Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #45, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Divisors on matroids and their volumes

Christopher Eur*

Department of Mathematics, University of California, Berkeley.

Abstract. The classical volume polynomial in algebraic geometry measures the degrees of ample (and nef) divisors on a smooth projective variety. We introduce an analogous volume polynomial for matroids, give a complete combinatorial formula, and show that it is a valuation under matroid polytope subdivisions. For a realizable matroid, we thus obtain an explicit formula for the classical volume polynomial of the associated wonderful compactification; in particular, we obtain another formula for volumes of generalized permutohedra. We then introduce a new invariant called the shifted rank-volume of a matroid as a particular specialization of its volume polynomial, and discuss its algebro-geometric and combinatorial properties in connection to graded linear series on blow-ups of projective spaces.

Keywords: matroids, Chow rings, volume polynomials, Newton-Okounkov bodies, generalized permutohedra

1 Introduction

Recent intersection theoretic approach to matroids has led to the resolution of the long-standing conjecture of Rota on the log-concavity of the coefficients of chromatic polynomials, first proven for graphs by Huh in [19], for realizable matroids in [20], and for general matroids by Adiprasito, Huh, and Katz in [1]. Among the key tools in [19, 20] for establishing log-concavity is the Teissier-Khovanskii inequality for intersection numbers of nef divisors on smooth projective varieties, which can be understood as a phenomenon of convexity: The Newton-Okounkov body $\Delta(D)$ of a nef divisor D is a convex body whose volume is the self-intersection number of D, and its existence reduces the Teissier-Khovanskii inequality to the Brunn-Minkowski inequality for volumes of convex bodies.

For general matroids, a combinatorial version of Teissier-Khovanskii inequality is proven in [1] by establishing Hodge theory analogues for the Chow ring of a matroid *without* explicit use of convex bodies. Noting that matroids can be considered as tropical linear varieties (see [4]), the results of [1] suggest an existence for an analogue of Newton-Okounkov bodies for tropical linear varieties, and perhaps tropical varieties in general. Our results here can be seen as a first step towards such a direction with various ramifications to both algebraic geometry and combinatorics of matroids.

*ceur@math.berkeley.edu

The volume polynomial in classical algebraic geometry measures the self-intersection number of a nef divisor on a smooth projective variety, or equivalently the volume of its Newton-Okounkov body. In essence, it is defined by choosing a presentation of the cohomology ring of the variety; here we define analogously the volume polynomial VP_M for a matroid M.

Let M be a matroid of rank $r = d + 1$ on a ground set E with lattice of flats $\mathscr{L}_M$, and denote $\overline{\mathscr{L}_M} := \mathscr{L}_M \setminus \{\emptyset, E\}$. The Chow ring of a matroid defined below plays the role of the cohomology ring of a smooth projective variety of dimension d.

Definition 1.1. The **Chow ring** of a simple matroid M is the graded ring

$$A^\bullet(M) := \frac{\mathbb{R}[x_F : F \in \overline{\mathscr{L}_M}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \ni i} x_F - \sum_{G \ni j} x_G \mid i, j \in E\rangle}$$

In analogy with Chow rings in algebraic geometry, we call elements of $A^1(M)$ **divisors** on a matroid M. Like the cohomology rings of smooth manifolds, each graded piece $A^i(M)$ of the Chow ring $A^\bullet(M)$ is a finite dimensional $\mathbb{R}$-vector space and is nonzero only for $0 \le i \le d$. It satisfies Poincaré duality, with the degree map

$$\deg_M : A^d(M) \xrightarrow{\sim} \mathbb{R} \quad \text{where } \deg_M(x_{F_1} x_{F_2} \cdots x_{F_d}) = 1 \text{ for every maximal chain } F_1 \subsetneq \cdots \subsetneq F_d \text{ in } \overline{\mathscr{L}_M}.$$

Given a graded algebra R with such Poincaré duality property, a standard procedure of Macaulay's inverse system in commutative algebra then provides a way to encode R by a single polynomial called the cogenerator from which R can be recovered (see Proposition 2.4).

Definition 3.1. Let M be a matroid of rank $r = d + 1$. The **volume polynomial** $VP_M(\underline{t}) \in \mathbb{R}[t_F : F \in \overline{\mathscr{L}_M}]$ is the cogenerator of $A^\bullet(M)$.

When M is realizable, VP_M agrees with the classical volume polynomial of the wonderful compactification Y_M of the complement of the associated hyperplane arrangement $\mathcal{A}_M$ of M. While VP_M is initially defined purely algebraically, we prove a completely combinatorial formula for VP_M, which follows from our first main theorem on the intersection numbers of divisors on a matroid.

Theorem 3.2. Let $M = (E, \mathcal{B})$ be a matroid, $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ a chain of flats in $\mathscr{L}_M$ of ranks $r_i := \operatorname{rk} F_i$, and $d_1, \ldots, d_k$ be positive integers such that $\sum_i d_i = d := \operatorname{rk} M - 1$. Denote by $\widetilde{d_i} := \sum_{j=0}^{i} d_j$ (where $d_0 := 0$). Then

$$\deg_M(x_{F_1}^{d_1} \cdots x_{F_k}^{d_k}) = (-1)^{d-k} \prod_{i=1}^{k} \binom{d_i - 1}{\widetilde{d_i} - r_i} \mu^{\widetilde{d_i} - r_i}(M|F_{i+1}/F_i)$$

where $\mu^i(N)$ denotes the i-th unsigned coefficient of the reduced characteristic polynomial $\overline{\chi}_N(t) = \mu^0(N) t^{\operatorname{rk} N - 1} - \mu^1(N) t^{\operatorname{rk} N - 2} + \cdots \pm \mu^{\operatorname{rk} N - 1}(N)$ of a matroid N.

Corollary 3.3. Let the notations be as above. The coefficient of $t_{F_1}^{d_1} \cdots t_{F_k}^{d_k}$ in $VP_M(\underline{t})$ is

$$(-1)^{d-k} \binom{d}{d_1, \dots, d_k} \prod_{i=1}^{k} \binom{d_i - 1}{\widetilde{d}_i - r_i} \mu^{\widetilde{d}_i - r_i}(M|F_{i+1}/F_i).$$

We give two immediate applications of the formula for the volume polynomial. The first is an explicit formula in Proposition 4.1 for the volumes of generalized permutohedra, adding to the ones given by Postnikov in [25]. The second is to show that taking the Chow ring of a matroid respects the "type A" structure of matroids. More precisely,

Proposition 4.3. Consider $VP_M(\underline{t})$ as a polynomial in $\mathbb{R}[t_S : S \in 2^E]$. Then the map $M \mapsto VP_M \in \mathbb{R}[t_S : S \in 2^E]$ is a matroid valuation in the sense of [3].

While the volume polynomial VP_M has the same information as the Chow ring $A(M)$ of a matroid, it lends itself more naturally as an invariant of a matroid than the Chow ring. That $M \mapsto VP_M$ is a valuation already illustrates this effect. Moreover, any natural specialization of VP_M defines an invariant of a matroid. We present here one such example coming from the rank function of a matroid.

Definition 5.1. Define the **shifted rank-volume** of a matroid M to be

$$\operatorname{shRVol}(M) := VP_M(t_F := \operatorname{rk} F) = \deg_M \Big(\sum_{F \in \overline{\mathscr{L}_M}} (\operatorname{rk} F) x_F \Big)^{\operatorname{rk} M - 1}.$$

The shifted rank-volume of a matroid seems to be a genuinely new invariant, as it is unrelated to classical invariants such as the Tutte polynomial or the volume of the matroid polytope; see Remark 5.3. It seems to measure how close the matroid is to the uniform matroid.

Theorem 5.4. Let M be a realizable matroid of rank r on n elements. Then

$$\operatorname{shRVol}(M) \leq \operatorname{shRVol}(U_{r,n}) = (n-r)^{r-1} \quad \text{with equality iff } M = U_{r,n}.$$

The proof of Theorem 5.4 is algebro-geometric in nature, involving counting sections of line bundles. In a subsequent work we give a proof for arbitrary matroids [5], but it is not a combinatorial reflection of such algebro-geometric approach. This naturally leads to asking whether there exist an analogue of the theory of Newton-Okounkov bodies for linear tropical varieties.

This is an extended abstract for the author's full paper [14]. Worked-out examples and codes can be found at https://math.berkeley.edu/~ceur/research.html.

Notations. $|S|$ denotes the cardinality of a (finite) set S. We use k for a field, which we always assume algebraically closed, and a variety is an integral separated scheme over a field k. A binomial coefficient $\binom{n}{m}$ is understood to be zero if $m < 0$ or $m > n$.

2 Preliminaries

Wonderful compactifications and Chow rings of matroids

See [24] for a general reference on matroids. For accounts tailored towards Chow ring of matroids, we recommend [6].

For a matroid M on a ground set E of rank $r = d+1$, denote by $\mathscr{L}_M$ the lattice of flats, and $\overline{\mathscr{L}_M} := \mathscr{L} \setminus \{\emptyset, E\}$[1]. An open interval in a lattice $\mathscr{L}$ is denoted $(\ell_1, \ell_2) = \{\ell \in \mathscr{L} \mid \ell_1 < \ell < \ell_2\}$. Let $\chi_M(t)$ be the characteristic polynomial of M, then its **reduced characteristic polynomial** is $\overline{\chi}_M(t) := \chi_M(t)/(t-1)$.

If the matroid M matroid is realizable, say as vectors $\{v_i\}_{i\in E}$ spanning a k-vector space V, denote by $\mathcal{A}_M = \{H_i\}_{i\in E}$ the associated hyperplane arrangement in $\mathbb{P}(V^*)$ where $H_i := \{f \in \mathbb{P}(V^*) : f(v_i) = 0\}$. A flat F of rank c in M correspond to a c-codimensional linear space $H_F := \{f \in \mathbb{P}(V^*) : f(v_i) = 0 \ \forall i \in F\}$. The **wonderful compactification** Y_M is then obtained by consecutively blowing-up (strict transforms of) $\{H_F\}_{F\in\overline{\mathscr{L}_M}}$ in $\mathbb{P}(V^*)$ starting with H_F with highest $\operatorname{rk} F$. Y_M is a compactification of the hyperplane arrangement complement $C(\mathcal{A}_M) = \mathbb{P}(V^*) \setminus \mathcal{A}_M$ whose boundary $Y_M \setminus C(\mathcal{A}_M)$ consists of the exceptional divisors $\widetilde{H}_F$, which have simple-normal-crossings. See [11] for the original construction, or [16] for a survey written for combinatorialists. The intersection theory of the boundary divisors of Y_M is encoded in the matroid ([1, Theorem 5.12]), which leads to the definition of the Chow ring of an arbitrary matroid (not necessarily realizable), first appearing in [15] and further studied in [1].

Definition 2.1. The **Chow ring** of a (simple) matroid M is the graded ring

$$A^\bullet(M) := \frac{\mathbb{R}[x_F : F \in \overline{\mathscr{L}_M}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F\ni i} x_F - \sum_{G\ni j} x_G \mid i,j \in E\rangle}.$$

We call elements of $A^1(M)$ **divisors** on M.

Recently in [1], the ring $A^\bullet(M)$ has been shown to satisfy the whole Kähler package—Poincaré duality, hard Lefschetz property, and Hodge-Riemann relations. For our purposes, we only need the Poincaré duality.

Proposition 2.2. [1, §5.10] The Chow ring $A^\bullet(M)$ of a matroid M of rank $r = d+1$ is a finite graded $\mathbb{R}$-algebra satisfying: (i) there exists a linear isomorphism $\deg_M : A^d(M) \to \mathbb{R}$ uniquely determined by the property that $\deg_M(x_{F_1}x_{F_2}\cdots x_{F_d}) = 1$ for every maximal chain $F_1 \subsetneq \cdots \subsetneq F_d$ in $\overline{\mathscr{L}_M}$, and (ii) the pairings $A^i(M) \times A^{d-i}(M) \to A^d(M) \overset{\deg}{\simeq} \mathbb{R}$ are non-degenerate.

[1] As the notions we discuss will only depend on $\mathscr{L}_M$, we assume for simplicity that M is simple.

Volumes of divisors and cogenerators

For a general reference on intersection theory, see [13]. Here we follow the survey [12].

Let X be a d-dimensional smooth projective variety over an algebraically closed field k, and let $A^\bullet(X)$ be its Chow ring and denote by $\deg_X$ or $\int_X$ the degree map $A^d(X) \to \mathbb{Z}$ sending a class of a closed point to 1. For a Cartier divisor D on X, the **volume** of D is defined as

$$\mathrm{vol}(D) := \lim_{t\to\infty} \frac{h^0(X, \mathscr{O}(tD))}{t^d/d!}.$$

In other words, denoting by $R(D)_\bullet := \bigoplus_{t\geq 0} H^0(X, tD)$ the **section ring** of D, the volume measures the asymptotics of $\frac{\dim_k R(D)_t}{t^d/d!}$ as $t \to \infty$.

By standard relation between Hilbert polynomials and intersection multiplicities, volume of a very ample divisor D is the degree of X under the embedding given by D. can be geometrically interpreted as follows: If $m > 0$ is such that mD is very ample, then for general divisors $E_1, \dots, E_d$ in the complete linear system $|mD|$ we have $\mathrm{vol}(D) = \frac{1}{m^d} \deg_X[E_1 \cap E_2 \cap \cdots \cap E_d]$. That is, $\mathrm{vol}(D) = \int_X (c_1(D))^d$ if D is ample.

The volume of a divisor depends only on its numerical equivalence class. Thus, letting $N^1(X)$ be the group divisors modulo numerical equivalence generated by $\{\xi_1, \dots, \xi_r\}$ and $\mathrm{Nef}(X) \subset N^1(X)_\mathbb{R}$ the nef cone, the map vol : $\mathrm{Nef}(X) \to \mathbb{R}$ defines the **volume polynomial** $VP_X \in \mathbb{R}[t_1, \dots, t_r]$ where

$$VP_X(t_1, \dots, t_r) = \mathrm{vol}(t_1\xi_1 + \cdots + t_r\xi_r) \quad \text{whenever } t_1\xi_1 + \cdots + t_r\xi_r \in \mathrm{Nef}(X).$$

Remark 2.3. The volume of a divisor D can be realized as a volume of convex body $\Delta(D)$ called the **Newton-Okounkov body** of D. For background on Newton-Okounkov bodies, see [22] for a more geometric perspective with applications to big divisors and Néron-Severi groups, and [21] for an approach using semigroups and with a view towards generalized Kushnirenko-Bernstein theorem.

Macaulay's inverse system provide a purely algebraic approach to the notion of volume polynomial as the cogenerator of a graded ring with Poincaré duality.

Proposition 2.4. [10, §13.4.7] Suppose a graded finite k-algebra $A = \bigoplus_{i=0}^d A_i$ satisfies: (i) A is generated in A_1 with $A_0 = k$, (ii) there exists a k-linear isomorphism deg : $A_d \to k$, and (iii) $A_i \times A_{d-i} \to A_d \overset{deg}{\simeq} k$ is a non-degenerate pairing. Let $x_1, \dots, x_n$ generate A_1, so that $A \simeq k[\underline{x}]/I$ for some ideal I. Then there exists $P \in k[t_1, \dots, t_n]$ such that

$$I = \{f \in k[\underline{x}] \mid f(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}) \cdot P = 0\}.$$

Up to scaling by an element of k, this **cogenerator** P is

$$P = \deg\left((t_1x_1 + \cdots + t_nx_n)^d\right)$$

where we extend deg : $A_d \to k$ to $A_d[t_1, \dots, t_n] \to k[t_1, \dots, t_n]$.

3 The volume polynomial of a matroid

As the Chow ring $A^\bullet(M)_\mathbb{R}$ of a matroid satisfies Poincaré duality, that is, the conditions of Proposition 2.4, we can define its cogenerator.

Definition 3.1. Let M be a matroid of rank $r = d+1$. The **volume polynomial** $VP_M \in \mathbb{R}[t_F : F \in \overline{\mathscr{L}_M}]$ is the cogenerator of $A^\bullet(M)_\mathbb{R}$, where VP_M is normalized so that the coefficient of any monomial $t_{F_1}t_{F_2}\cdots t_{F_d}$ corresponding to a maximal chain of flats in $\mathscr{L}_M$ is $d!$.

Equivalently, observe that via Proposition 2.4 the volume polynomial is $VP_M = \deg_M((\sum_{F\in\overline{\mathscr{L}_M}} x_F t_F)^d)$ (where $\deg_M : A^d(M) \to \mathbb{R}$ is extended to $A^d[t_F\text{'s}] \to \mathbb{R}[t_F\text{'s}]$).

The coefficient of $t_{F_1}^{d_1}\cdots t_{F_k}^{d_k}$ for $d_1+\cdots d_k = d = \operatorname{rk} M - 1$ in the volume polynomial VP_M is $\binom{d}{d_1,\dots,d_k}\deg_M(x_{F_1}^{d_1}\cdots x_{F_k}^{d_k})$. Thus, the knowing the volume polynomial amounts to knowing all the intersection numbers $\deg_M(x_{F_1}^{d_1}\cdots x_{F_k}^{d_k})$. Our main theorem is the combinatorial formula for all the intersection numbers.

Theorem 3.2. Let $M = (E,\mathcal{B})$ be a matroid, $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ a chain of flats in $\mathscr{L}_M$ of ranks $r_i := \operatorname{rk} F_i$, and $d_1,\dots,d_k$ be positive integers such that $\sum_i d_i = d := \operatorname{rk} M - 1$. Denote by $\widetilde{d_i} := \sum_{j=0}^i d_j$ (where $d_0 := 0$). Then

$$\deg(x_{F_1}^{d_1}\cdots x_{F_k}^{d_k}) = (-1)^{d-k}\prod_{i=1}^k \binom{d_i - 1}{\widetilde{d_i} - r_i}\mu^{\widetilde{d_i}-r_i}(M|F_{i+1}/F_i)$$

where $\mu^i(N)$ is the i-th unsigned coefficient of the reduced characteristic polynomial $\overline{\chi}_N(t) = \mu^0(N)t^{\operatorname{rk} N-1} - \mu^1(N)t^{\operatorname{rk} N-2} + \cdots + (-1)^{\operatorname{rk}(N)-1}\mu^{\operatorname{rk} N-1}(N)$ of a matroid N.

Corollary 3.3. Let the notations be as above. The coefficient of $t_{F_1}^{d_1}\cdots t_{F_k}^{d_k}$ in $\mathrm{VP}_M(\underline{t})$ is

$$(-1)^{d-k}\binom{d}{d_1,\dots,d_k}\prod_{i=1}^k \binom{d_i - 1}{\widetilde{d_i} - r_i}\mu^{\widetilde{d_i}-r_i}(M|F_{i+1}/F_i).$$

Remark 3.4 ($\overline{\mathcal{M}_{0,n}}$)**.** When M is realizable, VP_M agrees with the classical volume polynomial of the wonderful compactification Y_M. In particular, when $M = M(K_{n-1})$ is the matroid of the complete graph on $n-1$ vertices, its wonderful compactification Y_M is the Deligne-Mumford space $\overline{\mathcal{M}_{0,n}}$ of rational curves with n marked points ([23, §6.4]). The numerical cones of $\overline{\mathcal{M}_{0,n}}$ are believed to be complicated, as $\overline{\mathcal{M}_{0,n}}$ is not a Mori dream space in general ([18]). Nevertheless, our combinatorial formula for the volume polynomial allows for computation of the volume of any divisor in its nef cone.

Remark 3.5 (Computation)**.** Computing VP_M via Proposition 2.4 alone via Gröbner bases quickly becomes infeasible as the matroid becomes larger. For example, $M(K_6)$ has 203 flats, so that the Chow ring $A(M)$ has 201 variables.

The proof of Theorem 3.2 consists broadly of three steps. First, via the geometry of the Bergman fan of a matroid, one deduces the following key lemma.

Lemma 3.6. Let $M = (E, \mathcal{B})$ be a matroid, $\mathscr{F} : \varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ a chain of flats in $\mathscr{L}_M$. Write $x_{\mathscr{F}}$ for $x_{F_1}x_{F_2}\cdots x_{F_k}$. Then for any $1 \le i \le k$, we have

$$x_{\mathscr{F}} \cdot x_{F_i} = x_{\mathscr{F}} \cdot (-1) \left(\sum_{F \in (F_i, F_{i+1})} \frac{|F_{i+1} \setminus F|}{|F_{i+1} \setminus F_i|} x_F + \sum_{F \in (F_{i-1}, F_i)} \frac{|F \setminus F_{i-1}|}{|F_i \setminus F_{i-1}|} x_F \right)$$

The second step involves using this to expand $x_{F_1}^{d_1} \cdots x_{F_k}^{d_k}$ into square-free monomials. In doing so, one comes across the following quantity.

Definition 3.7. For M a loopless matroid on a ground set E of rank $d+1$ and $-1 \le i \le d+1$, define $\gamma(M, i)$ as follows. For $i = -1, d+1$, define $\gamma(M, -1) = -1$ and $\gamma(M, d+1) = 0$; for $0 \le i \le d$, define

$$\gamma(M, i) := \sum_{\mathscr{G} \in \mathscr{L}_M^{\le i}} (-1) \left(-\frac{|G_1 \setminus G_0|}{|G_1|} \right) \left(-\frac{|G_2 \setminus G_1|}{|G_2|} \right) \cdots \left(-\frac{|G_{i+1} \setminus G_i|}{|G_{i+1}|} \right)$$

where $\mathscr{L}_M^{\le i}$ consists of chains of flats $\mathscr{G} : \varnothing = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_i \subsetneq G_{i+1} = E$ such that $\operatorname{rk} G_j = j$ for $j = 0, \ldots, i$.

Lastly, one relates these quantities $\gamma(M, i)$ to the characteristic polynomial via Weisner's theorem [26, §3.9] and the cover-partition axiom for flats of a matroid.

Proposition 3.8. $\overline{\chi}_M(t) = \sum_{i=0}^{d} \gamma(M, i) t^{d-i}$ for a loopless matroid M of rank $r = d+1$.

4 First applications of the volume polynomial

We give some first applications of the volume polynomial of a matroid. In this section, we always set the ground set of a matroid to be $[n] := \{1, \ldots, n\}$ for some n.

Volumes of generalized permutohedra

A **generalized permutohedron** P is obtained by sliding the facets of the permutohedron. More precisely, for a submodular function $z_{(\cdot)} : 2^{[n]} \to \mathbb{R}$ (where $[n] \supset I \mapsto z_I$) on the boolean lattice $\mathscr{L}_{U_{n,n}}$, we define

$$P(\underline{z}) := \{(\underline{x}) \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = z_{[n]}, \sum_{i \in I} x_i \le z_I \ \forall I \subset [n]\}$$

which is a polytope of dimension at most $n-1$ in $\mathbb{R}^n$. It is characterized by the property that every edge is parallel to $e_i - e_j$ for some $i \ne j \in [n]$.

The Bergman fan $\Sigma_{A_{n-1}}$ of $U_{n,n}$ is the normal fan of the permutohedron, whose rays correspond to subsets of $[n]$. As nef torus invariant divisors on the toric variety $X_{\Sigma_{A_{n-1}}}$ exactly correspond to submodular functions with $z_\emptyset = z_{[n]} = 0$, combining Corollary 3.3 and well-known results from toric geometry on volumes of torus invariant divisors ([10, §13.4.3]) gives us:

Proposition 4.1. Let $z_{(\cdot)} : I \mapsto z_I \in \mathbb{R}$ be a submodular function such that $z_\emptyset = z_{[n]} = 0$, then the volume of the generalized permutohedron $P(\underline{z})$ is

$$(n-1)! \operatorname{Vol} P(\underline{z}) = \sum_{I_\bullet, \underline{d}} (-1)^{d-k} \binom{d}{d_1, \ldots, d_k} \prod_{i=1}^{k} \binom{d_i - 1}{\widetilde{d}_i - |I_i|} \binom{|I_{i+1}| - |I_i| - 1}{\widetilde{d}_i - |I_i|} z_{I_i}$$

where the summation is over chains $\emptyset \subsetneq I_1 \subsetneq \cdots \subsetneq I_k \subsetneq I_{k+1} = [n]$ and $\underline{d} = (d_1, \ldots, d_k)$ such that $\sum d_i = n-1$ and $\widetilde{d}_j := \sum_{i=1}^{j} d_i$.

Postnikov computed the volumes of generalized permutohedra that can be expressed as Minkowski sums of standard simplices ([25, Corollary 9.4]). Note that our formula above imposes no such restriction. It was not clear to the author how one recovers Proposition 4.1 from [25, Corollary 9.4] or vice versa.

Valuativeness of the volume polynomial

While the volume polynomial VP_M has the same information as the Chow ring $A(M)$, it lends itself more naturally as a valuation on a matroid when viewed as a map $M \mapsto VP_M \in \mathbb{R}[t_S : S \in 2^{[n]}]$. In this subsection, we illustrate this by showing that $M \mapsto VP_M$ is a valuation under matroid polytope subdivisions, which is a statement that does not make sense for $M \mapsto A(M)$.

We first give a brief sketch of matroid polytopes and matroid valuations; for more on matroid valuations, we point to [3]. Given a matroid M on $[n]$ of rank r with bases $\mathcal{B}$, its **matroid polytope** is defined as

$$\Delta(M) := \operatorname{Conv}(e_B = \sum_{i \in B} e_i \mid B \in \mathcal{B}) \subset \mathbb{R}^n$$

where $e_S := \sum_{i \in S} e_i$ for $S \subset [n]$ and e_i's are the standard basis of $\mathbb{R}^n$. Its vertices are the indicator vectors for the bases of M, and it follows from a theorem of Gelfand, Goresky, MacPherson, and Serganova that the faces of $\Delta(M)$ are also matroid polytopes ([17]). A **matroid subdivision** $\mathcal{S}$ of a matroid polytope Δ is a polyhedral subdivision $\mathcal{S} : \Delta = \bigsqcup_i \Delta(M_i)$ such that each $\Delta(M_i)$ is a matroid polytope of some matroid M_i (necessarily of rank r on with ground set $[n]$). Denote by $\operatorname{Int}(\mathcal{S})$ the faces of $\Delta(M_i)$'s that is not contained in the boundary of $\Delta(M)$. It is often of interest to see whether a valuation on matroids behave well via inclusion-exclusion with respect to matroid subdivisions:

Definition 4.2. Let R be an abelian group, and let $\mathcal{M} := \bigcup_{n\geq 0}\{\text{matroids on ground set } [n]\}$ be the set of all matroids. A map $\varphi : \mathcal{M} \to R$ is a **matroid valuation** (or is **valuative**) if for any $M \in \mathcal{M}$ and a matroid subdivision $\mathcal{S} : \Delta(M) = \bigsqcup_i \Delta(M_i)$ one has

$$\varphi(M) = \sum_{Q\in \mathrm{Int}(\mathcal{S})} (-1)^{\dim \Delta(M) - \dim Q}\varphi(M_i).$$

Many interesting functions on matroids are matroid valuations, for example the Tutte polynomial ([3, Corollary 5.7]). The formula in Corollary 3.3, combined with Hopf algebra structure for matroids (see [2]), allows one to prove that volume polynomial of a matroid is also a matroid valuation.

Proposition 4.3. $M \mapsto VP_M(t) \in \mathbb{R}[t_S : S \in 2^{[n]}]$ is a matroid valuation.

That the matroid volume polynomial behaves well with respect to matroid polytope subdivisions suggests that there may be a generalization of Chow ring of matroids to Coxeter matroids of arbitrary Lie type (where matroids are the type A case). For Coxeter matroids see [7].

5 The shifted rank-volume of a matroid

Let $M = (E, \mathcal{B})$ be a matroid. Following [1], a (strictly) submodular function $c_{(\cdot)} : 2^E \to \mathbb{R}$ gives a *combinatorially nef (ample)* divisor $D = \sum_{F\in\overline{\mathscr{L}_M}} c_F x_F \in A^1(M)_{\mathbb{R}}$. For realizable matroids, if the divisor $D \in A^1(M)_{\mathbb{R}}$ is combinatorially nef (ample) then as an element of $A^1(Y_M)_{\mathbb{R}}$ the divisor D is nef (ample) in the classical sense. As the rank function is a distinguished submodular function of a matroid[2], we define the following notions.

Definition 5.1. For a matroid M, define its **shifted rank-divisor** to be $D_M := \sum_{F\in\overline{\mathscr{L}_M}} (\operatorname{rk} F) x_F$, and define the **shifted rank-volume of a matroid** M to be the volume of its shifted rank-divisor:

$$\mathrm{shRVol}(M) := \deg \Big(\sum_{F\in\overline{\mathscr{L}_M}} (\operatorname{rk} F) x_F \Big)^{\operatorname{rk} M - 1}.$$

A slight modification of the proof of Proposition 4.3 implies that the shifted rank-volume of a matroid is a valuative invariant.

Corollary 5.2. The map $M \mapsto \mathrm{shRVol}(M)$ is a matroid valuation.

We do not know of a purely combinatorial meaning of the shifted rank-volume of a matroid. In fact, it seems to be a genuinely new invariant of a matroid.

[2]Technically one should impose $c_\emptyset = c_E = 0$ for the submodular function c, which the rank function does not satisfy (as most often $\operatorname{rk}(E) > 0$). This turns out not to be an issue here, but is the reason for the modifier “shifted" in the following definition.

Remark 5.3 (Relation to other invariants, or lack thereof)**.** We point to `Eur_DivMatVol.m2` that can be found at https://math.berkeley.edu/~ceur/research.html for computations supporting the statements below.

- Same Tutte polynomial does not imply same shRVol, and vice versa. The two graphs in Figure 2 of [9] have the same Tutte polynomial but their matroids are not isomorphic; their shifted rank-volumes are 1533457 and 1534702. There are many examples of matroids with same shRVol but with different Tutte polynomials.
- Same volume of the matroid polytope does not imply same shRVol, and vice versa.

The following theorem shows that the shifted rank-volume seems to measure how close the matroid is to the uniform matroids.

Theorem 5.4. Let M be a realizable matroid of rank r on n elements. Then

$$\mathrm{shRVol}(M) \leq \mathrm{shRVol}(U_{r,n}) = n^{r-1} \quad \text{with equality iff } M = U_{r,n}.$$

Proof. As Y_M is constructed as consecutive blow-ups of $\mathbb{P}^{r-1}$, let $\pi : Y_M \to \mathbb{P}^{r-1}$ be the blow-down map and let $\widetilde{H} = \pi^*(c_1(\mathscr{O}_{\mathbb{P}^{r-1}}(1)))$ be the pullback of the hyperplane class. Then the key observation is

$$D_M = n\widetilde{H} - \sum_{\operatorname{rk} F > 2} (|F| - \operatorname{rk}_M F) x_F.$$

Noting that $|F| - \operatorname{rk}_M F \geq 0$, we have that $D_M = D - E$ where $D = n\widetilde{H}$ is a nef divisor and $E = \sum_{\operatorname{rk} F>2}(|F| - \operatorname{rk}_M F)x_F$ is an effective divisor. Then the proof of the inequality then reduces to an almost trivial statement in algebraic geometry that $H^0(m(D-E)) \subset H^0(mD)$ for all $m \geq 0$ (or equivalently that $\Delta(D-E) \subset \Delta(D)$).

For the remainder of the statement, note that $\bigoplus_{m\geq 0} H^0(m(n\widetilde{H} - E))$ is a graded linear series on $\mathbb{P}^{r-1}$ of degree n hypersurfaces vanishing at certain loci dictated by E, where $E = 0$ exactly when $M = U_{r,n}$. Thus, one concludes that $\mathrm{shRVol}(U_{r,n}) =$ (degree of Veronese embedding by $\mathscr{O}_{\mathbb{P}^{r-1}}(n)) = n^{r-1}$ and that among realizable matroids the maximum n^{r-1} is achieved *uniquely by* $U_{r,n}$. □

There is no analogue of counting sections $H^0(D)$ of a divisor D in the purely combinatorial setting of matroids or more generally in tropical geometry. As a result, the proof of the inequality in Theorem 5.4 for arbitrary matroids given in a subsequent work [5] is not a combinatorial reflection of the algebro-geometric proof for realizable matroids given above, and the validity of the statement that the maximum is achieved uniquely by the uniform matroid remains unclear. This thus further begs the question of whether there is a notion of Newton-Okounkov bodies for matroids, a.k.a. tropical linear varieties.

Acknowledgements

The author would like to thank Bernd Sturmfels for suggesting this problem, Justin Chen for many helpful discussions and his Macaulay2 matroids package [8], Alex Fink for generously providing the sketch of the proof for valuativeness of the volume polynomial, and Federico Ardila for inspiring the author to pursue this direction of research. The author is also grateful for helpful conversations with David Eisenbud, June Huh, Vic Reiner, Botong Wang, and Mengyuan Zhang.

References

[1] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". *Ann. of Math. (2)* **188**.2 (2018), pp. 381–452. Link.

[2] F. Ardila and M. Aguiar. "Hopf monoids and generalized permutahedra". 2017. arXiv:1709.07504.

[3] F. Ardila, A. Fink, and F. Rincón. "Valuations for matroid polytope subdivisions". *Canad. J. Math.* **62**.6 (2010), pp. 1228–1245. Link.

[4] F. Ardila and C. J. Klivans. "The Bergman complex of a matroid and phylogenetic trees". *J. Combin. Theory Ser. B* **96**.1 (2006), pp. 38–49. Link.

[5] S. Backman, C. Eur, and C. Simpson. "Chow rings of matroids, matroid quotients, and beyond". In preparation. 2019.

[6] M. Baker. "Hodge theory in combinatorics". *Bull. Amer. Math. Soc. (N.S.)* **55**.1 (2018), pp. 57–80. Link.

[7] A. V. Borovik, I. M. Gelfand, and N. White. *Coxeter Matroids*. Progress in Mathematics 216. Birkhäuser Boston, Boston, 2003. Link.

[8] J. Chen. "Matroids: a Macaulay2 package". *J. Softw. Algebra Geom.* **9** (2019), pp. 19–27. Link.

[9] J. Clancy, T. Leake, and S. Payne. "A note on Jacobians, Tutte polynomials, and two-variable zeta functions of graphs". *Exp. Math.* **24**.1 (2015), pp. 1–7. Link.

[10] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric Varieties*. Graduate Studies in Mathematics 124. American Mathematical Society, Providence, RI, 2011.

[11] C. De Concini and C. Procesi. "Wonderful models of subspace arrangements". *Selecta Math. (N. S.)* **1**.3 (1995), pp. 459–494. Link.

[12] L. Ein, R. Lazarsfeld, M. Mustata, M. Nakamaye, and M. Popa. "Asymptotic invariants of line bundles". *Pure Appl. Math. Q.* **1**.2, Special Issue: In memory of Armand Borel. (2005), pp. 379–403. Link.

[13] D. Eisenbud and J. Harris. *3264 and All That—A Second Course in Algebraic Geometry*. Cambridge University Press, Cambridge, 2016.

[14] C. Eur. “Divisors on matroid and their volumes”. 2018. arXiv:1803.07103.

[15] E. M. Feichtner and S. Yuzvinsky. “Chow rings of toric varieties defined by atomic lattices”. *Invent. Math.* **155**.3 (2004), pp. 515–536. Link.

[16] E. M. Feichtner. “De Concini-Procesi wonderful arrangement models: a discrete geometer’s point of view.” *Combinatorial and Computational Geometry*. Cambridge University Press, 2005, pp. 333–360.

[17] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. “Combinatorial geometries, convex polyhedra, and Schubert cells”. *Adv. in Math.* **63**.3 (1987), pp. 301–316. Link.

[18] J. L. González and K. Karu. “Some non-finitely generated Cox rings”. *Compos. Math.* **152**.5 (2016), pp. 984–996. Link.

[19] J. Huh. “Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs”. *J. Amer. Math. Soc.* **25**.3 (2012), pp. 907–927. Link.

[20] J. Huh and E. Katz. “Log-concavity of characteristic polynomials and the Bergman fan of matroids”. *Math. Ann.* **354**.3 (2012), pp. 1103–1116. Link.

[21] K. Kaveh and A. G. Khovanskii. “Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory”. *Ann. of Math. (2)* **176**.2 (2012), pp. 925–978. Link.

[22] R. Lazarsfeld and M. Mustata. “Convex bodies associated to linear series”. *Ann. Sci. Éc. Norm. Supér. (4)* **42**.5 (2009), pp. 783–835. Link.

[23] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics 161. American Mathematical Society, Providence, RI, 2015.

[24] J. Oxley. *Matroid Theory*. 2nd ed. Oxford Graduate Texts in Mathematics 21. Oxford University Press, Oxford, 2011.

[25] A. Postnikov. “Permutohedra, associahedra, and beyond”. *Int. Math. Res. Not. IMRN* 6 (2009), pp. 1026–1106. Link.

[26] R. P. Stanley. *Enumerative Combinatorics. Volume 1*. 2nd ed. Cambridge Studies in Advanced Mathematics 49. Cambridge University Press, Cambridge, 2012.