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Candidate for the crystal $B(-\infty)$ for the queer Lie superalgebra

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Abstract. It is shown that the direct limit of the semistandard decomposition tableau model for polynomial representations of the queer Lie superalgebra exists, which is believed to be the crystal for the upper half of the corresponding quantum group. An extension of this model to describe the direct limit combinatorially is given. Furthermore, it is shown that the polynomials representations may be recovered from the limit in most cases.

Keywords: crystal, decomposition tableau, queer Lie superalgebra

1 Introduction

In the 1990s, Kashiwara began the study of crystals, a combinatorial skeleton of a quantum group representation $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra. Kashiwara showed that irreducible highest weight representations have crystal bases $B(\lambda)$ and that the lower half of the quantum group has a crystal basis $B(\infty)$ [9]. Furthermore, it was shown that the direct limit of $B(\lambda)$ is isomorphic to $B(\infty)$ and one can recover $B(\lambda)$ by cutting a part of $B(\infty)$ by taking the tensor product with a specific crystal $\mathcal{R}_\lambda$. Using the direct limit, numerous combinatorial models for $B(\infty)$ have been developed such as (marginally) large tableaux [3, 8] and rigged configurations [15].

For Lie superalgebras, there are two natural analogs of $\mathfrak{gl}(n)$. The first is the basic Lie superalgebra $\mathfrak{gl}(m|n)$, where crystal bases have been constructed for the polynomial representations [2] and Kac modules [13]. The other is the queer superalgebra $\mathfrak{q}(n)$. The tensor powers of the fundamental representation form a semisimple category [7], the irreducible representations are called the polynomial representations, and crystal bases of these irreducible representations have been constructed using semistandard decomposition tableaux [6, 5]. One important feature of the polynomial representations is that their characters are Schur P-functions. Recently, a local characterization of the crystals

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for polynomial representations of $\mathfrak{q}(n)$ was given [1, 4] in analogy to the Stembridge axioms [16].

The goal of this extended abstract is to construct the direct limit of the polynomial representations. From the model, it is easier for us to consider the direct limit of the lowest weight elements, which we believe to be the $\mathfrak{q}(n)$-analog of $B(-\infty)$ (the crystal basis of the upper half of $U_q(\mathfrak{q}(n))$). In order to show this is indeed $B(-\infty)$, one requires a recognition theorem of $B(-\infty)$ similar to [12, Prop. 3.2.3] and the existence of $B(-\infty)$. Instead, we follow the construction of (marginally) large tableaux for semistandard decomposition tableaux $\mathrm{SDT}(\lambda)$ by showing the we can construct a directed system and the $\mathfrak{q}(n)$-crystal operators respect enlarging the shape. We then identify elements in each $\mathrm{SDT}(\lambda)$ based on their distance from the lowest weight element and take a distinguished representative.

We then describe how we can recover $\mathrm{SDT}(\lambda)$ from our limit crystal $\mathrm{SDT}(-\infty)$ using a dual version of $\mathcal{R}_\lambda$. In the case where λ corresponds to a strict partition that has maximal length, we show that the resulting tensor product $\mathrm{SDT}(-\infty) \otimes \mathcal{R}_\lambda^\vee$ recovers $\mathrm{SDT}(\lambda)$. We also discuss how to extend this to the general case, including a possible construction of crystal bases of dual representations.

This extended abstract is organized as follows. In Section 2, we give the requisite background. In Section 3, we give our main results.

2 Background

2.1 Crystals for the superalgebra $\mathfrak{q}(n)$

Let $I_0 = \{1, \dots, n-1\}$ and $I = I_0 \sqcup \{\overline{1}\}$. Denote the standard basis vectors of $\mathbf{Z}^n$ by $\epsilon_1, \dots, \epsilon_n$ and define $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for each $i \in I_0$. Set

$$\Lambda^- = \left\{ \lambda = -\lambda_1\epsilon_1 - \cdots - \lambda_n\epsilon_n \in \mathbf{Z}^n_{\le 0} : \begin{array}{l} \lambda_i \ge \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies} \\ \lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \dots, n \end{array} \right\}.$$

Equip Λ^- with a partial order $\lambda \le \mu$ if and only if $\mu - \lambda \in \Lambda^-$. An element $\lambda = -\lambda_1\epsilon_1 - \cdots - \lambda_n\epsilon_n$ in Λ^- will be henceforth be identified with the strict partition $w_0\lambda = (\lambda_n, \dots, \lambda_1)$.

Definition 2.1 ([10, Definition 1.2.1]). An *abstract $\mathfrak{gl}(n)$-crystal* is a set $\mathcal{B}$ together with maps $e_i, f_i \colon \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{\mathbf{0}\}$, $\varphi_i, \varepsilon_i \colon \mathcal{B} \longrightarrow \mathbf{Z} \sqcup \{-\infty\}$, for $i \in I_0$, and $\mathrm{wt} \colon \mathcal{B} \longrightarrow \mathbf{Z}^n$ satisfying the following conditions:

1. $\mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i$ if $i \in I_0$ and $e_i b \ne \mathbf{0}$;
2. $\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$ if $i \in I_0$ and $f_i b \ne \mathbf{0}$;

3. for any $i \in I_0$ and $b \in \mathcal{B}$, we have $\varphi_i(b) = \varepsilon_i(b) + \mu_i - \mu_{i+1}$ for $\mathrm{wt}(b) = \sum_{i \in I_0} \mu_i \epsilon_i$;
4. for any $i \in I_0$ and $b, b' \in \mathcal{B}$, we have $f_i b = b'$ if and only if $b = e_i b'$;
5. for any $i \in I_0$ and $b \in \mathcal{B}$ such that $e_i b \neq \mathbf{0}$, we have $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(e_i b) = \varphi_i(b) + 1$;
6. for any $i \in I_0$ and $b \in \mathcal{B}$ such that $f_i b \neq \mathbf{0}$, we have $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(f_i b) = \varphi_i(b) - 1$;
7. for any $i \in I_0$ and $b \in \mathcal{B}$ such that $\varphi_i(b) = -\infty$, we have $e_i b = f_i b = \mathbf{0}$.

Definition 2.2 ([6, Definition 1.9])**.** An *abstract $\mathfrak{q}(n)$-crystal* is an abstract $\mathfrak{gl}(n)$-crystal $\mathcal{B}$ together with maps $e_{\bar{1}}, f_{\bar{1}} \colon \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{\mathbf{0}\}$ such that

1. $\mathrm{wt}(\mathcal{B}) \subset \mathbf{Z}^n_{\geq 0}$;
2. $\mathrm{wt}(e_{\bar{1}} b) = \mathrm{wt}(b) + \alpha_1$ provided $e_{\bar{1}} b \neq \mathbf{0}$;
3. $\mathrm{wt}(f_{\bar{1}} b) = \mathrm{wt}(b) - \alpha_1$ provided $f_{\bar{1}} b \neq \mathbf{0}$;
4. for any $b, b' \in \mathcal{B}$, $f_{\bar{1}} b = b'$ if and only if $b = e_{\bar{1}} b'$;
5. if $3 \leq i \leq n-1$, we have
   (a) the operators $e_{\bar{1}}$ and $f_{\bar{1}}$ commute with e_i and f_i, and
   (b) if $e_{\bar{1}} b \in \mathcal{B}$, then $\varepsilon_i(e_{\bar{1}} b) = \varepsilon_i(b)$ and $\varphi_i(e_{\bar{1}} b) = \varphi_i(b)$.

Let $\mathcal{B}$ and $\mathcal{C}$ be abstract $\mathfrak{q}(n)$-crystals. A *crystal morphism* is a map $\psi \colon \mathcal{B} \longrightarrow \mathcal{C} \sqcup \{\mathbf{0}\}$ such that

1. if $b \in \mathcal{B}$ and $\psi(b) \in \mathcal{C}$, then for all $i \in I_0$,

$$\mathrm{wt}\big(\psi(b)\big) = \mathrm{wt}(b), \qquad \varepsilon_i\big(\psi(b)\big) = \varepsilon_i(b), \qquad \varphi_i\big(\psi(b)\big) = \varphi_i(b);$$

2. for $b \in \mathcal{B}$ and $i \in I$, we have $\psi(e_i b) = e_i \psi(b)$ provided $\psi(e_i b) \neq \mathbf{0}$ and $e_i \psi(b) \neq \mathbf{0}$;
3. for $b \in \mathcal{B}$ and $i \in I$, we have $\psi(f_i b) = f_i \psi(b)$ provided $\psi(f_i b) \neq \mathbf{0}$ and $f_i \psi(b) \neq \mathbf{0}$.

A morphism ψ is called *strict* if ψ commutes with e_i and f_i for all $i \in I$. Moreover, a morphism $\psi \colon \mathcal{B} \longrightarrow \mathcal{C} \sqcup \{\mathbf{0}\}$ is called an *embedding* if the induced map $\psi \colon \mathcal{B} \longrightarrow \mathcal{C} \sqcup \{\mathbf{0}\}$ is injective and is called an *isomorphism* if the induced map $\psi \colon \mathcal{B} \sqcup \{\mathbf{0}\} \longrightarrow \mathcal{C} \sqcup \{\mathbf{0}\}$ is a bijection.

Again let $\mathcal{B}$ and $\mathcal{C}$ be abstract $\mathfrak{q}(n)$-crystals. The *tensor product* $\mathcal{B} \otimes \mathcal{C}$ is defined to be the Cartesian product $\mathcal{B} \times \mathcal{C}$ equipped with crystal operations defined, for $i \in I_0$, by

$$e_i(b \otimes c) = \begin{cases} e_i b \otimes c & \text{if } \varphi_i(c) < \varepsilon_i(b), \\ b \otimes e_i c & \text{if } \varphi_i(c) \geq \varepsilon_i(b), \end{cases} \qquad e_{\overline{1}}(b \otimes c) = \begin{cases} b \otimes e_{\overline{1}} c & \text{if } e_{\overline{1}} b = f_{\overline{1}} b = \mathbf{0}, \\ e_{\overline{1}} b \otimes c & \text{otherwise.} \end{cases}$$

$$f_i(b \otimes c) = \begin{cases} f_i b \otimes c & \text{if } \varphi_i(c) \leq \varepsilon_i(b), \\ b \otimes f_i c & \text{if } \varphi_i(c) > \varepsilon_i(b), \end{cases} \qquad f_{\overline{1}}(b \otimes c) = \begin{cases} b \otimes f_{\overline{1}} c & \text{if } e_{\overline{1}} b = f_{\overline{1}} b = \mathbf{0}, \\ f_{\overline{1}} b \otimes c & \text{otherwise.} \end{cases}$$

Remark 2.3. This is equivalent to the rule in [4]. Moreover, this is the reverse convention of the tensor product to that given in [5].

Following the method of [11, p. 74], one can construct direct limits in the category of abstract $\mathfrak{q}(n)$-crystals. Indeed, let $\{\mathcal{B}_j\}_{j \in J}$ be a directed system of crystals and let $\psi_{k,j} \colon \mathcal{B}_j \longrightarrow \mathcal{B}_k$, $j \leq k$, be a crystal morphism (with $\psi_{j,j}$ being the identity map on $\mathcal{B}_j$) such that $\psi_{k,j}\psi_{j,i} = \psi_{k,i}$. Let $\vec{\mathcal{B}} = \varinjlim \mathcal{B}_j$ be the the direct limit of this system and let $\psi_j \colon \mathcal{B}_j \longrightarrow \vec{\mathcal{B}}$. Then $\vec{\mathcal{B}}$ has a crystal structure induced from the crystals $\{\mathcal{B}_j\}_{j \in J}$. Indeed, for $\vec{b} \in \vec{\mathcal{B}}$ and $i \in I$, define $e_i\vec{b}$ to be $\psi_j(e_i b_j)$ if there exists $b_j \in \mathcal{B}_j$ such that $\psi_j(b_j) = \vec{b}$ and $e_i(b_j) \neq 0$. This definition does not depend on the choice of b_j. If there is no such b_j, then set $e_i\vec{b} = 0$. The definition of $f_i\vec{b}$ is similar. Moreover, the functions wt, ε_i, and φ_i on $\mathcal{B}_j$ extend to functions on $\vec{\mathcal{B}}$.

2.2 Semistandard decomposition tableaux

This section summarizes the results of [5] using the conventions of [4].

Definition 2.4. Let $\lambda = (\lambda_n, \dots, \lambda_1)$ be a strict partition. Define $|\lambda| = \lambda_1 + \cdots + \lambda_n$ and $\ell(\lambda)$ to be the number of $1 \leq i \leq n$ such that $\lambda_i \neq 0$.

1. The *shifted Young diagram of shape λ* is an array of boxes in which the i-th row has λ_{n+1-i} cells, and is shifted $i-1$ units to the right with respect to the top row.
2. A word $u = u_1 u_2 \cdots u_N$ is a *hook word* if there exists $1 \leq k \leq N$ such that
$$u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_N.$$
3. A *semistandard decomposition tableau of shifted shape λ* is a filling T of λ with letters from $\{1, 2, \dots, n\}$ such that
   (a) the word v_i formed by reading the i-th row from left to right is a hook word of length λ_{n-i+1}, and
   (b) v_i is a hook subword of maximal length in $v_{i+1}v_i$ for $1 \leq i \leq \ell(\lambda) - 1$.

4. Set $\mathrm{read}(T)$ to be the word obtained by reading T in rows from right to left starting at the top.

5. For $\lambda \in \Lambda^-$, let $\mathrm{SDT}(\lambda)$ denote the set of all semistandard decomposition tableaux of shape $w_0\lambda$.

Definition 2.5. Let T be a semistandard decomposition tableau of shape $w_0\lambda$.

1. Suppose $i \in I_0$. Create a subword w of $\mathrm{read}(T)$ consisting of only the letters i and $i+1$ such that pairs of letters of the form $(i+1, i)$ (in that order) are sequentially removed. The remaining subword is a list of i's followed by $(i+1)$'s.

   (a) If there is no such letter, then $e_i T = \mathbf{0}$. Otherwise, $e_i T$ is the tableau obtained from T by changing the $(i+1)$-box corresponding to the leftmost $i+1$ in the subword above to an i-box.

   (b) If there is no such letter, then $f_i T = \mathbf{0}$. Otherwise, $f_i T$ is the tableau obtained from T by changing the i-box corresponding to the rightmost i remaining in the subword above to an $i+1$ box.

2. If $i = -1$, consider the subword w of $\mathrm{read}(T)$ consisting of only the letters 1 and 2.

   (a) If the leftmost letter in w is 1, then $e_{\overline{1}} T = 0$. Otherwise $e_{\overline{1}} T$ is the tableau obtained from T by changing the 2-box corresponding to the leftmost 2 in w to a 1-box.

   (b) If the leftmost letter in w is 2, then $f_{\overline{1}} T = 0$. Otherwise $f_{\overline{1}} T$ is the tableau obtained from T by changing the 1-box corresponding to the leftmost 1 in w to a 2-box.

For a $\lambda \in \Lambda^-$ with $\ell(\lambda) = N$, define $L^\lambda \in \mathrm{SDT}(\lambda)$ to be the tableau whose i-th row from the bottom contains only the letter i.

Example 2.6. Let $n = 5$ and $w_0\lambda = (7, 5, 3, 2, 1)$. Then

$$L^\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ \hline & 4 & 4 & 4 & 4 & & \\ \cline{2-5} & & 3 & 3 & 3 & & \\ \cline{3-5} & & & 2 & 2 & & \\ \cline{4-5} & & & & 1 & & \\ \cline{5-5} \end{array} .$$

Theorem 2.7 ([5, Theorem 2.5])**.** *For $\lambda \in \Lambda^-$, the set $\mathrm{SDT}(\lambda)$ together with the operators defined in Definition 2.5 form an abstract $\mathfrak{q}(n)$-crystal isomorphic to the crystal of the irreducible highest weight $\mathfrak{q}(n)$-module with highest weight $w_0\lambda$. Moreover, the unique lowest weight vector in $\mathrm{SDT}(\lambda)$ is L^λ.*

3 Main results

3.1 Auxiliary crystals

We begin by defining some auxiliary crystals that we will use in describing the direct limit and recovering the polynomial representations from the direct limit. There are analogous crystals in the Kac–Moody setting perform similar roles.

Lemma 3.1. *Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$.

1. *Define* $\mathcal{T}_\lambda = \{t_\lambda\}$ *with operations*

$$e_i t_\lambda = e_{\overline{1}} t_\lambda = f_i t_\lambda = f_{\overline{1}} t_\lambda = \mathbf{0}, \qquad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \qquad \mathrm{wt}(t_\lambda) = \lambda,$$

   for $i \in I_0$. *Then* $\mathcal{T}_\lambda$ *is an abstract* $\mathfrak{q}(n)$*-crystal.*

2. *Define* $\mathcal{R}_\lambda^\vee = \{r_\lambda^\vee\}$ *with operations*

$$\begin{gathered} e_i r_\lambda^\vee = e_{\overline{1}} r_\lambda^\vee = f_i r_\lambda^\vee = f_{\overline{1}} r_\lambda^\vee = \mathbf{0}, \\ \varepsilon_i(r_\lambda^\vee) = 0, \ \varphi_i(r_\lambda^\vee) = \lambda_i - \lambda_{i+1}, \\ \mathrm{wt}(r_\lambda^\vee) = \lambda, \end{gathered}$$

   for $i \in I_0$. *Then* $\mathcal{R}_\lambda^\vee$ *is an abstract* $\mathfrak{q}(n)$*-crystal. Moreover,* $\mathcal{R}_\lambda^\vee \cong \mathcal{R}_\mu^\vee$ *as abstract* $\mathfrak{q}(n)$*-crystals where* $\mu = \lambda + (1^n)$.

3.2 Candidate for $B(-\infty)$

Now we describe our candidate for $B(-\infty)$ by taking the direct limit of the crystals corresponding to the polynomial representations taken as lowest weight representations.

Definition 3.2. A semistandard decomposition tableau T for $\mathfrak{q}(n)$ is called *dual large* if

1. T has n rows,
2. for all $1 \le i \le n$, the number of leftmost i-boxes in row $n - i + 1$ is strictly greater than the total number of boxes in row $n - i + 2$.

Example 3.3. Consider the following semistandard decomposition tableaux for $\mathfrak{q}(3)$:

dual large:

3	3	3	3
	2	2	2
		1	

not dual large:

3	3	3	3
	2	1	2
		1	

Definition 3.4. A semistandard decomposition tableau T for $\mathfrak{q}(n)$ is called *dual marginally large* if it is large and for all $1 \le i \le n$, the number of leftmost i-boxes in row $n-i+1$ is greater than the total number of boxes in row $n-i+2$ by exactly one. Denote the set of all dual marginally large semistandard tableaux for $\mathfrak{q}(n)$ by $\mathrm{SDT}(-\infty)$.

Example 3.5. Consider the following semistandard decomposition tableaux for $\mathfrak{q}(3)$:

$$\text{not dual marginally large: } \begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline & 2 & 2 & 2 \\ \cline{2-4} & & 1 & \\ \cline{3-3} \end{array} \qquad \text{dual marginally large: } \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline & 2 & 2 \\ \cline{2-3} & & 1 \\ \cline{3-3} \end{array}$$

Definition 3.6. Let $T \in \mathrm{SDT}(-\infty)$ for $\mathfrak{q}(n)$.

1. Suppose $i \in I_0$. Create a subword w of $\mathrm{read}(T)$ consisting of only the letters i and $i+1$ such that pairs of letters of the form $(i+1, i)$ (in that order) are sequentially removed. The remaining subword is a list of i's followed by $(i+1)$'s.
   (a) Let T' be the tableau obtained from T by changing the $(i+1)$-box corresponding to the leftmost $i+1$ in the subword above to an i-box. If T' is dual marginally large, then $T' = e_i T$. Otherwise, let T'' be the tableau obtained from T' by adding a $(n-k+1)$-box in row k, for each $1 \le k \le n-i+1$. Then $T'' = e_i T$.
   (b) If there is no such letter, then $f_i T = \mathbf{0}$. Otherwise, let T' be the tableau obtained from T by changing the i-box corresponding to the rightmost i remaining in the subword above to an $i+1$ box. If T' is dual marginally large, then $T' = f_i T$. Otherwise, let T'' be the tableau obtained from T' by removing a $(n-k+1)$-box in row k, for each $1 \le k \le n-i+1$. Then $T'' = f_i T$.
2. Both $e_{\bar{1}}$ and $f_{\bar{1}}$ are defined exactly as in Definition 2.5, except for the need to maintain the dual marginally large condition as in (1a) and (1b) above.

Example 3.7. Let $n = 3$ and

$$T = \begin{array}{|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & 2 \\ \hline & 2 & 2 & 1 & \\ \cline{2-4} & & 1 & & \\ \cline{3-3} \end{array} \in \mathrm{SDT}(-\infty).$$

Then $\mathrm{read}(T) = 233331221$. After pairing off all possible $(2,1)$, there is no 1 remaining and the leftmost 2 remaining corresponds to the 2 in 233331221. Hence $f_1 T = \mathbf{0}$, but

$$e_1 T = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & 3 & 2 \\ \hline & 2 & 2 & 1 & 1 & \\ \cline{2-5} & & 1 & & & \\ \cline{3-3} \end{array} \quad \text{and} \quad e_{\bar{1}} T = \begin{array}{|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & 1 \\ \hline & 2 & 2 & 1 & \\ \cline{2-4} & & 1 & & \\ \cline{3-3} \end{array}.$$

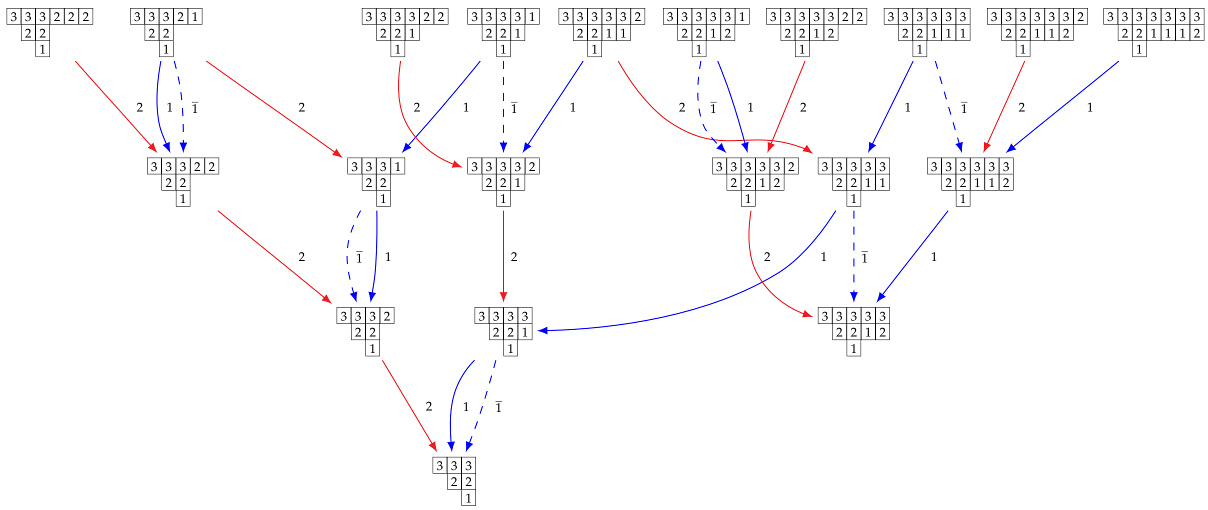

Figure 3.1: A bottom portion of the $\mathfrak{q}(3)$-crystal $\mathrm{SDT}(-\infty)$ containing the lowest weight element $L^{-\infty}$ created using SageMath [14].

Note that a 3-box needed to be added to the first row and a 2-box needed to be added to the second row to maintain the dual marginally large condition in $e_1 T$.

After pairing off all possible $(3,2)$ in $\mathrm{read}(T)$, the leftmost 3 and rightmost 2 remaining are the highlighted letters in 233331221. Hence

$$e_2 T = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & 2 & 2 \\ \hline \end{array} \atop \begin{array}{c|c|c|c|} & 2 & 2 & 1 \\ \hline \end{array} \atop \begin{array}{c|c|} & 1 \\ \hline \end{array} \qquad \text{and} \qquad f_2 T = \begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline \end{array} \atop \begin{array}{c|c|c|c|} & 2 & 2 & 1 \\ \hline \end{array} \atop \begin{array}{c|c|} & 1 \\ \hline \end{array} .$$

Note that a 3-box needed to be added to the first row of $e_2 T$ and removed from the first row of $f_2 T$ to maintain the dual marginally large condition.

A diagram of the crystal graph $\mathrm{SDT}(-\infty)$ up to height 3 is included in Figure 3.1.

Lemma 3.8. *Suppose $\lambda \leq \mu$. Then there exists a $\mathfrak{q}(n)$-crystal embedding*

$$v_{\lambda,\mu} \colon \mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda} \longrightarrow \mathrm{SDT}(\lambda+\mu) \otimes \mathcal{T}_{-\lambda-\mu}$$

such that $L^{\lambda} \otimes t_{-\lambda} \mapsto L^{\lambda+\mu} \otimes t_{-\lambda-\mu}$.

The method of proof is to make such an embedding explicit. Indeed, define $v_{\lambda,\mu}$ as follows. Let $T \in \mathrm{SDT}(\lambda)$. Then $v_{\lambda,\mu}(T)$ is obtained from T by adding enough i-boxes to row $n-i+1$, for each $1 \leq i \leq n$ so that the result has the requisite shape. The result now follows from careful comparison between $\mathrm{read}(T)$ and $\mathrm{read}\big(v_{\lambda,\mu}(T)\big)$.

Corollary 3.9. *The collection $\{\mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda}\}_{\lambda \in \Lambda^-}$ together with the inclusion maps from Lemma 3.8 form a directed system.*

To prove the corollary, one makes repeated use of Lemma 3.8 applied to diagrams of the following form:

$$\begin{array}{ccc} \mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda} & \xrightarrow{\upsilon_{\lambda,\mu}} & \mathrm{SDT}(\lambda+\mu) \otimes \mathcal{T}_{-\lambda-\mu} \\ & \searrow^{\upsilon_{\lambda,\mu+\xi}} & \downarrow \upsilon_{\lambda+\mu,\xi} \\ & & \mathrm{SDT}(\lambda+\mu+\xi) \otimes \mathcal{T}_{-\lambda-\mu-\xi}. \end{array}$$

Theorem 3.10. *The set $\mathrm{SDT}(-\infty)$ together with e_i, f_i from Definition 3.6 is an abstract $\mathfrak{q}(n)$-crystal such that*

$$\mathrm{SDT}(-\infty) \cong \varinjlim_{\lambda \in \Lambda^-} \mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda}.$$

Define $L^{-\infty}$ to be the decomposition tableau $L^{-n\epsilon_1-(n-1)\epsilon_2-\cdots-\epsilon_n}$.

3.3 Recovering $\mathrm{SDT}(\lambda)$ from $\mathrm{SDT}(-\infty)$

Our construction is parallel to the $\mathfrak{gl}(n)$-crystal construction of $B(\lambda)$ from $B(\infty)$ by essentially undoing the direct limit construction and adjusting $\varepsilon_i(b)$ to be the number of times we can apply e_i before getting $\mathbf{0}$. See Figure 3.2 for an example. However, we note that whenever $\lambda_1 - \lambda_2 = 0$, we obtain a connected component that is too large as we should have $e_{\overline{1}}(L^{-\infty} \otimes r_\lambda^\vee) = \mathbf{0}$. Thus, we would require a modification to the tensor product rule, but we can obtain $\mathrm{SDT}(\lambda)$ when $\lambda_i < \lambda_{i+1}$ for all $i \in I_0$.

Theorem 3.11. *Let $\lambda \in \Lambda^-$ such that $\lambda_i < \lambda_{i+1}$ for all $i \in I_0$. As $\mathfrak{q}(n)$-crystals using the modified tensor product rule, the connected component of $\mathrm{SDT}(-\infty) \otimes \mathcal{R}^\vee_{w_0\lambda}$ generated by $L^{-\infty} \otimes r^\vee_{w_0\lambda}$ is isomorphic to $\mathrm{SDT}(\lambda)$.*

Let us discuss how to extend Theorem 3.11 to more general cases. Consider the examples in Figure 3.3. For $\lambda = -\epsilon_1 - \epsilon_2$, we note that the connected component we obtain after also setting $e_{\overline{1}}(L^{-\infty} \otimes r_\lambda^\vee) = \mathbf{0}$ is isomorphic to $\mathrm{SDT}(\lambda)$. Therefore, a suitably modified tensor product rule will yield $\mathrm{SDT}(\lambda)$ when λ may contain zero entries. Furthermore, we would expect a modified tensor product rule to yield dual representations. For instance, if we consider $\lambda = -\epsilon_1$, note that after setting $e_{\overline{1}}(e_2 e_1 L^{-\infty} \otimes r_\lambda^\vee) = \mathbf{0}$, we would obtain the dual version of $\mathrm{SDT}(-\epsilon_1 - \epsilon_2)$.

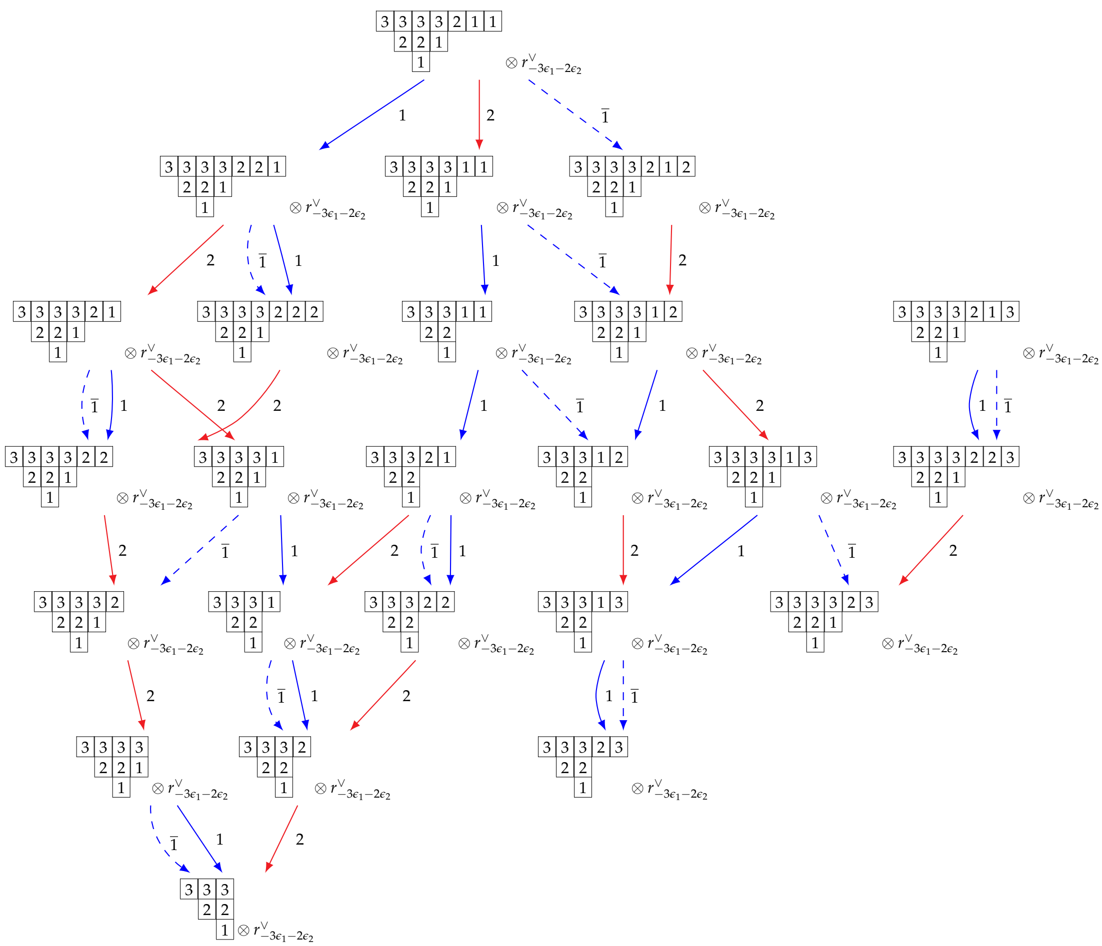

Figure 3.2: The $\mathfrak{q}(3)$-crystal SDT(λ) with $\lambda = -3\epsilon_1 - \epsilon_2$ created using SageMath [14]. Compare with [5, Figure 1].

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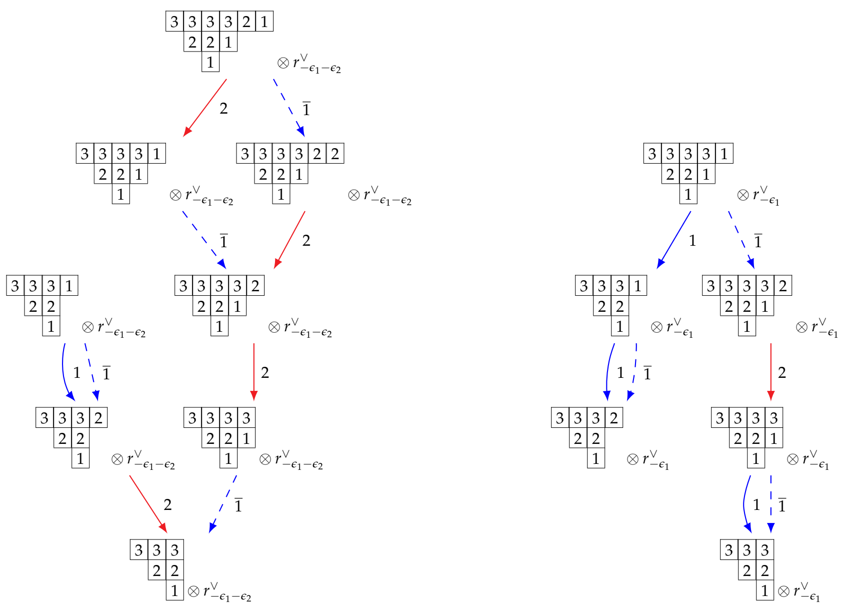

Figure 3.3: Left: The $\mathfrak{q}(3)$-crystal connected component of $\mathrm{SDT}(-\infty) \otimes \mathcal{R}^{\vee}_{-\epsilon_1}$ (resp. $\mathrm{SDT}(-\infty) \otimes \mathcal{R}^{\vee}_{-\epsilon_1-\epsilon_2}$) generated by $L^{-\infty} \otimes r^{\vee}_{-\epsilon_1}$ (resp. $L^{-\infty} \otimes r^{\vee}_{-\epsilon_1-\epsilon_2}$).

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