Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #53, 12 pp.
Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

On partition identities of Capparelli and Primc

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Abstract. We show that, up to multiplication by a factor $\frac{1}{(cq;q)_\infty}$, the weighted words version of Capparelli's identity is a particular case of the weighted words version of Primc's identity. We prove this first using recurrences, and then bijectively. We also give finite versions of both identities.

Résumé. Nous prouvons que, à multiplication par un facteur $\frac{1}{(cq;q)_\infty}$ près, la version mots pondérés de l'identité de Capparelli est un cas particulier de la version mots pondérés de l'identité de Primc. Nous prouvons cela d'abord en utilisant des récurrences, puis bijectivement. Nous donnons aussi des versions finies des deux identités.

Keywords: integer partitions, partition identities, representation theory, recurrences, bijection

1 Introduction and statement of results

1.1 Historical background

A *partition* λ of a positive integer n is a non-increasing sequence of natural numbers whose sum is n, the partitions of 4 being $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. The number n is called the *weight* of λ. Let us recall, for $n \in \mathbb{N} \cup \{\infty\}$, the classical q-series notation

$$(a;q)_n := \prod_{k=0}^{n-1} (1-aq^k).$$

Connections between partition identities and representation theory have been a major subject of interest over the last few decades, beginning with Lepowsky and Wilson's representation theoretic proof of the famous Rogers-Ramanujan identities [12, 13].

Theorem 1 (The Rogers-Ramanujan identities). *Let $i = 0$ or 1. Then*

$$\sum_{n \geq 0} \frac{q^{n^2+(1-i)n}}{(q;q)_n} = \frac{1}{(q^{2-i};q^5)_\infty (q^{3+i};q^5)_\infty}. \tag{1.1}$$

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Lepowsky and Wilson showed that, after multiplying both sides of (1.1) by $(-q;q)_\infty$, the right-hand side is the principally specialised Weyl-Kac character formula for level 3 standard modules of $A_1^{(1)}$ [11], and the left-hand side corresponds to bases constructed from vertex operators.

The Rogers-Ramanujan identities can also be seen as combinatorial identities on partitions.

Theorem 2 (Rogers-Ramanujan, combinatorial version)**.** *Let $i = 0$ or 1. For all non-negative integers n, the number of partitions of n into parts differing by at least 2 and having at most i ones is equal to the number of partitions of n into parts congruent to $\pm(2-i)$ modulo 5.*

The approach of Lepowsky and Wilson was then extended and modified by several authors to treat other levels and other Lie algebras, leading to many interesting new Rogers-Ramanujan type identities which were previously unknown to combinatorialists. For some examples, see [5, 14, 16, 18] and the references therein. On the other hand, combinatorialists have been working on combinatorial proofs and refinements of these new identities, see for example [2, 6, 8].

The purpose of this paper is to establish a connection between two seemingly unrelated partition identities from representation theory: Capparelli's identity and Primc's identity. Let us now present these two theorems in detail.

1.2 Capparelli's identity

A good example of the interplay between combinatorics and representation theory is Capparelli's identity, which was conjectured by Capparelli in [4] by studying the Lie algebra $A_2^{(2)}$ at level 3.

Theorem 3 (Capparelli)**.** *Let $C(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n such that $\lambda_s > 1$ and for all i,*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 \text{ if } \lambda_i + \lambda_{i+1} \equiv 0 \mod 3, \\ 4 \text{ otherwise.} \end{cases}$$

Let $D(n)$ denote the number of partitions of n into distinct parts not congruent to $\pm 1 \pmod 6$. Then for every positive integer n, $C(n) = D(n)$.

The first proof was given by Andrews in [2] and used q-trinomial coefficients and recurrences. The identity was then proved bijectively, refined and generalized by Alladi, Andrews and Gordon [1] using the method of weighted words. Soon after, it was reproved via representation theoretic techniques by Capparelli [5] and by Tamba-Xie [19]. In [15], Meurman and Primc later showed that Capparelli's identity can be recovered by studying the $(1,2)$-specialisation of the character formula of the level 1 modules in $A_1^{(1)}$.

The principle of the weighted words approach of Alladi, Andrews and Gordon in [1] is to prove a "non-dilated" version of Capparelli's identity on coloured partitions, which recovers the original identity under certain transformations called dilations. In addition to providing a refinement of Capparelli's identity, the advantage of this method is that one can perform other dilations and obtain infinitely many new combinatorial identities.

Instead of explaining their original result, we present a new equivalent formulation which will make the connection with Primc's identity clearer.

Let us consider partitions into natural numbers in three colours, a, c, and d, with the ordering

$$1_a < 1_c < 1_d < 2_a < 2_c < 2_d < \cdots , \tag{1.2}$$

satisfying the difference conditions in the matrix

$$C = \begin{array}{c c} & \begin{array}{ccc} a & c & d \end{array} \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}. \tag{1.3}$$

Here the entry (x, y) in the matrix C gives the minimal difference between successive part of colour x and y (recall that in a partition, parts are always in non-increasing order).

The weighted words version of Capparelli's identity can be stated as follows.

Theorem 4 (Capparelli's identity, weighted words version)**.** *Let $C(n; i, j)$ denote the number of partitions of n into coloured integers satisfying the difference conditions in matrix C, having i parts coloured a and j parts coloured d. We have*

$$\sum_{n,i,j \geq 0} C(n; i, j) a^i d^j q^n = (-q; q)_\infty (-aq; q^2)_\infty (-dq; q^2)_\infty. \tag{1.4}$$

Note that to obtain an infinite product, one cannot keep track of the number of parts coloured c.

Under the dilations

$$q \to q^3, \quad a \to aq^{-1}, \quad d \to dq,$$

which correspond to the following transformations of the coloured integers

$$k_a \to (3k-1)_a, \quad k_c \to (3k)_c, \quad k_d \to (3k+1)_d,$$

the order (1.2) becomes the natural ordering

$$2_a < 3_c < 4_d < 5_a < 6_c < 7_d < \cdots ,$$

and the difference conditions in the matrix C of (1.3) become the difference conditions defining the partitions counted by $C(n)$ in Theorem 3. Under the same dilations, the infinite product in (1.4) becomes the generating function for the partitions counted by $D(n)$. With the two extra parameters a and b, this gives the following refinement of Capparelli's identity.

Corollary 5 (Alladi-Andrews-Gordon)**.** *Let $C(n;i,j)$ and $D(n;i,j)$ denote the number of partitions counted by $C(n)$ and $D(n)$, respectively, in Theorem 3, having i parts congruent to* 1 *modulo* 3 *and j parts congruent to* 2 *modulo* 3*. Then for all $n,i,j \in \mathbb{N}$, $C(n;i,j) = D(n;i,j)$.*

1.3 Primc's identity

We now describe Primc's identity and its weighted word version.

In [17], Primc established a connection between the difference conditions in certain vertex operator constructions and energy functions of certain perfect crystals. He further developed his ideas in [16] to prove new partition identities arising from crystal base theory. His approach relies not only on the Weyl-Kac character formula as was done by Lepowsky and Wilson, but also on the crystal base character formula of Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki [10].

Here, we focus on one of the identities of [16]. Consider partitions into natural numbers in four colours a,b,c,d, with the ordering

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots, \tag{1.5}$$

satisfying the difference conditions in the matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}. \tag{1.6}$$

Primc conjectured that after the transformations

$$k_a \to (2k-1)_a, \quad k_b \to (2k)_b, \quad k_c \to (2k)_c, \quad k_d \to (2k+1)_d, \tag{1.7}$$

corresponding to the dilations

$$q \to q^2, \quad a \to q^{-1}, \quad c \to 1, \quad d \to q, \tag{1.8}$$

the generating function for these partitions is equal to $\frac{1}{(q;q)_\infty}$.

In [8], Lovejoy and the author proved the following weighted words version of Primc's theorem.

Theorem 6 (Dousse-Lovejoy, weighted words version of Primc's identity)**.**
Let $P(n;k,\ell,m)$ denote the number of four-coloured partitions of n with the ordering (1.5) and matrix of difference conditions (1.6), having k parts coloured a, ℓ parts coloured c, and m parts coloured d. Then

$$\sum_{n,k,\ell,m \geq 0} P(n;k,\ell,m) q^n a^k c^\ell d^m = \frac{(-aq;q^2)_\infty (-dq;q^2)_\infty}{(q;q)_\infty (cq;q^2)_\infty}.$$

After performing the dilations (1.8), the infinite product above indeed becomes $\frac{1}{(q;q)_\infty}$.

1.4 Statement of results

The goal of this paper is to establish a connection between Capparelli's and Primc's identities. To do so, we consider only their weighted words versions (Theorems 4 and 6). Indeed, we have just seen that this is more general and that the original theorems can be recovered under particular dilations.

From a representation theoretic point of view, the two identities do not seem to be related a priori: Primc's identity comes from the study of crystal bases of $A_1^{(1)}$, while Capparelli's identity does not seem related to crystal bases and originated from a vertex operator construction on the level 3 modules of $A_2^{(2)}$. The matrix C from Theorem 4 also appeared in Primc's paper [16] as the energy matrix of an "almost perfect" $\mathfrak{sl}(2,\mathbb{C})^{\sim}$–crystal, together with identities for the particular dilations $q \to q^2$, $a \to q^{-1}$, $d \to q$ and $q \to q^3$, $a \to q^{-2}$, $d \to q^2$. However, Primc said that these identities are *'not related to the crystal base theory, at least not in any obvious way.'* Indeed, they were proved by him and Meurman in [15] by using the vertex operator algebra construction for the basic $\mathfrak{sl}(2,\mathbb{C})^{\sim}$-module.

Combinatorially, the difference conditions of Capparelli's and Primc's identity do not seem to be related either, and don't even involve the same number of colours.

However we show that, up to a $\frac{1}{(cq;q)_\infty}$ factor, Capparelli's identity is actually the particular case $b = c$ of Primc's identity. Therefore Theorem 4 (and thus also the identities mentioned by Primc) is actually connected to the crystal base theory, as a particular case of Theorem 6.

Let us state our main theorem. Define $G_k^C(q;a,c,d)$ to be the generating function for partitions into coloured integers (1.2) satisfying the difference conditions from Capparelli's identity (1.3), with the added condition that the largest part is at most k. In the same way, define $G_k^P(q;a,b,c,d)$ to be the generating function for partitions into coloured integers (1.5) satisfying the difference conditions from Primc's identity (1.6), with the added condition that the largest part is at most k. In these generating functions, the power of a (resp. b,c,d) counts the number of parts coloured a (resp. b,c,d) in the partition.

Theorem 7. *For all positive integers k, we have*

$$\frac{G_k^C(q;a,c,d)}{(cq;q)_k} = G_k^P(q;a,c,c,d).$$

Remark. In Theorem 4, one needs to set the variable c to be equal to 1 (i.e. not keep track of the number of parts coloured c) to obtain an infinite product generating function. Similarly, in Theorem 6, one needs to set the variable b to be equal to 1 (i.e. not keep track of the number of parts coloured b). However we see here that the generating functions $G_k^C(q;a,c,d)$ and $G_k^P(q;a,c,c,d)$ are equal even when keeping track of all the colour variables a,c,d from Capparelli's identity.

In terms of partitions, Theorem 7 can be expressed in the following way. Let $\mathcal{C}$ (resp. $\mathcal{P}$) to be the set of coloured partitions satisfying the order (1.2) (resp. (1.5)) and difference conditions (1.3) (resp. (1.6)).

Theorem 8 (Combinatorial version)**.** *Let $\mathcal{C}(n;k;i,j,\ell)$ denote the number of partition pairs (λ,μ) of total weight n, where $\lambda \in \mathcal{C}$ and μ is an unrestricted partition coloured c, having in total i parts coloured a, j parts coloured c, ℓ parts coloured d, and largest part at most k. Let $\mathcal{P}(n;k;i,j,\ell)$ denote the number of partitions $\lambda \in \mathcal{P}$ of weight n, having i parts coloured a, j parts coloured b or c, ℓ parts coloured d, and largest part at most k. Then for all positive integers n and k and all non-negative integers i,j,ℓ,*

$$\mathcal{C}(n;k;i,j,\ell) = \mathcal{P}(n;k;i,j,\ell).$$

Thanks to Theorem 7, Capparelli's identity is now a corollary of Primc's identity. Indeed

$$\begin{aligned}
\sum_{n,i,j \geq 0} C(n;i,j) a^i d^j q^n &= \lim_{k\to\infty} G_k^C(q;a,1,d) \\
&= \lim_{k\to\infty} (q;q)_k G_k^P(q;a,1,1,d) \\
&= (q;q)_\infty \sum_{n,i,j,\ell \geq 0} P(n;i,\ell,j) q^n a^i d^j \\
&= \frac{(-aq;q^2)_\infty (-dq;q^2)_\infty}{(q;q^2)_\infty}.
\end{aligned}$$

By Euler's identity $\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty$, this is the same as (1.4).

Even though the variable c (resp. b) needs to be set equal to 1 in Capparelli's (resp. Primc's) identity to obtain an infinite product, Theorem 7 highlights the importance of these variables. Therefore it is interesting to find a formula for the generating functions of Capparelli's and Primc's identities with all colour variables. Moreover, finding finite versions of partition identities has been a subject of interest in the recent years (see, e.g., [3] and [9]). We now present a finite version of both theorems with all colour variables.

Theorem 9 (Finite version of Primc's identity)**.** *We have, for every positive integer k,*

$$G_k^P(q;a,b,c,d) = \left(1 - bq^{k+1}\right) \sum_{j=0}^{k+1} \frac{u_j(a,b,c,d) q^{\binom{k+1-j}{2}}}{(q;q)_{k+1-j}},$$

where for all $n \geq 0$,

$$u_{2n}(a,b,c,d) = (1-b) \sum_{\ell=0}^{n} \frac{(-aq^{2\ell+1};q^2)_{n-\ell}(-dq^{2\ell+1};q^2)_{n-\ell}}{(bq^{2\ell};q^2)_{n-\ell+1}(cq^{2\ell+1};q^2)_{n-\ell}} \frac{q^{2\ell}}{(q;q)_{2\ell}},$$

and

$$u_{2n+1}(a,b,c,d) = (b-1)\sum_{\ell=0}^{n} \frac{(-aq^{2\ell+2};q^2)_{n-\ell}(-dq^{2\ell+2};q^2)_{n-\ell}}{(bq^{2\ell+1};q^2)_{n-\ell+1}(cq^{2\ell+2};q^2)_{n-\ell}} \frac{q^{2\ell+1}}{(q;q)_{2\ell+1}}.$$

Theorem 10 (Finite version of Capparelli's identity)**.** *We have, for every positive integer k,*

$$G_k^C(q;a,c,d) = (cq;q)_{k+1}\sum_{j=0}^{k+1} \frac{u_j(a,c,c,d)q^{\binom{k+1-j}{2}}}{(q;q)_{k+1-j}},$$

where the sequence $(u_n(a,b,c,d))_{n\in\mathbb{N}}$ is defined as in Theorem 9.

Therefore, when $b=1$, Theorem 9 becomes

Corollary 11. *We have, for every positive integer k,*

$$G_k^P(q;a,1,c,d) = \left(1-q^{k+1}\right)\sum_{j=0}^{\lfloor \frac{k}{2}\rfloor} \frac{(-aq;q^2)_j(-dq;q^2)_j}{(q^2;q^2)_j(cq;q^2)_j} \frac{q^{\binom{k+1-j}{2}}}{(q;q)_{k+1-j}},$$

and Theorem 10 becomes

Corollary 12. *We have, for every positive integer k,*

$$G_k^C(q;a,1,d) = (q;q)_{k+1}\sum_{j=0}^{\lfloor \frac{k}{2}\rfloor} \frac{(-aq;q^2)_j(-dq;q^2)_j}{(q^2;q^2)_j(cq;q^2)_j} \frac{q^{\binom{k+1-j}{2}}}{(q;q)_{k+1-j}}.$$

It is then easy to recover the infinite product form by performing the change of variable $j = \lfloor \frac{k}{2} \rfloor - j$, letting k tend to infinity, and using the fact that

$$\lim_{k\to\infty}\sum_{j=0}^{\lfloor \frac{k}{2}\rfloor} \frac{q^{\binom{2j}{2}}}{(q;q)_{2j}} = \lim_{k\to\infty}\sum_{j=0}^{\lfloor \frac{k}{2}\rfloor} \frac{q^{\binom{2j+1}{2}}}{(q;q)_{2j+1}} = (-q;q)_\infty.$$

The proofs of the finite versions can be found in [7]. In the next two sections, we give the main ideas of the recurrence proof of Theorem 7 and the bijective proof of Theorem 8.

2 Idea of the proof of Theorem 7 using recurrences

In this section, we give a sketch of proof of Theorem 7 using recurrences based on the difference condition from matrices C from (1.3) and P from (1.6). The complete proof can be found in [7]. In the following, we will often omit the variables q,a,b,c,d in $G_k^C(q;a,c,d)$ and $G_k^P(q;a,b,c,d)$.

We start with Capparelli's identity. By using the order (1.2) and the difference conditions from (1.3), we do a classical combinatorial reasoning which consists in removing the largest part of the partition, and obtain three equations:

$$G^C_{k_d} - G^C_{k_c} = dq^k \left(aq^k G^C_{(k-2)_d} + G^C_{(k-1)_c} \right), \tag{2.1a}$$
$$G^C_{k_c} - G^C_{k_a} = cq^k G^C_{(k-1)_c}, \tag{2.1b}$$
$$G^C_{k_a} - G^C_{(k-1)_d} = aq^k G^C_{(k-2)_d}. \tag{2.1c}$$

Combining these equations, we get the following recurrence equation for $k \geq 1$:

$$\begin{aligned} G^C_{k_d} &= \left(1 + cq^k\right) G^C_{(k-1)_d} + \left(aq^k + dq^k + adq^{2k}\right) G^C_{(k-2)_d} \\ &\quad + adq^{2k-1} \left(1 - cq^{k-1}\right) G^C_{(k-3)_d}. \end{aligned} \tag{2.2}$$

Together with the initial conditions

$$G^C_{0_d} = 1, \qquad G^C_{-1_d} = 1, \qquad G^C_{-2_d} = 0, \tag{2.3}$$

the recurrence equation (2.2) completely determines $G^C_{k_d}(q; a, c, d)$ for $k \geq 1$.

Let us now introduce the sequence $(H_k(q; a, b, c, d))$ defined by the following recurrence equation for $k \geq 0$:

$$\begin{aligned} \left(1 - cq^k\right)\left(1 - bq^{k+1}\right) H_k(q; a, b, c, d) &= (1 - bcq^{2k}) H_{k-1}(q; a, b, c, d) \\ &\quad + (aq^k + dq^k + adq^{2k}) H_{k-2}(q; a, b, c, d) \\ &\quad + adq^{2k-1} H_{k-3}(q; a, b, c, d), \end{aligned} \tag{2.4}$$

and the initial conditions

$$H_{-1}(q; a, b, c, d) = 1, \qquad H_{-2}(q; a, b, c, d) = 0, \qquad H_{-3}(q; a, b, c, d) = \frac{(b-1)cq}{ad}.$$

This completely determines $(H_k(q; a, b, c, d))$.

We can relate $G^C_{k_d}(q; a, c, d)$ to $H_k(q; a, c, c, d)$ with the following lemma.

Lemma 1. *For all $k \geq 0$,*

$$\frac{G^C_{k_d}(q; a, c, d)}{(cq; q)_{k+1}} = H_k(q; a, c, c, d).$$

The proof simply consists in showing that $\left(\frac{G^C_{k_d}(q;a,c,d)}{(cq;q)_{k+1}}\right)$ satisfies the same recurrence relation and initial conditions as $(H_k(q; a, c, c, d))$.

Let us now turn to Primc's identity. Using again a combinatorial reasoning on the largest part of partitions satisfying the difference conditions from (1.6), we obtain a recurrence satisfied by $G^P_{k_d}(q;a,b,c,d)$ for $k \geq 2$:

$$\begin{aligned}(1-cq^k)G^P_{k_d} &= \frac{1-bcq^{2k}}{1-bq^k}G^P_{(k-1)_d} \\ &\quad + \frac{aq^k+dq^k+adq^{2k}}{1-bq^{k-1}}G^P_{(k-2)_d} + \frac{adq^{2k-1}}{1-bq^{k-2}}G^P_{(k-3)_d},\end{aligned} \tag{2.5}$$

and initial conditions

$$G^P_{-1_d} = 1-b, \qquad G^P_{0_d} = 1, \qquad G^P_{1_d} = \frac{bq}{1-bq} + \frac{(1+aq)(1+dq)}{1-cq}.$$

This completely determines $\left(G^P_{k_d}\right)$.

As for Capparelli's identity, we relate $G^P_{k_d}(q;a,b,c,d)$ and $H_k(q;a,b,c,d)$.

Lemma 2. *For all $k \geq 0$,*

$$\frac{G^P_{k_d}(q;a,b,c,d)}{1-bq^{k+1}} = H_k(q;a,b,c,d).$$

Finally, combining Lemmas 1 and 2 in which we set $b=c$, we get that for all $k \geq 0$,

$$\frac{G^C_{k_d}(q;a,c,d)}{(cq;q)_{k+1}} = H_k(q;a,c,c,d) = \frac{G^P_{k_d}(q;a,c,c,d)}{1-cq^{k+1}}.$$

Simplifying completes the proof of Theorem 7.

3 Bijective proof of Theorem 8

In this section, we give a bijection between partition pairs counted by $\mathcal{C}(n;k;i,j,\ell)$ and partitions counted by $\mathcal{P}(n;k;i,j,\ell)$. Due to space constraints, we omit some justifications which can be found in [7].

Let (λ,μ) be a partition pair of total weight n, where $\lambda \in \mathcal{C}$ and μ is an unrestricted partition coloured c, having in total i parts coloured a, j parts coloured c, ℓ parts coloured d, and largest part at most k. We transform (λ,μ) into a partition in $\mathcal{P}$ by following the steps below.

To make the bijection easier to follow, we will illustrate each step on the example

$$\begin{aligned}\lambda &= 8_d + 8_a + 6_c + 5_c + 3_d + 1_a, \\ \mu &= 8_c + 8_c + 7_c + 5_c + 3_c + 2_c + 2_c + 1_c + 1_c.\end{aligned}$$

Step 0**:** Change the colour of all the parts of μ to b. We obtain a partition pair (λ, μ'). On our example, we get

$$\begin{aligned}\lambda &= 8_d + 8_a + 6_c + 5_c + 3_d + 1_a,\\ \mu' &= 8_b + 8_b + 7_b + 5_b + 3_b + 2_b + 2_b + 1_b + 1_b.\end{aligned}$$

This process is clearly reversible.

Step 1**:** Insert the parts of μ' in the partition λ according to the order (1.5) of Primc's identity. Call ν_1 the resulting partition. In our example, we obtain

$$\nu_1 = 8_d + 8_b + 8_b + 8_a + 7_b + 6_c + 5_c + 5_b + 3_d + 3_b + 2_b + 2_b + 1_b + 1_b + 1_a.$$

This process is also clearly reversible, as one can simply separate the b-parts from the rest to recover the partitions λ and μ'.

The partition ν_1 satisfies the difference conditions in the matrix

$$M_1 = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \end{array}, \tag{3.1}$$

together with the following additional conditions for all $m \geq 1$:

(C_1) m_a and $(m-1)_a$ cannot both be parts of ν_1,

(C_2) m_c and m_a cannot both be parts of ν_1,

(C_3) m_c and $(m-1)_d$ cannot both be parts of ν_1,

(C_4) m_d and $(m-1)_d$ cannot both be parts of ν_1.

Note that in ν_1, the c-parts can only appear once, while the b-parts can repeat.

Step 2**:** By the difference conditions satisfied by ν_1, if m_a or m_d appears in ν_1 (they can both appear at the same time), then m_c cannot appear, but m_b can appear arbitrarily many times. If there are such m_b's, transform them all into m_c's. Call ν_2 the resulting partition. In our example, we obtain

$$\nu_2 = 8_d + 8_c + 8_c + 8_a + 7_b + 6_c + 5_c + 5_b + 3_d + 3_c + 2_b + 2_b + 1_c + 1_c + 1_a.$$

This process is again reversible: to obtain ν_1 from ν_2, change all the m_c's which appear at the same time as a m_a or m_d into m_b's.

The partition ν_2 satisfies the difference conditions in the matrix

$$M_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}, \tag{3.2}$$

together with the following additional conditions for all $m \geq 1$:

(C_1') m_d and m_b cannot both be parts of ν_2,

(C_2') m_c can repeat if and only if it appears at the same time as m_d or m_a,

(C_3') m_c and $(m-1)_d$ cannot both be parts of ν_2.

Step 3**:** If in ν_2 there is a part m_c followed by an arbitrary number of parts m_b, then change all these parts to m_c. Call ν_3 the resulting partition. In our example, we obtain

$$\nu_3 = 8_d + 8_c + 8_c + 8_a + 7_b + 6_c + 5_c + 5_c + 3_d + 3_c + 2_b + 2_b + 1_c + 1_c + 1_a.$$

This step is also reversible. To obtain ν_2 from ν_3, search for all the parts m_c that repeat but do not appear at the same time as m_a or m_d, and change the colour of all but the first of these c-parts to b.

The partition ν_3 belongs to $\mathcal{P}$. Therefore, we have established a bijection between the partition pairs counted by $\mathcal{C}(n;k;i,j,\ell)$ and the partitions counted by $\mathcal{P}(n;k;i,j,\ell)$. Theorem 8 is proved.

Acknowledgements

This article is an extended abstract of [7]. The author thanks the referees for their helpful comments which contributed to improving the readability of this paper.

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