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Restricted Tutte polynomials for some periodic oriented forests on infinite square lattice

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Abstract. For any finite graph, the Tutte polynomial is the generating function of spanning trees counted by their numbers of active external, respectively internal, edges. We consider two restrictions of this definition, either summing over a subset of spanning trees or counting only the activities in a subset of edges. Adding to the (infinite) square lattice one projective vertex in a (rational) direction $\vec{\theta}$, we define the restricted Tutte polynomial $T_{\vec{\theta},W\times H}(q,t)$ summing over some periodic spanning forests of period $W \times H$ and considering only activities on edges of the fundamental domain. Those polynomials are symmetric in q and t by self-duality of square lattice. Our main result is a family of bijections indexed by a finite number of $\vec{\theta}$ proving that $(T_{\vec{\theta},W\times H}(q,1))_{\vec{\theta}}$ does not depend on $\vec{\theta}$. Auto-duality preserving the number of trees per period and their common slope, we obtain refinements $(T_{\vec{\theta},W\times H}(w,z;q,t))_{\vec{\theta}}$ still symmetric in q and t.

Keywords: Sandpile model, Square lattice, Tutte polynomial, Symmetric q,t-numbers

This work is motivated by results on the sandpile model presented in Section 5 at the end of this document. We focus first on the combinatorial result on an analogue of Tutte polynomial for the infinite square lattice.

1 Tutte polynomial

For any finite connected graph $G = (V, E)$, the Tutte polynomial [10] is a classical graph invariant defined as follow:

$$T_G(q,t) := \sum_T q^{\mathsf{ext}(T)} t^{\mathsf{int}(T)}$$

where T runs over the set $\mathcal{T}_G$ of spanning trees of G and $\mathsf{ext}(T)$, respectively $\mathsf{int}(T)$, is the soon defined external, respectively internal, (Tutte) activity. Tutte activities depend on an arbitrary permutation of the edges of G also denoted as a total order $<_E$. There are $|E|!$ orders/permutations. The *fundamental cycle* of an external edge $e \notin T$ of the

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spanning tree T is the unique cycle in $T \cup \{e\}$ and is noted $C_e(T)$. An external edge $e \notin T$ is *(externally) active* if minimal according to $<_E$ among edges on $C_e(T)$. An internal edge $e \in T$ is *(internally) active* if minimal according to $<_E$ in the *fundamental cutset* that is the set of edges between the two connected components of $T \setminus \{e\}$. The external, respectively internal, activity is the number of external, respectively internal, active edges. The activities of a given tree may change with the order $<_E$ but the Tutte polynomial does not [10]. While Tutte shows this results taking account both activities, we propose a lightened proof for its marginal $T_G(q, 1)$.

Our lightened proof also relies on the effect of same exchange of two consecutive edges in $<_E$: the (elementary) transposition τ_i that exchanges the two consecutive edges e_i, e_{i+1} in the order $<_E := (e_i)_{i=1,\ldots|E|}$ leading to $<_{\tau_i.E} := e_1, \ldots e_{i-1}, e_{i+1}, e_i, e_{i+2}, \ldots e_{|E|}$. A pair of edges (e_j, e_k) is a *critical pair* in tree T for the order $<_E$ if e_j is an external edge, e_k belongs to $C_{e_j}(T)$, and e_j and e_k are the two minimal edges in this cycle. We define a map $\Phi_{\tau_i,<_E}$ on spanning trees by $\Phi_{\tau_i,<_E}(T) = (T \setminus \{e_i\}) \cup \{e_{i+1}\}$ if (e_i, e_{i+1}) is a critical pair for order $<_E$, $\Phi_{\tau_i,<_E}(T) = (T \setminus \{e_{i+1}\}) \cup \{e_i\}$ if (e_{i+1}, e_i) is a critical pair for order $<_E$ and $\Phi_{\tau_i,<_E}(T) = T$ otherwise. When non trivial, this update is a case of the exchange property of graphic matroids.

Lemma 1. *For any order $<_E$, any elementary transposition $\tau_i = (i, i+1)$, the involution $\Phi_{\tau_i,<_E}$ maps a tree T of external activity k for order $<_E$ to a tree $\Phi_{\tau_i,<_E}(T)$ with the same external activity k for order $<_{\tau_i.E}$.*

Proof. Following Tutte proof's beginning, the activities are preserved by the exchange when e_i and e_{i+1} are both internal or both external. Otherwise, there are still preserved if the internal edge is not on the fundamental cycle of the external edge. In the later case, if there are not both minimal in the fundamental cycle, there is an internal edge that is smaller than both in the cycle since there are consecutive in $<_E$. Then, the exchange preserves the external activity. It remains the case where the edges are the two minimal edges of the cycle. Suppose (e_i, e_{i+1}) is a critical pair without loss of generality. Then $\Phi_{\tau_i,<_E}$ maps the tree T to the tree $T' := T\Delta\{e_i, e_{i+1}\}$, the symmetric difference of edges. This map exchanges e_i and e_{i+1} in the tree and in the order. Then $C_{e_i}(T)$ and $C_{e_{i+1}}(T')$ are identical, so the activities of the external edges e_i or e_{i+1} in T or $T\Delta\{e_i, e_{i+1}\}$ are the same. An internal edge $e \notin \{e_i, e_{i+1}\}$ of T remains internal in T' so it does not change external activity. It remains to consider the external edge e not in $\{e_i, e_{i+1}\}$. We consider the two connected components of the vertices on the cycle $C_{e_i}(T)$ after the deletion of e_i and e_{i+1} then discuss if the fundamental cycle $C_e(T)$ of the external edge e contains one vertex of each component or not. If it does not, this cycle is the same in T and T' so activity is preserved. If it does, this cycle $C_e(T)$ in T becomes $C_e(T') := C_e(T)\Delta C_{e_i}(T)$ in T', hence all the new edges in $C_e(T')$ belongs to $C_{e_i}(T)$. Hence the two exchanged set of edges in the fundamental cycles of e in T and T' are $C_e(T)/C_{e_i}(T)$ and $C_{e_i}(T)/C_e(T)$ whose minimal edges are e_i or e_{i+1}. Since e_i and e_{i+1} remain consecutive in the order, it

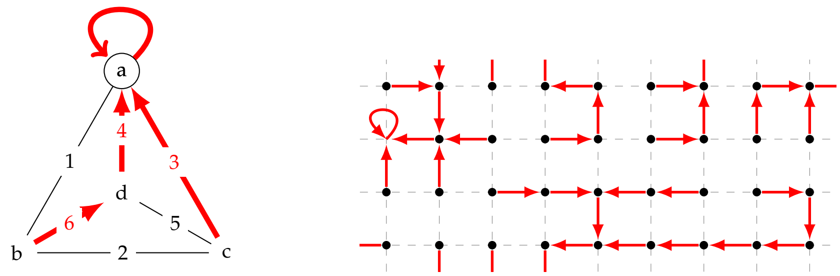

Figure 1: Example of weakly acyclic parenthood function on K_4 (left) and $\mathbb{Z}^2$ (right). Here $\mathrm{Ray}(b) = \{bd, da, aa\}$

means that the activity of e is the same in T and T'. □

1.1 Spanning tree as parenthood function

Let $G = (V, E)$ be a finite simple connected graph. We define object similar to oriented cycle rooted spanning forests (OCRSF) [7]. We describe the orientation by a function in a formalism closer to infinite matroids in graphs [1]. The *neighborhood* of $v \in V$ noted $\mathcal{V}(v)$ the set of vertices adjacent to v in G. We call *parenthood function* an endofunction R that maps for any vertex $v \in V$ a vertex in $\mathcal{V}(v) \cup \{v\}$.

Let R be a parenthood function.

Definition 2. *The* ray *of $u \in V$ in R is the set of edges in the orbit $(u, R(u), R^2(u) \dots)$ of u in R:* $\mathrm{Ray}(u) := (R^i(u)R^{i+1}(u))_{i \geq 0}$ *(see Figure 1).*

Definition 3. *An edge $uv \in E$ is* external *for R if $R(u) \neq v$ and $R(v) \neq u$. The* fundamental cycle *of this edge is the symmetrical difference of* $\mathrm{Ray}(u)$ *and* $\mathrm{Ray}(v)$*: $C_{uv}(R) = \mathrm{Ray}(u)\Delta\mathrm{Ray}(v)$.*

A rooted spanning tree is a parenthood function where each vertex maps to its father in the tree and where the root maps to itself. In this case, Definition 3 matches with the usual definition of fundamental cycle. A k-cycle in R is a sequence vertices $(v_1, v_2, \dots, v_k)$ such that $v_{i+1} = R(v_i)$ for $1 \leq k - 1$ and $v_1 = R(v_k)$. Then, rooted spanning trees match with the parenthood functions with exactly one cycle of length 1 that codes the root (see Figure 1). The bijection from parenthood functions to spanning trees consist on removing the loop on the root and removing the orientation. The rooted spanning forests are the functions whose cycles have size 1.

Definition 4. *A parenthood function is said weakly acyclic if all its cycles have length* 1.

We extend the definition of fundamental cycle for rooted spanning forests using this definition. Thus we can define the external activity for any rooted spanning forests using this extension.

1.2 Parenthood functions for infinite graphs

Let $G = (V, E)$ be a infinite simple connected graph and R be an weakly acyclic parenthood function. In this setup, rays may be infinite. For each external edge, we associate its fundamental cycle using Definition 3. Let uv be an external edge. Either u and v are in the same connected component of R then $C_{uv}(R)$ is finite, or $\mathrm{Ray}(u)$ and $\mathrm{Ray}(v)$ are finite then $C_{uv}(R)$ is finite, or $\mathrm{Ray}(u)$ and/or $\mathrm{Ray}(v)$ are infinite then $C_{uv}(R)$ is infinite. In the finite cases, the activity of uv is decidable for any order on the edges. In the infinite case, the activity of uv is not straight forward unless we have some guarantees on the order and R.

To avoid considerations on orders on the infinite set of edges, we only consider orders on edges that verify the following property.

Definition 5 ($k-$order assumption)**.** *Let k be an integer. A pair of an order $<_E$ and a set $\mathcal{F}$ of weakly acyclic parenthood functions verifies the $k-$order assumption if for any $f \in \mathcal{F}$ and any vertex $u \in V$, the minimal edge of $Ray_f(u)$ with respect to $<_E$ is among the k firsts edges of the ray.*

Under the $k-$order assumption, we can compute effectively the activity of any external edge still by its minimality in its fundamental cycle.

1.3 Restriction for summability

Let $G = (V, E)$ be a infinite simple connected graph. We want to define an analogue of Tutte polynomial on G using the weakly acyclic parenthood functions to study the distribution of the external activity. However, the sum would be infinite and the external activity of a weakly acyclic parenthood function might also be infinite. Let $\mathcal{F}$ be a finite set of weakly acyclic parenthood function of G and E' a finite subset of edges of E. Under the $k-$order assumption we can define an polynomial

$$T_{\mathcal{F},E',<_E}(q) := \sum_{F\in\mathcal{F}} q^{\mathsf{ext}_{E'}(F)}$$

encoding the distribution of external activity on $\mathcal{F}$ restricted to the edges of E'.

2 Restricted Tutte polynomial on the square lattice

From this section, we work on the infinite square lattice indexed by $\mathbb{Z}^2$ where $(x', y') \in \mathcal{V}(x, y) \iff |x' - x| + |y' - y| = 1$. Generic restrictions of Tutte polynomials defined in previous Section 1.3 allow to consider some well-defined polynomials on this graph. We consider spanning forests of $\mathbb{Z}^2$. First we give two restrictions and an order to explicitly compute the distribution of the external activity on finite families of spanning forests. Secondly, by observing that the square lattice is self-dual, we define a family of restricted polynomials on the distribution of both activities.

A natural finite subset spanning forests of $\mathbb{Z}^2$ is the set of periodic spanning forests of given fundamental domain.

Definition 6. *A spanning forest F of $\mathbb{Z}^2$ is* periodic of period $(W, H) \in \mathbb{N}_+^2$ *if for any edge $uv \in F$, the edge obtained by the translation of vector (kW, lH) is in F for any $(k, l) \in \mathbb{Z}^2$.*

From motivation in Section 5, we only consider spanning forests without finite tree. These forests are said *admissible* and consist on infinite periodic branches and finite subtrees are grafted on these branches (see Figure 2). We denote by $\mathcal{F}^{W\times H}$ the set of admissible periodic spanning forests of period (W, H). The notion of fundamental cycle is not well defined in this context unless we give an orientation on spanning trees as in Section 1.2. On finite graph, the orientation of edges is given thanks to the root. Here we orient the infinite branches towards a "projective" root at the infinity in a direction that can be seen as the merging point of infinite branches.

Let $\vec{\theta} \in \mathbb{Z}^2$ be a non null vector coding the direction of the root and $\vec{\theta}^\perp = (-\theta_y, \theta_x)$. We define a total order $<_{E_\theta}$ on edges of the square lattice:

$$e_i <_{E_\theta} e_j \iff (\langle \vec{e_i}, \vec{\theta} \rangle, \langle \vec{e_i}, \vec{\theta}^\perp \rangle) <_{\mathsf{lex}} (\langle \vec{e_j}, \vec{\theta} \rangle, \langle \vec{e_j}, \vec{\theta}^\perp \rangle)$$

where edges are geometrically identified by its middle point in the usual embedding, $\langle ., . \rangle$ denotes the usual scalar product and $<_{\mathsf{lex}}$ is the usual lexicographic order giving priority to the first coordinate. Let F be an admissible periodic spanning forest. We associate a parenthood function $\vec{F}$ coding the orientation of the edges. For any vertex u on a finite subtree, $\vec{F}(u)$ is its father as in finite graphs. Infinite branches can have two orientations. In order to verify the $k-$order assumption, we orient the edges such that infinite branches are periodically increasing for $<_{E_\theta}$: let $M > 0$ be the period of the infinite branch, for any u on the infinite branch $(\vec{F}^{iM}(u)\vec{F}^{iM+1}(u))_{i>0}$ is increasing. Then $\vec{F}$ is an acyclic parenthood function and is said *admissible for $\vec{\theta}$*.

Lemma 7. *The order $<_{E_\theta}$ and the set of admissible periodic parenthood function for $\vec{\theta}$ of period (W, H) verify the $k-$order assumption with $k = 2WH$.*

The natural finite subset of edges on which we want to compute the activity is the set of edges in a fundamental domain of the forests. We denote by $\mathcal{E}_{W\times H} := \{e_i \mid \vec{e_i} = (x, y)$ where $0 \le x \le W + 1/2$ and $0 \le y \le H + 1/2\}$ this subset of $2WH$ edges that contains one copy of each edge in the torus $W \times H$, in the usual copy of the torus containing the origin and sometimes called the fundamental domain.

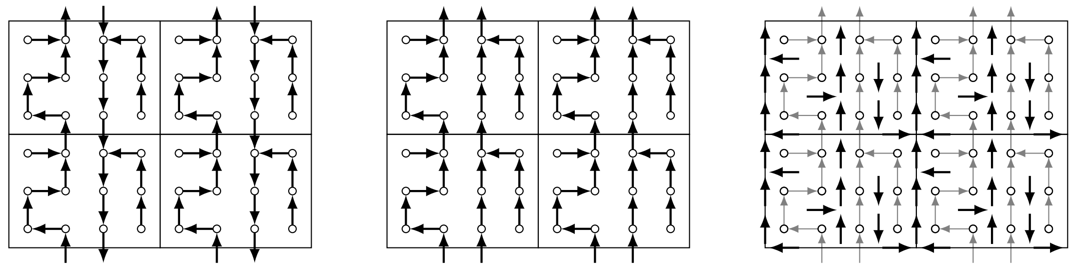

Figure 2: Examples of acyclic periodic parenthood function of period $(4,3)$ (left non admissible, center admissible, right dual)

The external activity of $F \in \mathcal{F}^{W\times H}$ with respect to $<_{E_\theta}$ restricted on $\mathcal{E}_{W\times H}$ is the well defined external activity of $\vec{F}$ on the edges of $\mathcal{E}_{W\times H}$. Now we can define and compute effectively the following polynomial encoding the distribution of external activity: $T_{\mathcal{F}^{W\times H},\mathcal{E}_{W\times H},<_{E_\theta}}(q) := \sum_{F\in\mathcal{F}^{\mathcal{W}\times\mathcal{H}}} q^{\mathsf{ext}_{\mathcal{E}_{W\times H}}(F)}$.

The set of admissible spanning forests is stable by duality (see Figure 2). We can define and compute the internal activity of an admissible periodic spanning forest as the external activity of its dual forest.

Definition 8 (Bivariate restricted Tutte polynomial)**.**

$$T_{\vec{\theta},W\times H}(q,t) := T_{\mathcal{F}^{W\times H},\mathcal{E}_{W\times H},<_{E_\theta}}(q,t) = \sum_{F\in\mathcal{F}^{\mathcal{W}\times\mathcal{H}}} q^{\mathsf{ext}_{\mathcal{E}_{W\times H}}(F)} t^{\mathsf{int}_{\mathcal{E}_{W\times H}}(F)}.$$

Proposition 9. *By design this polynomial is symmetric in q and t: $T_{\vec{\theta},W\times H}(q,t) = T_{\vec{\theta},W\times H}(t,q)$*

Theorem 10. *For any two rational directions $\vec{\theta}$ and $\vec{\theta}'$ we have $T_{\vec{\theta},W\times H}(q,1) = T_{\vec{\theta}',W\times H}(q,1)$.*

By symmetry, $T_{\vec{\theta},W\times H}(1,q)$ is the same polynomial independent of $\vec{\theta}$ *but* the full polynomial $T_{\vec{\theta},W\times H}(q,t)$ may depend on $\vec{\theta}$, see Section 4 for examples. Section 3 gives elements of proof for Theorem 10. This proof follows the same line as in the finite case where a bijection similar to Lemma 1 allows to prove the invariance by an elementary transposition on the order $<_E$ of edges. The main difficulty is that turning the order $<_{E_\theta}$ into the order $<_{E_{\theta+d\theta}}$, even for small $d\theta$ requires an infinite number of elementary transpositions on the infinite number of edges so one has to ensure that updates commute and then directly describe the result of an infinite sequence of elementary updates. The periodicity of forests allows to do so.

3 Distribution independent of $\vec{\theta}$: elements of proof

Let F be a periodic admissible forest of $\mathcal{F}^{W\times H}$. We show that there exists a finite set of directions $(\theta_i)_{i\in\mathbb{Z}_k}$ indexed counterclockwise such that the restricted external activity

$\text{ext}_{\mathcal{E}_{W\times H}}(\vec{F})$ is piecewise constant on every $[\theta_i, \theta_{i+1})$ and that for any θ_i, there exists bijectively a forest $F' \in \mathcal{F}^{W\times H}$ such that $\text{ext}_{\mathcal{E}_{W\times H}}(\vec{F})$ for $<_{E_{\theta_i}}$ equals $\text{ext}_{\mathcal{E}_{W\times H}}(\vec{F}')$ for $<_{E_{\theta_{i+1}}}$.

We denote by $\vec{\pi}(\vec{F}) \in W\mathbb{Z} \times H\mathbb{Z}$ the common period vector of any infinite branch of $\vec{F}$ with respect to $W \times H$. Let e be an external edge of $\vec{F}$. Let $\mathcal{H} = \text{Hull}(C_e(\vec{F}))$ be the *convex hull* of the middle points of the edges of $C_e(\vec{F})$ (Figure 3). Note that if $C_e(\vec{F})$ is finite, then the corners of $\mathcal{H}$ are finitely many. Otherwise $C_e(\vec{F})$ contains infinite periodic branches then $\mathcal{H}$ is a super sets of every $\mathcal{H} + k\vec{\pi}(\vec{F})$ where $k > 0$. The convex hull $\mathcal{H}$ has two semi infinite sides of slope $\vec{\pi}(\vec{F})$ with endpoint in $C_e(\vec{F})$. Moreover, these *two* endpoints are on the infinite branches, so they are internal edges of $\vec{F}$. Since the path lengths from the endpoints of e to infinite branches is bounded by $2WH$, $\mathcal{H}$ has finitely many corners.

Lemma 11. *For any e, $\text{Hull}(C_e(\vec{F}))$ has finitely many corners.*

We note $(h_i)_{1\leq i\leq k}$ the corners of $\mathcal{H}$ indexed counterclockwise where h_1 and h_k are the endpoint of the infinite sides when required. Based on the corners of the convex hull we can show the following results.

Lemma 12. *Let e be an external edge of $\vec{F}$. If an edge is minimal on $C_e(\vec{F})$ for the direction $\vec{\theta}$, then it is a corner of $\text{Hull}(C_e(\vec{F}))$. Reciprocally if an edge is a corner of $C_e(\vec{F})$ that is not an endpoint of a semi infinite side, then there exists a direction for which it is minimal.*

Since the order defined from $\vec{\theta}$ comes from usual scalar product, the minimality of $f \in C_e(\vec{F})$ for the direction $\vec{\theta}$ means that $C_e(\vec{F})$ is in the halfplane $D_{f,\vec{\theta}} := \{e \in E(\mathbb{Z}^2) \mid \langle e - f, \vec{\theta}\rangle \geq 0\}$. This compatibility gives the previous lemma. In particular, for any consecutive corners h_i and h_{i+1} of $\mathcal{H}$, h_{i+1} is minimal for the direction $\overrightarrow{h_i h_{i+1}}^{\perp}$. Then an external edge can be active if it is a corner of its fundamental cycle.

When $C_e(\vec{F})$ is infinite, e cannot be active for the direction $\vec{\pi}(\vec{F})^{\perp}$ since it is not on the infinite sides. For this direction, the repetitions of each edge of infinite branches of $\vec{F}$ are decreasing. We define $\overleftarrow{F}$ the oriented forest obtained by reversing the orientation of the edges of the infinite branches.

Lemma 13. *$\vec{F}$ and $\overleftarrow{F}$ have the same set of active external edges for the direction $\vec{\pi}(\vec{F})^{\perp}$.*

Indeed none of the external edges of infinite fundamental cycles is active and the finite fundamental cycles are preserved in $\overleftarrow{F}$.

We call a triplet (a_e, e, b_e) a critical triplet if e is external and if (a_e, e, b_e) are consecutive corners of $\text{Hull}(C_e(\vec{F}))$. Each critical triplet defines two directions $\overrightarrow{a_e e}^{\perp}$ and $\overrightarrow{e b_e}^{\perp}$ and e is active for a direction θ if and only if $\theta \in [\overrightarrow{a_e e}^{\perp}, \overrightarrow{e b_e}^{\perp})$. The set of critical triplets is finite up to translations. We denote by $\Theta_{\vec{F}}$ the finite set of these directions in addition to the direction $\vec{\pi}(\vec{F})^{\perp}$. We select among $\Theta_{\vec{F}}$ the $(\theta_i)_{1\leq i\leq k}$ such that $(\langle\vec{\pi}(\vec{F}), \vec{\theta}_i\rangle, \langle\vec{\pi}(\vec{F})^{\perp}, \vec{\theta}_i\rangle) >_{lex} (0,0)$. We index them counterclockwise with $\theta_k \equiv \vec{\pi}(\vec{F})^{\perp}$ where $\vec{u} \equiv \vec{v} \iff \exists\alpha > 0$ s.t. $\vec{u} = \alpha\vec{v}$.

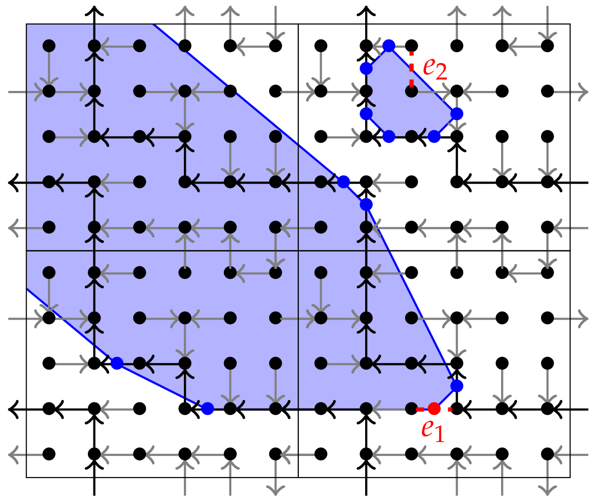

Figure 3: Periodic oriented forest $\vec{F}$. e_1 is a corner of $\mathrm{Hull}(C_{e_1}(\vec{F}))$ and e_2 is not a corner of $\mathrm{Hull}(C_{e_2}(\vec{F}))$. The corners of the convex hulls are the blue and red dots.

Lemma 14. *For any $1 \leq i < k$, the set of active external edges is invariant for any direction $\theta \in [\theta_i, \theta_{i+1})$.*

We consider θ_{i+1} for $1 \leq i < k$. For any critical triplet (a_e, e, b_e), the pair (a_e, e) (resp. (e, b_e)) is a critical pair for the direction θ_{i+1} if $\overrightarrow{a_e e}^{\perp} \equiv \theta_{i+1}$ (resp. $\overrightarrow{e b_e}^{\perp} \equiv \theta_{i+1}$). Then, we can show that these critical pairs of $\vec{F}$ are pairwise disjoint and we note $Pc_{\vec{F},\theta_{i+1}}$ the set of the critical pairs.

If $\theta_{i+1} = \vec{\pi}(\vec{F})^{\perp}$, then we replace $\vec{F}$ by $\overleftarrow{F}$ while preserving the critical pairs thanks to Lemma 13. We can assume that $\theta_{i+1} \not\equiv \vec{\pi}(\vec{F})^{\perp}$.

We construct step by step $\tilde{F} = F \Delta Pc_{\vec{F},\theta_{i+1}}$ the forest obtained by swapping the edges of each critical pairs in F. On each step, we swap periodically one critical pair. Let (e, f) be a critical pair. We can define $F' = F \Delta \cup_{v \in W\mathbb{Z} \times H\mathbb{Z}} \{e + v, f + v\}$. It does not create any finite cycle or finite tree, so F' and $\tilde{F}$ are admissible periodic forests.

A swap may change the direction of infinite branches of F (see Figure 4). We skip here the details to only sketch the proof. We can show that the set of critical pairs of F' for the direction θ_{i+1} is the same as F. And the newly created external edge of F' has the same activity for θ_{i+1} than the previous external edge of F for the direction θ_i. Finally, the other external edges keep their activity through this swap.

Since we preserve the critical pairs and the activity of all external edges that do not appear in critical pairs, the order of the swap is not relevant. We can do all the swaps in parallel. Moreover, since the critical pairs are preserved, this is an involution.

Corollary 15. *The external activity of $\tilde{F}$ for the direction θ_{i+1} is the same as the external activity of F for θ_i.*

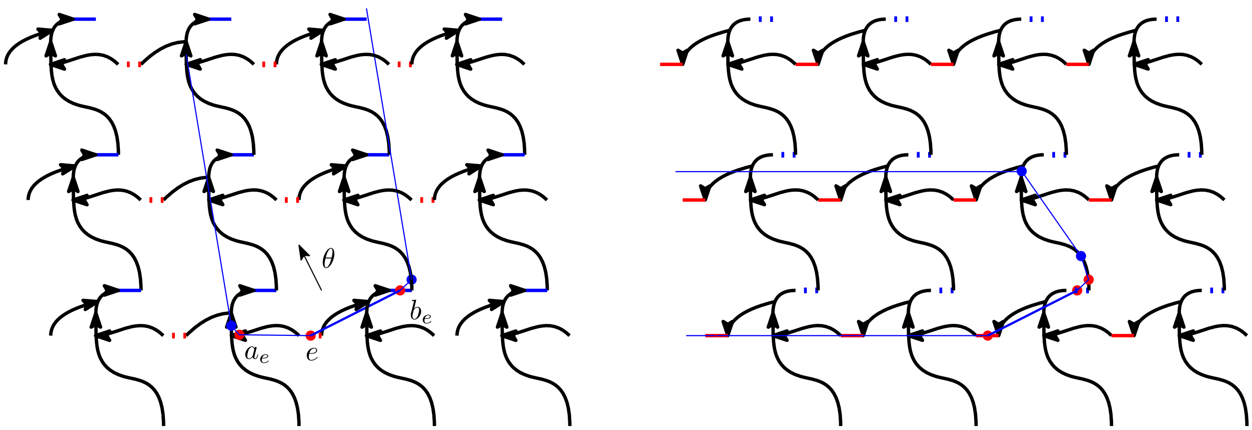

Figure 4: Example of critical pair swap that changes the infinite branches

Let $\Theta = \cup_{F \in \mathcal{F}^{W\times H}} \Theta_{\vec{F}}$. Since $\mathcal{F}^{W\times H}$ is finite, Θ is finite. We reuse the notation $\Theta = (\theta_i)_{i\in\mathbb{Z}_{|\Theta|}}$ indexed counterclockwise. For any F and any θ_{i+1}, we map bijectively a $\tilde{F}$ such that the external activity of $\tilde{F}$ for the direction θ_{i+1} is the same as the external activity of F for θ_i.

So for any rational directions θ and θ', we map step by step for each F a unique $F_{\theta\to\theta'}$ preserving the external activity, ending the proof of Theorem 10.

4 Restricted Tutte polynomials $T_{\vec{\theta},W\times H}(q,t)$ for $H=1$

When $H = 1$, the admissible periodic forests different from horizontal cycle on the tore are naturally in bijection with the spanning trees of the wheel graph W_{W+1}. Each cycle (of length 1) is map to an edge toward the center in the wheel graph, and the other edges maps to edges on the cycle of the wheel. Thus the $T_{\vec{\theta},W\times H}(1,1) - 1$ counts the number of these trees.

This case gives one of the smallest counter-example against the invariance on θ of $T_{\vec{\theta},W\times H}(q,t)$: $T_{\vec{0},3\times1}(q,t) = q^3t^3 + 3q^2 + 3qt + 3t^2 + 3q + 3t + 1$ and $T_{\vec{\pi/2},3\times1}(q,t) = q^3t^3 + 3q^2t + 3qt^2 + 3q + 3t + 4$.

We can refine these polynomials taking account of the slope of the infinite branches using $T_{\vec{\theta},W\times H}(w,z;q,t)$ where for any (i,j) coprime and for any $k > 0$, the polynomial coefficient $[w^{ki}z^{kj}]T_{\vec{\theta},W\times H}(w,z;q,t))_{\vec{\theta}}$ count the Tutte polynomial restricted to forests with k cycles on the torus and where the slope of infinite branches is $\pm(iW, jH)$. For instance $T_{\vec{0},3\times1}(w,z;q,t) = w + 3z(1+q+t) + 3z^2(q^2+t^2) + z^3q^3t^3$. The polynomial $T_{\vec{\theta},W\times H}(w,z;1,1)$ can be extracted from a determinantal formula (see [7, 4]). Polynomials for some (W,H) can be found on https://www.labri.fr/perso/hderycke/biperiodic_forests.

5 Application for the sandpile model

The sandpile model [5], also called Chip-Firing Game, is a discrete model for diffusion on graphs. Let $G = (V, E)$ be a simple connected finite graph. A *configuration* $\eta = (\eta_v)_{v \in V}$ for G is a function from V to $\mathbb{N}$. A vertex v is *unstable* in configuration η if η_v is at least the degree of v. The *toppling* of an unstable vertex v moves along each edge incident to v a grain from v to the opposite endpoint, so v losses its degree in grains and each neighbour gains one grain. An extra vertex s, called the sink, is added to the vertices and some edges E_s connect this sink to some other vertices leading to $G_s := (V \cup \{s\}, E \cup E_s)$. A configuration is stable if all vertices are stable, except possibly at the sink s.

Given a configuration η on G_s, we topple the unstable vertices except the sink until none remains. This algorithm is called *stabilization* and terminates on a stable configuration noted $\mathrm{stab}(\eta)$ whatever the order on the toppled vertices is. When η is stable, we denote by $\mathrm{dhar}(\eta)$ the result of the toppling of the sink s followed by a stabilization. The application dhar is called the Dhar operator. A *recurrent configuration* is a stable configuration that is a fixed point of the Dhar operator. Note that for such configuration, all vertices topple exactly once during this algorithm. This is called the *Dhar criterion*.

Various schedulings of this Dhar criterion provided many bijections from recurrent configurations in G_s to spanning trees of G_s, e.g. [9, 2], where one records for each vertex distinct from the sink the edge crossed by the grain allowing toppling.

We consider the generating function $G_s(q) := \sum_u q^{\mathsf{level}(u)}$ where u runs over recurrent configurations and $\mathsf{level}(u) := \left(\sum_{v \neq s} u_v\right) - |E|$ is up to a constant the number of grains in configuration u. This generating function $G_s(q)$ is also the generating function of spanning trees according to the Tutte external activity [8] so an evaluation at $t = 1$ of the bivariate Tutte polynomial $T_{G_s}(q, t)$.

In a recent work [4], the authors proposed a generalisation of the notion of recurrent configuration for the infinite square lattice, usually denoted $\mathbb{Z}^2$. Other approaches exist, for example in [6] where recurrent configurations are defined as the image of some spanning trees via an extension of a bijection of the finite case [9]. First, we focus on periodic stable configurations of period $W \times H$, where $(W, H) \in \mathbb{N}_+^2$. Our aim was to define the notion of recurrent configuration for such periodic configurations via an extension of Dhar criterion.

Selecting one vertex, say the origin, as the sink, will break the periodicity. Our intuitive choice was to add the sink as an extra "projective" vertex of the square lattice at infinity in a direction $\vec{\theta}$ as in Section 2. With this choice, the definition of a toppling of the sink now at infinity becomes challenging. We consider the half-plane $D_{p,\vec{\theta}} := \{v \in \mathbb{Z}^2 \mid \langle v - p, \vec{\theta} \rangle \geq 0\}$ where p is a vertex. Intuitively, this half-plane is made of vertices at least as close as p to the sink at infinity in direction $\vec{\theta}$.

Definition 16 (Weak Dhar Criterion)**.** *A periodic stable configuration is* recurrent for the direction $\vec{\theta}$ *if for any* p*, after a forced toppling of the half-plane* $D_{p,\vec{\theta}}$*, all other vertices in the complement of* $D_{p,\vec{\theta}}$ *topple (once).*

Proposition 17 ([4])**.** *There exists an algorithm performing weak Dhar criterion on any periodic stable configuration for any rational angle* $\vec{\theta}$ *and the result do not depend on the choice of the vertex* p *for the half-plane* $D_{p,\vec{\theta}}$*.*

As in the finite case, the algorithm performing the weak Dhar criterion leads to a bijection, extending the one in [2] with some admissible parenthood function for $\vec{\theta}$. Let $\mathcal{F}_{+}^{W\times H}$ the subset of admissible spanning forests which infinite branches are strongly positive.

Proposition 18 ([4])**.** *Recurrent configurations of period* $W \times H$ *defined by weak Dhar criterion with projective sink in direction* $\vec{\theta}$ *are in bijection with admissible spanning forests of* $\mathcal{F}_{+}^{W\times H}$*, hence excluding those of slope orthogonal to* $\vec{\theta}$*.*

The details of the proof, see [3], initially depends on the vertex p defining the half-plane $D_{p,\vec{\theta}}$. But the scheduling of weak Dhar criterion, initially only periodic in the orthogonal direction $\vec{\theta^{\perp}}$ becomes also periodic in the direction $\vec{\theta}$ and then independent of p. This ultimate periodic behaviour leads via an adaptation of the used finite case bijection, to the expected admissible forests.

We consider a new version of the weak Dhar criterion that will allow to extends the set of recurrent configurations to obtain a bijection with all the admissible spanning forests, hence the number of recurrent configurations of a given period $W \times H$ will no more depend on $\vec{\theta}$.

The bijection from Proposition 18 links the number of grains on recurrent configurations in direction $\vec{\theta}$ with the activities of the edges on spanning forests with respect to $\vec{\theta}$ by placing grains on the endpoints of each edge. Lemma 14 says that the activity per edges is piece-wise constant with respect to $\vec{\theta}$. Similar refinement exists for previous bijection but with a larger still finite $\Theta' \supset \Theta$. In other words, the bijection is stable for small variations of the direction $\vec{\theta}$. Using the same notation, we assume $\theta \in [\theta'_i, \theta'_{i+1})$ and define $\theta^{+} = (\theta + \theta'_{i+1})/2$.

Proposition 19. *For any direction* $\vec{\theta}$*, the new weak Dhar criterion for* $\vec{\theta^{+}}$ *defines a bijection between recurrent configurations of period* $W \times H$ *and all the admissible spanning forests of the same period.*

The polynomials studied in Section 2 satisfies $T_{\vec{\theta},W\times H}(q,1) = \sum_{u} q^{\mathsf{level}(u)}$ where the left member is the generating function of recurrent configurations of period $W \times H$ defined for the new weak Dhar criterion $\vec{\theta^{+}}$ and the level is the number of grains on $u^{W\times H}$ on the torus (or one period) minus $2WH$ that is the number of edges on the torus. This identity relies on the fact that the used bijection, like in the finite case [2], turns the level of a configuration into the external activity of the spanning tree for an order of edges related to $\vec{\theta}$. It was a priori unclear that this polynomial summing over a strict subset of spanning trees does not depend on θ and our main result states that it is.

References

[1] H. Bruhn and R. Diestel. "Infinite matroids in graphs". *Discrete Math.* **311**.15 (2011), pp. 1461–1471. Link.

[2] R. Cori and Y. L. Borgne. "The sand-pile model and Tutte polynomials". *Adv. in Appl. Math.* **30**.1–2 (2003), pp. 44–52. Link.

[3] H. Derycke. "Combinatorics on some stabilisations in the Abelian Sandpile Model on the square lattice Z^2". (French). PhD thesis. Université de Bordeaux, 2018. Link.

[4] H. Derycke and Y. Le Borgne. "A definition and counting of biperiodic recurrent configurations in the sandpile model on $\mathbf{Z}^2$". 17e Journées Montoises d'Informatique Théorique. 2018. Link.

[5] D. Dhar. "Self-organized critical state of sandpile automaton models". *Phys. Rev. Lett.* **64**.14 (1990), pp. 1613–1616. Link.

[6] S. Gamlin and A. Járai. "Anchored burning bijections on finite and infinite graphs". *Electron. J. Probab.* **19** (2014), 23 pp. Link.

[7] R. Kenyon. "Determinantal spanning forests on planar graphs". 2017. arXiv:1702.03802.

[8] C. M. López. "Chip firing and the Tutte polynomial". *Ann. Combin.* **1**.1 (1997), pp. 253–259. Link.

[9] S. Majumdar and D. Dhar. "Equivalence between the Abelian sandpile model and the $q \to 0$ limit of the Potts model". *Physica A: Stat. Mech. Appl.* **185**.1–4 (1992), pp. 129–145. Link.

[10] W. T. Tutte. "A Contribution to the Theory of Chromatic Polynomials". *Canad. J. Math.* **6** (1954), pp. 80–91. Link.