, m, n)^k$ is **transitive** if $\langle u_1, \ldots, u_k \rangle$ acts transitively on $\{\omega_m^i e_j\}$.
- (In wreath product, every factorization of Coxeter element is transitive.)
- In $G(m, m, n)$, a factorization of a Coxeter element is transitive if and only if the projection into $\mathfrak{S}_n$ is transitive.

Main theorem in $G(m, m, n)$

- Say $(u_1, \ldots, u_k) \in G(m, m, n)^k$ is **transitive** if $\langle u_1, \ldots, u_k \rangle$ acts transitively on $\{\omega_m^i e_j\}$.

Theorem (L–Morales (2019))

Let $b_{r_1,\ldots,r_k}$ be the number of transitive factorizations of a Coxeter element in $G = G(m, m, n)$ as a product of k elements of G with fixed space dimensions $r_1, \ldots, r_k$, respectively. Then

$$\sum_{r_1,\ldots,r_k \geq 0} b_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} =$$

$$|G|^{k-1}/n^k \cdot \sum_{p_1,\ldots,p_k \geq 1} M^n_{p_1,\ldots,p_k} \frac{P_{p_1}(x_1)}{m^{p_1-1}(p_1-1)!} \cdots \frac{P_{p_k}(x_k)}{m^{p_k-1}(p_k-1)!},$$

where $P_0(x) = 1$, $P_1(x) = x$, and for $i > 1$,
$P_i(x) = (x - (i-1)(m-1)) \cdot (x-1)(x-m-1) \cdots (x-(i-2)m-1)$,
and $M^n_{p_1,\ldots,p_k}$ is once again the same thing.

Proof idea: same projection.

Exceptional CRGs

- There are 26 other irreducible complex reflection groups for which this question makes sense ("well generated"): 13 of rank 2, five of rank 3, ..., one of rank 8

- Ask same question, but *what basis to use*?
- For $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, roots of $(x-1)_i^{(m)}$ are $1, m+1, \ldots, (n-1)m+1$
- For $G(m,m,n)$, roots of $P_i(x)$ are $1, \ldots, m(i-2)+1; (i-1)(m-1)$
- Both cases: these are the **coexponents** of the group

Exceptional CRGs

- There are 26 other irreducible complex reflection groups for which this question makes sense ("well generated"): 13 of rank 2, five of rank 3, ..., one of rank 8

Theorem (L–Morales (2019))

For any well generated CRG G of rank $n = 2$ or 3 with one exception (G_{25}), let c be a fixed Coxeter element in G, and let $a_{r_1,\dots,r_k}$ be the number of factorizations of c as a product of k elements of G with fixed space dimensions $r_1, \dots, r_k$, respectively. Then

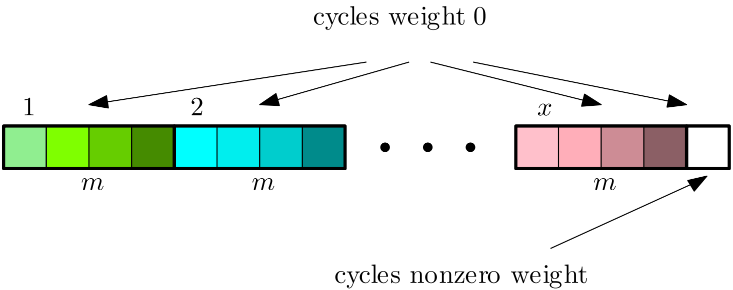
$$\sum_{\substack{r_1,\dots,r_k \\ r_i \geq 0}} a_{r_1,\dots,r_k} x_1^{r_1} \cdots x_k^{r_k} = |G|^{k-1} \sum_{\substack{p_1,\dots,p_k \\ p_i \geq 0}} M^n_{p_1,\dots,p_k} P_{p_1}(x_1) \cdots P_{p_k}(x_k),$$

where the $P_i(x)$ is a polynomial defined in terms of the coexponents *and degrees of the group.*

Choice of basis unambiguous for G_{25}, G_{32}, but coefficients are different; for other cases, choice of basis is not clear

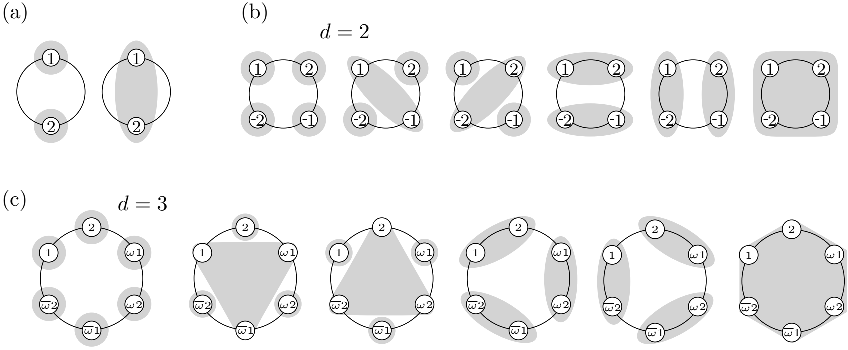
Thanks!

Thanks for listening!



Speculative: encode this as maps?

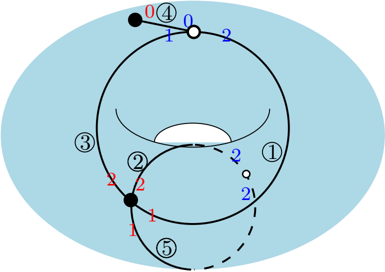
- Genus-0 factorizations of n-cycle in $\mathfrak{S}_n \longleftrightarrow$ noncrossing partitions, Catalan numbers
- Genus-0 factorizations of Coxeter element in $G(d,1,n) \longleftrightarrow$ *symmetric noncrossing partitions*, type B Catalan $= \binom{2n}{n}$

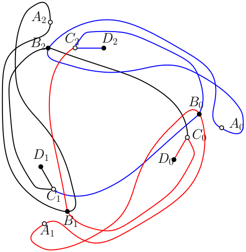


Speculative: encode this as maps?

- Genus-0 factorizations of n-cycle in $\mathfrak{S}_n \longleftrightarrow$ noncrossing partitions, Catalan numbers
- Genus-0 factorizations of Coxeter element in $G(d,1,n) \longleftrightarrow$ *symmetric noncrossing partitions*, type B Catalan $= \binom{2n}{n}$
- It is easy to attach *weighted maps* or *maps with symmetry*:

$$\begin{bmatrix} & \omega^2 & & & \\ & & \omega^2 & & \\ & & & & \omega \\ & & & 1 & \\ \omega & & & & \end{bmatrix} \cdot \begin{bmatrix} & & & \omega^2 & \\ & & & & \omega^2 \\ \omega & & & & \\ & & 1 & & \\ & \omega^2 & & & \end{bmatrix} = \begin{bmatrix} & & & & \omega \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$$





but genus is not "right". Other ideas?

Refining by cycle type

- For $\mathfrak{S}_n$, cycle type = orbit of fixed space

Theorem (Bernardi–Morales (2013))

$a_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ *counts factorizations with given cycle types of factors. One has*

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(k)}} a_{\lambda^{(1)},\ldots,\lambda^{(k)}} p_{\lambda^{(1)}}(\mathbf{x}_1)\cdots p_{\lambda^{(k)}}(\mathbf{x}_k) =$$

$$(n!)^{k-1} \sum_{\mu^{(1)},\ldots,\mu^{(k)}} \frac{M^{n-1}_{\ell(\mu^{(1)})-1,\ldots,\ell(\mu^{(k)})-1}}{\binom{n-1}{\ell(\mu^{(i)})-1}\cdots\binom{n-1}{\ell(\mu^{(k)})-1}} m_{\mu^{(1)}}(\mathbf{x}_1)\cdots m_{\mu^{(k)}}(\mathbf{x}_k)$$

- For $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, orbit of fixed space = cycle type of weight 0

Refining by cycle type

- For $\mathfrak{S}_n$, cycle type = orbit of fixed space
- For $(\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, orbit of fixed space = cycle type of weight 0

Theorem (L–Morales 2019)

$a^{(m)}_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ *counts factorizations with given weight-0 cycle type. One has*

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(k)}} a^{(m)}_{\lambda^{(1)},\ldots,\lambda^{(k)}} \prod_{i=1}^{k} p_{\lambda^{(i)}}(1, \underbrace{x_1^{(i)},\ldots,x_1^{(i)}}_{m}, \underbrace{x_2^{(i)},\ldots,x_2^{(i)}}_{m}, \ldots) =$$

$$|G|^{k-1} \sum_{\mu^{(1)},\ldots,\mu^{(k)}} \frac{M^{n-1}_{q_1-1,\ldots,q_k-1}}{\prod_{i=1}^{k}\binom{n-1}{q_i-1}} m_{\mu^{(1)}}(\mathbf{x}_1)\cdots m_{\mu^{(k)}}(\mathbf{x}_k),$$

where the sum on the RHS is over partitions $\mu^{(i)}$ of size at most n, not all of size n, and $q_j = \begin{cases} \ell(\mu^{(j)}) & \text{if } |\mu^{(j)}| = n \\ \ell(\mu^{(j)})+1 & \text{otherwise} \end{cases}$.

Refining by weight

Theorem (L–Morales (2019))

For $m > 1$, let $G = G(m,1,n)$ and let c be the Coxeter element in G. For $i = 1, \ldots, k$, let $\mathbf{r}_i = (r_{i,0}, \ldots, r_{i,m-1})$ be a tuple of nonnegative integers, and let $a^{(m)}_{\mathbf{r}_1,\ldots,\mathbf{r}_k}$ be the number of factorizations $c = u_1 \cdots u_k$ of c as a product of k factors such that u_i has exactly $r_{i,j}$ cycles of weight j for each $j = 0, \ldots, m-1$. Let $\mathbf{x}_i$ denote the variable set $\{x_{i,0}, \ldots, x_{i,m-1}\}$. Then

$$\sum_{\mathbf{r}_1,\ldots,\mathbf{r}_k} a^{(m)}_{\mathbf{r}_1,\ldots,\mathbf{r}_k} \mathbf{x}_1^{\mathbf{r}_1} \cdots \mathbf{x}_k^{\mathbf{r}_k} = |G|^{k-1} \sum_{t:\ t^m=1} t^{-1} \cdot$$
$$\sum_{p_1,\ldots,p_k \geq 1} M^{n-1}_{p_1-1,\ldots,p_k-1} \cdot \prod_i \binom{\left(x_{i,0} + t x_{i,1} + \cdots + t^{m-1} x_{i,m-1}\right)/m}{p_i}.$$