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Resolving Stanley's conjecture on k-fold acyclic complexes

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Abstract. In 1993 Stanley showed that if a simplicial complex is acyclic over some field, then its face poset can be decomposed into disjoint rank 1 boolean intervals whose minimal faces together form a subcomplex. Stanley further conjectured that complexes with a higher notion of acyclicity could be decomposed in a similar way using boolean intervals of higher rank. We provide an explicit counterexample to this conjecture. We also prove both a weaker version and a special case of the original conjecture.

Keywords: Simplicial complexes, f-vectors, homology, face poset

1 Introduction

The interplay between combinatorial and topological properties of simplicial complexes has been a subject of great interest for researchers for many decades (see, e.g., [1, 2, 3, 6, 9, 11, 13, 15, 16, 18, 19]). One particularly beautiful result due to Stanley [18, Theorem 1.2] connects the homology of the geometric realization of a complex to a well-behaved decomposition of its face poset. In particular, if a simplicial complex Δ is acyclic over some field $\mathbb{k}$, then its face poset can be written as the disjoint union of rank 1 boolean intervals such that the minimal faces of these intervals together form a subcomplex of Δ. Stanley [18, Proposition 2.1] and Duval [6, Theorem 1.1] generalized this result. Stanley further conjectured [18, Conjecture 2.4] that complexes with a higher notion of acyclicity possess similar decompositions into boolean intervals of higher rank.

Definition 1. *A simplicial complex is k-**fold acyclic** if* $\operatorname{link}_\Delta \sigma$ *is acyclic (over a field $\mathbb{k}$) for all $\sigma \in \Delta$ such that $|\sigma| < k$.*

Conjecture 2 ([18, Conjecture 2.4])**.** *Let Δ be a k-fold acyclic simplicial complex. Then Δ can be decomposed into disjoint rank k boolean intervals, the minimal faces of which together form a subcomplex.*

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Our main result is an explicit counterexample to Conjecture 2. Our construction relies on reducing to relative complexes and follows ideas similar to those recently developed in [7] and [12].

Central to this problem is the f-polynomial $f(\Delta, t)$ of a d-dimensional simplicial complex, which we define as

$$f(\Delta, t) = \sum_{\sigma \in \Delta} t^{|\sigma|} = f_{-1} + f_0 t + f_1 t^2 + \cdots + f_d t^{d+1}$$

where $f_i = f_i(\Delta)$ is the number of i-dimensional faces of Δ. Stanley's result [18, Theorem 1.2] shows that if Δ is acyclic, then $f(\Delta, t) = (1+t) f(\Gamma, t)$ where Γ is a subcomplex. Earlier, Kalai [13] showed this equality holds for some complex Γ, not necessarily a subcomplex of Δ. Using results from Kalai's algebraic shifting [14], Stanley [18, Proposition 2.3] further showed that the f-polynomial of a k-fold acyclic complex can be written as

$$f(\Delta, t) = (1+t)^k f(\Gamma, t)$$

for some complex Γ (which is not necessarily a subcomplex of Δ). If it had been true, Conjecture 2 would have provided a combinatorial witness for this Γ. We prove a weaker version the original conjecture in Theorem 9, which provides a witness Γ as a subcomplex of Δ.

In Section 2, we review definitions and relevant background material. In Section 3, we provide the construction of our counterexample. In Section 4, we prove a weaker version of Conjecture 2, replacing boolean intervals with boolean trees. In Section 5, we prove a special case of the original conjecture. We end with a section on open questions.

2 Preliminaries

We let $[n]$ denote the set $\{1, \ldots, n\}$. A **simplicial complex** Δ on $[n]$ is a subset of $2^{[n]}$ such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are **faces**, and maximal faces are **facets**. If $F_1, \ldots, F_j$ are the facets of Δ, we will often write $\Delta = \langle F_1, \ldots, F_j \rangle$, since the facets uniquely determine Δ. The **dimension** of a face σ is $\dim \sigma = |\sigma| - 1$ and the dimension of Δ is $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$. A complex is **pure** if all facets have the same dimension. For a pure complex, a **ridge** is a face of one dimension lower than the facets. Unless otherwise specified, we assume throughout that $\dim \Delta = d$.

A **subcomplex** of Δ is a simplicial complex Γ such that $\Gamma \subseteq \Delta$. If $W \subseteq [n]$, then the **induced subcomplex** on W is $\Delta|_W := \{\sigma \in \Delta \mid \sigma \subseteq W\}$. Given a face $\sigma \in \Delta$, the **link** and **star** of σ in Δ are

$$\begin{aligned} \operatorname{link}_\Delta \sigma &= \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\} \\ \operatorname{star}_\Delta \sigma &= \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\} \end{aligned}$$

which we will often denote as $\operatorname{link}\sigma$ and $\operatorname{star}\sigma$ if there is no possibility of confusion. Given two complexes Δ_1 and Δ_2 on disjoint vertex sets, their **join** is $\Delta_1 \star \Delta_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$. If Δ_1 is a simplex on $[k]$, then this join is the k**-fold cone** of Δ_2. We note that $\operatorname{star}\sigma = \langle\sigma\rangle \star \operatorname{link}\sigma$.

Throughout we fix a base field $\Bbbk$. The notation $\tilde{H}_i(X;\Bbbk)$ denotes the i^{th} reduced homology group of the complex X with coefficients in $\Bbbk$. Since we have fixed $\Bbbk$, we drop it from the notation and instead write $\tilde{H}_i(X)$. The **(reduced) Betti numbers** of a complex Δ are $\tilde{\beta}_i = \dim_{\Bbbk} \tilde{H}_i(\Delta)$. A complex is **acyclic** (over $\Bbbk$) if all of its reduced homology groups are zero.

We note that 1-fold acyclicity is equivalent to acyclicity, so [18, Theorem 1.2] is the $k = 1$ case of Conjecture 2. When $k > 1$, k-fold acyclicity is not topological; for example, a triangle is 3-fold acyclic but its barycentric subdivision is 1-fold acyclic but not even 2-fold acyclic.

The construction of our counterexample relies on **relative simplicial complexes**; given a simplicial complex Δ and a subcomplex Γ, the relative complex $\Phi = (\Delta, \Gamma)$ is the set of all of the faces of Δ that are not faces of Γ.

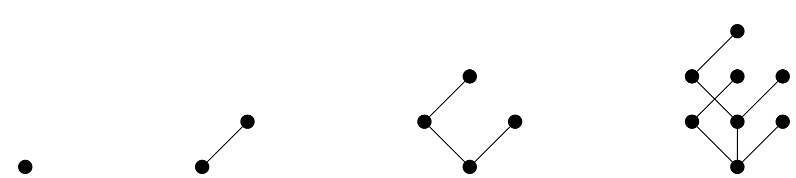
Given a poset P and two elements $x, y \in P$, the **interval** from x to y is $[x, y] = \{z \in P \mid x \leq z \leq y\}$. If $[x, y] = \{x, y\}$, then we say that y **covers** x, which is denoted $x \lessdot y$. An interval I is a **rank** k **boolean interval** if $I \cong 2^{[k]}$. A **boolean interval decomposition** of P is a collection $\mathcal{B}$ of disjoint boolean intervals in P such that

$$P = \bigsqcup_{I \in \mathcal{B}} I.$$

Such a decomposition is a **rank** k **boolean interval decomposition** if all intervals in the decomposition are of rank k. We also refer to this as a rank k boolean decomposition.

Definition 3. *A **boolean tree** of rank i is a subposet T_i of a poset P, that has a unique minimal element r, and is defined recursively as follows. Any subposet with exactly one element is a boolean tree of rank 0. Now assume T_1 and T_2 are two disjoint boolean trees of rank $(i-1)$, each with minimal elements r_1 and r_2 respectively, such that r_2 covers r_1 in P. Then $T_1 \cup T_2$ is a boolean tree of rank i, with r_1 as its unique minimal element.*

The following are examples of boolean trees of ranks zero through three. Boolean trees have the same size and ranks as a boolean interval, but relations between certain elements may be missing.



A **(rank k) boolean tree decomposition** of a poset is defined the same as a (rank k) boolean interval decomposition, except that boolean intervals are replaced with boolean trees.

Definition 4. *A simplicial complex Δ is a* ***stacked simplicial complex*** *if Δ is pure of dimension d with a facet order $F_1, \ldots, F_j$ such that for each $i \in [j-1]$, $\langle F_1, \ldots, F_i \rangle \cap \langle F_{i+1} \rangle$ is a $(d-1)$-simplex. Such an order is known as a* ***stacked shelling****.*

3 Construction

To construct our counterexample, we require complexes to glue together, and a way to glue them. First, we need a gluing lemma to maintain k-fold acyclicity. We use a variant of [7, Theorem 3.1], which allows us to construct a counterexample to Conjecture 2 by reducing the problem to finding a relative complex (Δ, Γ) with appropriate properties. Second, we must actually find such a pair (Δ, Γ). We begin with the gluing lemma.

Lemma 5. *Let Δ_1 and Δ_2 be simplicial complexes such that Δ_1 is j-fold acyclic, Δ_2 is k-fold acyclic, and $\Delta_1 \cap \Delta_2$ is ℓ-fold acyclic. Then $\Delta_1 \cup \Delta_2$ is m-fold acyclic, where $m = \min\{j, k, \ell\}$.*

This lemma follows from a Mayer-Vietoris sequence. It is used to preserve k-fold acyclicity in the following theorem, which is a k-fold acyclic version of [7, Theorem 3.1].

Theorem 6. *Let $\Phi = (\Delta, \Gamma)$ be a relative complex such that*

1. *Δ and Γ are k-fold acyclic;*
2. *Γ is an induced subcomplex of Δ; and*
3. *Φ cannot be written as a disjoint union of rank k boolean intervals.*

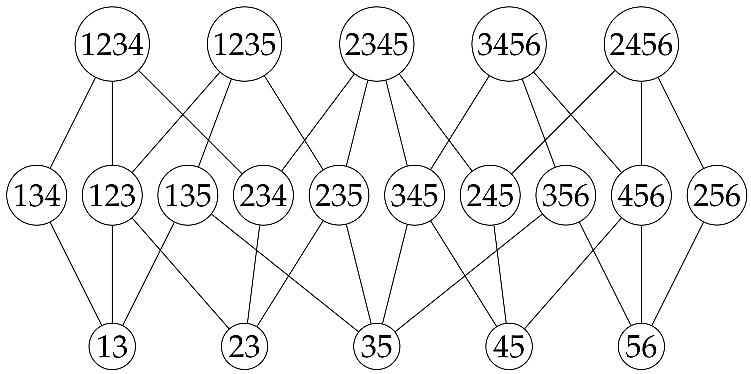
Let ℓ be the total number of faces of Γ and let $N > \ell/2^k$. If $\Omega = \Omega_N$ is the complex formed by gluing N copies of Δ together along Γ, then Ω is a k-fold acyclic complex that cannot be written as a disjoint union of rank k boolean intervals.

The proof follows from Lemma 5 and the pigeonhole principle.

We now start the construction of our counterexample, beginning with the following relative complex Ψ, which is inspired by the complex in [7, Remark 3.6]. We have shortened the notation so instead of writing $\{1, 2, 3, 4\}$ we write 1234, for example.

$$\Sigma = \langle 1234, 1235, 2345, 2456, 3456 \rangle$$
$$\Upsilon = \langle 125, 124, 246, 346 \rangle$$
$$\Psi = (\Sigma, \Upsilon)$$

Both Σ and Υ are 2-fold acyclic and the face poset of Ψ cannot be decomposed into disjoint rank 2 boolean intervals. Since Υ is not an induced subcomplex of Σ, we cannot immediately apply Theorem 6 to produce a counterexample to Conjecture 2. However, this complex is the foundation of our counterexample and will be referred to repeatedly in our construction. The face poset of Ψ is given below, for the reader to verify that it cannot be decomposed into disjoint rank 2 boolean intervals.



Our goal is to create a new pair (Δ, Γ) that does meet the conditions of Theorem 6. We now consider the following complex, Γ. It is straightforward to check that Γ is 2-fold acyclic. In particular, Γ is a simplicial 3-ball with no interior vertices.

$$\Gamma = \langle ABCE, BCEF, BCDF, ABCG, BCGH, BCDH, ABEG, BEFG, BFHG \rangle$$

Within Γ there are the following six pairs of triangles:

$$\begin{array}{lll} \{ABC, BCD\}, & \{ABE, BEF\}, & \{ABG, BGH\}, \\ \{CDF, CEF\}, & \{CDH, CGH\}, & \{EFG, FGH\}. \end{array} \tag{3.1}$$

To each of the edges AB, CD, EF, GH in Γ we add a vertex, forming four triangles which are not in Γ:

$$ABI, CDJ, EFK, GHL. \tag{3.2}$$

For any two triangles from (3.2) there is a unique pair of triangles in (3.1) so that the four triangles together form a complex isomorphic to Υ. For example, the two triangles $\{ABI, CDJ\}$ from (3.2) together with $\{ABC, BCD\}$ form a complex isomorphic to Υ. Given these four triangles, we glue a copy of Σ to Γ along this Υ in the natural way.

We obtain Δ as the result of gluing six copies of Σ to Γ in this way, one for each choice of two triangles from (3.2). For clarity, we list all the facets of Δ that are not in Γ.

$$\begin{array}{ccccc} ABCJ, & ABIJ, & BCIJ, & BCDI, & CDIJ, \\ ABEK, & ABIK, & BEIK, & BEFI, & EFIK, \\ ABGL, & ABIL, & BGIL, & BGHI, & GHIL, \\ CDFK, & CDJK, & CFJK, & CEFJ, & EFJK, \\ CDHL, & CDJL, & CHJL, & CGHJ, & GHJL, \\ EFGL, & EFKL, & FGKL, & FGHK, & GHKL. \end{array} \tag{3.3}$$

It is straightforward to verify that Δ is 2-fold acyclic and that Γ is an induced subcomplex of Δ. It only remains to be shown that (Δ, Γ) is not decomposable into rank 2 boolean intervals; this is done using a similar argument as in the proof of Theorem 6.

Theorem 7. $\Phi = (\Delta, \Gamma)$ *is not rank* 2 *boolean decomposable.*

Since $f(\Gamma) = (1, 8, 22, 24, 9)$, Theorem 6 immediately implies that Ω_{17} is a counterexample to Conjecture 2, since $17 > 64/4$.

Remark 8. *A linear program [5] verifies that* $\Omega = \Omega_3$ *is a counterexample to Conjecture 2. The f-polynomial of this counterexample is* $f(\Omega_3, t) = 1 + 20t + 136t^2 + 216t^3 + 99t^4 = (1 + t)^2(1 + 18t + 99t^2)$. *This is the smallest known counterexample to Conjecture 2.*

4 Boolean Trees

While Conjecture 2 is false, we will use this section to prove a weakened version of it by replacing boolean intervals with boolean trees. We will rely on algebraic shifting, developed by Kalai in [14], and iterated homology, developed by Duval and Rose in [9] and Duval and Zhang in [10]. We refer the reader to these sources for more details.

Theorem 9. *Let* Δ *be k-fold acyclic. Then* Δ *can be written as a disjoint union of boolean trees of rank k. Furthermore, the minimal faces of these boolean trees together form a subcomplex* Δ'.

The proof is similar to the proof of [10, Corollary 3.5]. It follows from [9, Theorem 4.1], [10, Theorem 3.2], and [14, Theorem 4.2].

The subcomplex Δ' in Theorem 9 is a combinatorial witness to the subcomplex in [18, Proposition 2.3]. This shows that the correct generalization of Stanley's acyclic matching is to boolean trees rather than boolean intervals.

We note the similarity between this case and the so-called Partitionability Conjecture (see, e.g., [8], [17]). A complex Δ is **partitionable** if its face poset can be written as the disjoint union of boolean intervals whose maximal faces are the *facets* of Δ. Though there exist Cohen-Macaulay complexes which are not partitionable [7], all Cohen-Macaulay complexes do have a similar decomposition if "boolean interval" is replaced in the definition of partitionable with "boolean tree" [10, Theorem 5.4].

5 d-fold Acyclic Complexes

In this section, we will show that Conjecture 2 holds for d-fold acyclic complexes where $d = \dim \Delta$. We first show that Conjecture 2 holds for stacked complexes. We then show that d-dimensional d-fold acyclic complexes must be stacked. Thus Conjecture 2 holds when $k = \dim \Delta$.

Our interest in this case was sparked by the following result.

Theorem 10 (Duval–Klivans–Martin, unpublished)**.** *If Δ is 2-dimensional and 2-fold acyclic, then Δ is stacked.*

Theorem 10 together with the following proposition shows that Conjecture 2 holds if $\dim \Delta \leq 2$.

Proposition 11. *Let Δ be a d-dimensional stacked simplicial complex. Then Δ is d-fold acyclic and Δ can be written as the disjoint union of rank d boolean intervals, the minimal elements of which form a subcomplex $\Delta' \subseteq \Delta$. In other words, Conjecture 2 holds for stacked complexes.*

The proof of this proposition follows from a straightforward induction on dimension. The following are two technical results needed in the proof of Theorem 14.

Lemma 12. *Let Δ be d-dimensional and d-fold acyclic. Then the f-polynomial of Δ is $f(\Delta, t) = (1+t)^d(1+nt)$ where n is the number of facets of Δ.*

Lemma 13. *Let Δ be d-dimensional and d-fold acyclic. Then Δ is pure and its facet-ridge graph is connected.*

Lemma 12 follows immediately from [18, Proposition 2.3], and Lemma 13 follows from straightforward elementary arguments.

Theorem 14. *If Δ is d-dimensional and d-fold acyclic, then Δ is stacked.*

Using the above lemmas, Theorem 14 is proved by first producing a partitioning of a given d-fold acyclic complex Δ and then showing that this partitioning gives rise to a stacked shelling.

Combining Theorem 14 and Proposition 11, we see that a d-dimensional complex Δ is stacked if and only if it is d-fold acyclic. This leads immediately to our main result of this section.

Corollary 15. *Conjecture 2 holds when $k = \dim \Delta$.*

6 Open Questions

While our construction gives a counterexample to Conjecture 2, our result in Theorem 9 provides an explicit witness to the structure of the f-polynomials of k-fold acyclic complexes. Perhaps the most interesting questions in light of Remark 8 are in determining any additional conditions that would make the conjecture hold. We know that Ω_3 is the lowest dimensional counterexample possible, but we have no reason to suspect that is in other senses the smallest.

Question 1. *What is a minimal counterexample to Conjecture 2 with respect to the total number of faces, vertices, or facets, respectively?*

Though our counterexample is three-dimensional, it cannot be embedded into $\mathbb{R}^3$. It is unknown if this non-embedding is necessary to be a counterexample.

Question 2. *Is it possible to find a counterexample to Conjecture 2 that embeds into $\mathbb{R}^3$? In general, is it possible to find a d-dimensional counterexample that embeds into $\mathbb{R}^d$?*

It is also unknown whether complexes with additional topological or combinatorial structure could be counterexamples.

Question 3. *Do all k-fold acyclic simplicial balls have a rank k boolean interval decomposition? If they do, must there be a decomposition so that the bottoms of these intervals forms a subcomplex?*

Although a bit further afield from the techniques developed in this paper, one can ask about random simplicial complexes.

Question 4. *For a fixed triple of k, d, v, there are finite k-fold acyclic complexes of dimension d with v vertices. Sampling from this set with the uniform distribution, what is the probability the chosen complex has a rank k boolean decomposition? What is the limiting probability as v goes towards ∞?*

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