

# Total Nonnegativity and Evaluations of Hecke Algebra Characters

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**Abstract.** Let  $\mathfrak{S}_{[i,j]}$  be the subgroup of the symmetric group  $\mathfrak{S}_n$  generated by adjacent transpositions  $(i, i + 1), \dots, (j - 1, j)$ . We give a combinatorial rule for evaluating induced sign characters of the type  $A$  Hecke algebra  $H_n(q)$  at all elements of the form  $\sum_{w \in \mathfrak{S}_{[i,j]}} T_w$  and at all products of such elements. By a result of Stembridge, our result connects  $H_n(q)$  trace evaluation to immanants of totally nonnegative matrices.

**Keywords:** Hecke algebra trace, total nonnegativity

## 1 Introduction

Related to the study of totally nonnegative matrices, those matrices having only nonnegative minors, is the study of polynomials  $p(x_{1,1}, \dots, x_{n,n})$  satisfying  $p(a_{1,1}, \dots, a_{n,n}) \geq 0$  for every totally nonnegative  $n \times n$  matrix  $A = (a_{i,j})$ . We call these *totally nonnegative (TNN) polynomials*. In particular, work of Lusztig [10] implies that if we view  $\mathbb{Z}[x] := \mathbb{Z}[x_{1,1}, \dots, x_{n,n}]$  as a free  $\mathbb{Z}$ -module, then certain elements which are related to the dual canonical basis of the quantum group  $\mathcal{O}_q(SL_n(\mathbb{C}))$  are TNN polynomials.

In practice, it is sometimes possible to use cluster algebras and a computer to demonstrate that a polynomial is TNN by expressing it as a subtraction-free rational expression in matrix minors [5]. On the other hand, no simple characterization of TNN polynomials is known. To improve our understanding of TNN polynomials, one might begin by investigating the *immanant subspace*  $\text{span}_{\mathbb{Z}}\{x_{1,w_1} \cdots x_{n,w_n} \mid w \in \mathfrak{S}_n\}$  of  $\mathbb{Z}[x]$ , where  $\mathfrak{S}_n$  is the symmetric group, especially the generating functions

$$\text{Imm}_{\theta}(x) := \sum_{w \in \mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n} \tag{1.1}$$

for class functions  $\theta : \mathfrak{S}_n \rightarrow \mathbb{Z}$ . Or, since some published results concern Hecke algebra traces, linear functions  $\theta_q : H_n(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  satisfying  $\theta_q(gh) = \theta_q(hg)$ , one might investigate these.

In particular, let  $\{\tilde{C}_w(q) \mid w \in \mathfrak{S}_n\}$  be the (modified, signless) *Kazhdan-Lusztig basis* of  $H_n(q)$ , defined by

$$\tilde{C}_w(q) := q^{\frac{\ell(w)}{2}} C'_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v,$$

where  $\{T_w \mid w \in \mathfrak{S}_n\}$  is the natural basis of  $H_n(q)$ ,  $\{P_{v,w}(q) \mid v, w \in \mathfrak{S}_n\}$  are the Kazhdan-Lusztig polynomials [8], and  $\leq$  denotes the Bruhat order. Specializing at  $q^{\frac{1}{2}} = 1$  we have  $T_v \mapsto v$  and  $H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$ . For  $1 \leq a < b \leq n$ , let  $s_{[a,b]} \in \mathfrak{S}_n$  be the *reversal* whose one-line notation is  $1 \cdots (a-1)b(b-1) \cdots (a+1)a(a+2) \cdots n$ . Stembridge [14] showed that for any linear function  $\theta : \mathbb{Z}[\mathfrak{S}_n] \rightarrow \mathbb{Z}$ , the immanant  $\text{Imm}_\theta(x)$  is TNN if for all sequences  $J_1, \dots, J_r$  of subintervals of  $[1, n]$ , we have

$$\theta(\tilde{C}_{s_{J_1}}(1) \cdots \tilde{C}_{s_{J_r}}(1)) \geq 0. \quad (1.2)$$

Furthermore, by Lindström's Lemma and its converse [2], a combinatorial interpretation of the above expression would immediately yield a combinatorial interpretation of the number  $\text{Imm}_\theta(A)$  for  $A$  a TNN matrix. Now if  $\theta_q : H_n(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  specializes at  $q^{\frac{1}{2}} = 1$  to  $\theta$ , then (1.2) is clearly a consequence of the condition

$$\theta_q(\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_r}}(q)) \in \mathbb{N}[q], \quad (1.3)$$

and a combinatorial interpretation of the coefficients of the resulting polynomial would yield combinatorial interpretations of the earlier expressions. Haiman [6, Appendix] observed that (1.3) in turn follows from the condition that for all  $w \in \mathfrak{S}_n$  we have

$$\theta_q(\tilde{C}_w(q)) \in \mathbb{N}[q], \quad (1.4)$$

since products of Kazhdan-Lusztig basis elements belong to  $\text{span}_{\mathbb{N}[q]} \{\tilde{C}_w(q) \mid w \in \mathfrak{S}_n\}$ .

Stembridge [13] and Haiman [6] proved that for  $\theta$  equal to any irreducible character  $\chi^\lambda$  of  $\mathfrak{S}_n$  ( $\chi_q^\lambda$  of  $H_n(q)$ ) the evaluations (1.2) and (1.4) belong to  $\mathbb{N}$  and  $\mathbb{N}[q]$ , respectively. They conjectured the same [14], [6] for functions  $\phi^\lambda$  ( $\phi_q^\lambda$ , respectively), called *monomial traces*, related to irreducible characters by the inverse Kostka numbers. None of these results or conjectures included a combinatorial interpretation. To better understand TNN polynomials of the form (1.1), it would be desirable to solve the following problem.

**Problem 1.** *Give combinatorial interpretations of all of the expressions in (1.2) or (1.3) when  $\theta_q$  varies over all elements of any basis of the  $H_n(q)$  trace space.*

So far, only some special cases have such interpretations. In the case that  $w$  avoids the patterns 3412 and 4231, the Kazhdan-Lusztig basis element  $\tilde{C}_w(q)$  is closely related to a product of the form appearing in (1.3). Combinatorial interpretations of the corresponding expressions (1.3) and (1.4) were given in [3] for  $\theta_q \in \{\chi_q^\lambda \mid \lambda \vdash n\}$ , and for  $\theta_q$  belonging to several other bases of the  $H_n(q)$  trace space, including the basis  $\{\epsilon_q^\lambda \mid \lambda \vdash n\}$  of induced sign characters. Also in this case, combinatorial interpretations for  $\theta_q = \phi_q^\lambda$  were given only when  $\lambda$  has at most two parts, or when  $\lambda$  has rectangular shape and  $q = 1$  [14, Thm. 2.8]. In the case that all permutations  $s_{J_1}, \dots, s_{J_r}$  in (1.3) are adjacent

transpositions  $(s_1, \dots, s_{n-1})$ , combinatorial interpretations of the corresponding expressions in (1.3) were given in [7] for  $\theta_q = \epsilon_q^\lambda$ .

We solve **Problem 1** for the trace space basis  $\{\epsilon_q^\lambda \mid \lambda \vdash n\}$  and state our solution in **Section 3**. In **Section 2** we introduce our computational tools: the quantum matrix bialgebra, combinatorial structures called star networks, and our general evaluation theorem which links the two.

## 2 The quantum matrix bialgebra and star networks

Define the *quantum matrix bialgebra* (See, e.g., [11])  $\mathcal{A} = \mathcal{A}(n, q)$  to be the associative algebra with unit 1 generated over  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  by  $n^2$  variables  $x = (x_{1,1}, \dots, x_{n,n})$ , subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \quad (2.1)$$

for all indices  $1 \leq i < j \leq n$  and  $1 \leq k < \ell \leq n$ . The counit map  $\varepsilon(x_{i,j}) = \delta_{i,j}$ , and coproduct map  $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$  give  $\mathcal{A}$  a bialgebra structure. While not a Hopf algebra,  $\mathcal{A}$  is closely related to the quantum group  $\mathcal{O}_q(SL_n(\mathbb{C})) \cong \mathbb{C} \otimes \mathcal{A} / (\det_q(x) - 1)$ , where

$$\det_q(x) \stackrel{\text{def}}{=} \sum_{v \in \mathfrak{S}_n} (-q^{\frac{1}{2}})^{\ell(v)} x_{1,v_1} \cdots x_{n,v_n} = \sum_{v \in \mathfrak{S}_n} (-q^{-\frac{1}{2}})^{\ell(v)} x_{v_1,1} \cdots x_{v_n,n} \quad (2.2)$$

is the  $(n \times n)$  *quantum determinant* of the matrix  $x = (x_{i,j})$ . (The second equality holds in  $\mathcal{A}$  but not in the noncommutative ring  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\langle x_{1,1}, \dots, x_{n,n} \rangle$ .) The antipode map of this Hopf algebra is  $\mathcal{S}(x_{i,j}) = (-q^{\frac{1}{2}})^{j-i} \det_q(x_{[n] \setminus \{j\}, [n] \setminus \{i\}})$ , where

$$[n] \stackrel{\text{def}}{=} \{1, \dots, n\}, \quad x_{L,M} \stackrel{\text{def}}{=} (x_{\ell,m})_{\ell \in L, m \in M}, \quad (2.3)$$

and  $\det_q(x_{L,M})$  is defined analogously to (2.2), assuming  $|L| = |M|$ . Specializing  $\mathcal{A}$  at  $q^{\frac{1}{2}} = 1$ , we obtain the commutative ring  $\mathbb{Z}[x_{1,1}, \dots, x_{n,n}]$ .

$\mathcal{A}$  has a natural  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -basis  $\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \mid a_{1,1}, \dots, a_{n,n} \in \mathbb{N}\}$  of monomials in which variables appear in lexicographic order, and the relations (2.1) provide an algorithm for expressing any other monomial in terms of this basis. The submodule  $\mathcal{A}_{[n],[n]}$  spanned by the monomials  $\{x^{u,v} \stackrel{\text{def}}{=} x_{u_1,v_1} \cdots x_{u_n,v_n} \mid u, v \in \mathfrak{S}_n\}$  has rank  $n!$  and natural basis  $\{x^{e,w} \mid w \in \mathfrak{S}_n\}$ .

To evaluate induced sign characters at elements  $\tilde{C}_{s_{j_1}}(q) \cdots \tilde{C}_{s_{j_m}}(q)$  of  $H_n(q)$ , we will associate to each such element a graph called a *star network*, a related matrix  $B$ , and a map

$\sigma_B : \mathcal{A}_{[n],[n]} \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ . A generating function  $\text{Imm}_{\epsilon_q^\lambda}(x) \in \mathcal{A}_{[n],[n]}$  for  $\{\epsilon_q^\lambda(T_w) \mid w \in \mathfrak{S}_n\}$  will then allow us to compute

$$\epsilon_q^\lambda(\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q)) = \sigma_B(\text{Imm}_{\epsilon_q^\lambda}(x)) \quad (2.4)$$

and to combinatorially interpret the resulting polynomial.

For  $1 \leq a < b \leq n$ , let  $G_{[a,b]}$  be the directed planar graph on  $2n + 1$  vertices defined as follows.

1. In a column on the left,  $n$  vertices are labeled *source*  $1, \dots, \text{source } n$ , from bottom to top; in a column on the right,  $n$  more vertices are labeled *sink*  $1, \dots, \text{sink } n$ , from bottom to top.
2. For  $i = 1, \dots, a - 1$  and  $i = b + 1, \dots, n$  a directed edge begins at source  $i$  and terminates at sink  $i$ .
3. An interior vertex is placed between the sources and sinks. For  $i = a, \dots, b$ , a directed edge begins at source  $i$  and terminates at the interior vertex, and another directed edge begins at the interior vertex and terminates at sink  $i$ .

For  $a = 1, \dots, n$  we define  $G_{[a,a]}$  to be the similar directed planar graph on  $n$  sources and  $n$  sinks, with one edge from source  $i$  to sink  $i$  for  $i = 1, \dots, n$ . Call each of the above graphs a *simple star network*. Define a *star network* to be the concatenation of finitely many simple star networks. We write  $G \circ H$  for the network in which sink  $i$  of  $G$  is identified with source  $i$  of  $H$ , for  $i = 1, \dots, n$ . In figures we will not explicitly draw vertices or show edge orientations (assumed to be from left to right). For  $n = 4$ , there are seven simple star networks:  $G_{[1,4]}, G_{[2,4]}, G_{[1,3]}, G_{[3,4]}, G_{[2,3]}, G_{[1,2]}, G_{[1,1]} = \cdots = G_{[4,4]}$ . Drawing these and two more star networks  $G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$  and  $G_{[2,4]} \circ G_{[1,3]}$ , we have

$$\begin{array}{cccccccc} \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array}, & \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \\ \text{---} \end{array}, & \begin{array}{c} \text{---} \\ \diagdown \\ \diagup \\ \text{---} \end{array}, & \begin{array}{c} \times \\ \text{---} \\ \text{---} \end{array}, & \begin{array}{c} \text{---} \\ \times \\ \text{---} \end{array}, & \begin{array}{c} \text{---} \\ \times \\ \text{---} \end{array}, & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}; & \begin{array}{c} \text{---} \\ \diagdown \\ \diagup \\ \text{---} \\ \diagdown \\ \diagup \end{array}, & \begin{array}{c} \text{---} \\ \diagdown \\ \diagup \\ \text{---} \\ \diagdown \\ \diagup \end{array}. \end{array} \quad (2.5)$$

Let  $\pi = (\pi_1, \dots, \pi_n)$  be a sequence of source-to-sink paths in a star network  $G$ . We call  $\pi$  a *path family* if there exists a permutation  $w = w_1 \cdots w_n \in \mathfrak{S}_n$  such that  $\pi_i$  is a path from source  $i$  to sink  $w_i$ . In this case, we say more specifically that  $\pi$  has *type*  $w$ . We say that the path family *covers*  $G$  if it contains every edge exactly once.

One can enhance a star network by associating to each edge a *weight* belonging to some ring  $R$ , and by defining the *weight of a path* to be the product of its edge weights. If  $R$  is noncommutative, then one multiplies weights in the order that the corresponding edges appear in the path. For a *family*  $\pi = (\pi_1, \dots, \pi_n)$  of  $n$  paths in a planar network, one defines  $\text{wgt}(\pi) = \text{wgt}(\pi_1) \cdots \text{wgt}(\pi_n)$ . The (*weighted*) *path matrix*  $B = B(G) = (b_{i,j})$  of  $G$  is defined by letting  $b_{i,j}$  be the sum of weights of all paths in  $G$  from source  $i$  to

sink  $j$ . Thus the product  $b_{1,w_1} \cdots b_{n,w_n}$  is equal to the sum of weights of all path families of type  $w$  in  $G$  (covering  $G$  or not).

Assigning weights to the edges of  $G = G_{J_1} \circ \cdots \circ G_{J_m}$  can aid in the evaluation of a linear function  $\theta_q : H_n(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  at  $\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q)$  by relating this evaluation to the generating function

$$\text{Imm}_{\theta_q}(x) := \sum_{w \in \mathfrak{S}_n} q^{\frac{-\ell(w)}{2}} \theta_q(T_w) x_{1,w_1} \cdots x_{n,w_n} \in \mathcal{A}(n, q), \tag{2.6}$$

which specializes at  $q^{\frac{1}{2}} = 1$  to the generating function (1.1) in  $\mathbb{Z}[x]$ . In particular, write  $G_{J_p} = G_{[i_p, j_p]}$  and let  $\{z_{h,p,k} \mid 1 \leq p \leq m; i_p \leq h \leq j_p; 1 \leq k \leq 2\}$  be indeterminate weights satisfying

$$z_{h_2, p_2, k_2} z_{h_1, p_1, k_1} = \begin{cases} z_{h_1, p_1, k_1} z_{h_2, p_2, k_2} & \text{if } p_1 \neq p_2, \text{ or } k_1 \neq k_2, \\ q^{\frac{1}{2}} z_{h_1, p_1, k_1} z_{h_2, p_2, k_2} & \text{if } p_1 = p_2, k_1 = k_2, \text{ and } h_1 < h_2. \end{cases} \tag{2.7}$$

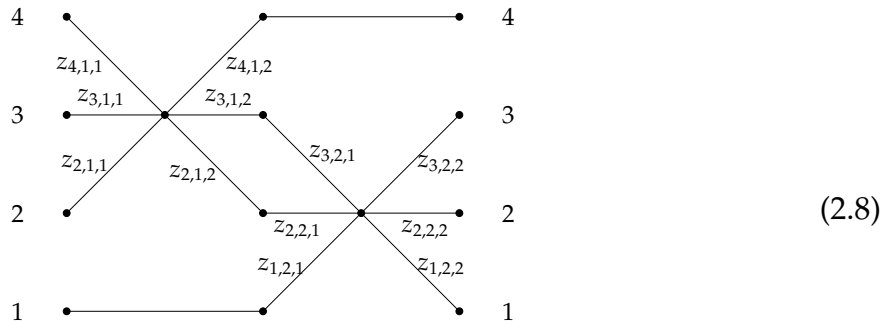
We assign weights to the edges of  $G_{J_p}$  as follows.

1. Assign weight 1 to the  $n - j_p + i_p - 1$  edges not incident upon the central vertex.
2. Assign weights  $z_{i_p, p, 1}, z_{i_p+1, p, 1}, \dots, z_{j_p, p, 1}$ , to the  $j_p - i_p + 1$  edges entering the central vertex, from bottom to top.
3. Assign weights  $z_{i_p, p, 2}, z_{i_p+1, p, 2}, \dots, z_{j_p, p, 2}$ , to the  $j_p - i_p + 1$  edges leaving the central vertex, from bottom to top.

Let  $Z_G$  be the quotient of the noncommutative ring

$$\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \langle z_{h,p,k} \mid p = 1, \dots, m; h_p = i_p, \dots, j_p; k = 1, 2 \rangle$$

modulo the ideal generated by the relations (2.7), and assume that  $q^{\frac{1}{2}}, q^{-\frac{1}{2}}$  commute with all other indeterminates. Let  $z_G$  be the product of all indeterminates  $z_{h,p,k}$ , in lexicographic order, and for  $f \in Z_G$ , let  $[z_G]f$  denote the coefficient of  $z_G$  in  $f$ . For example, the star network  $G_{[2,4]} \circ G_{[1,3]}$  has weighting



and monomial  $z_G = z_{1,2,1}z_{1,2,2}z_{2,1,1}z_{2,1,2}z_{2,2,1}z_{2,2,2}z_{3,1,1}z_{3,1,2}z_{3,2,1}z_{3,2,2}z_{4,1,1}z_{4,1,2}$ .

To complete the description of (2.4), we define a map which allows us to evaluate a linear functional  $\theta_q$  on certain  $H_n(q)$  elements via the corresponding immanant  $\text{Imm}_{\theta_q}(x) \in \mathcal{A}(n, q)$ . Given matrix  $B \in \text{Mat}_{n \times n}(Z_G)$ , let  $\sigma_B$  be the  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -linear map

$$\begin{aligned} \sigma_B : \mathcal{A}_{[n],[n]} &\rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \\ x_{1,v_1} \cdots x_{n,v_n} &\mapsto [z_G]b_{1,v_1} \cdots b_{n,v_n}, \end{aligned} \quad (2.9)$$

where  $[z_G]b_{1,v_1} \cdots b_{n,v_n}$  denotes the coefficient of  $z_G$  in  $b_{1,v_1} \cdots b_{n,v_n}$ , taken after  $b_{1,v_1} \cdots b_{n,v_n}$  is expanded in the lexicographic basis of  $Z_G$ . Note that the ‘‘substitution’’  $x_{i,j} \mapsto b_{i,j}$  is performed only for monomials of the form  $x^{e,v}$  in  $\mathcal{A}_{[n],[n]}$ : we define  $\sigma_B(x^{u,w})$  by first expanding  $x^{u,w}$  in the basis  $\{x^{e,v} \mid v \in \mathfrak{S}_n\}$ , and then performing the substitution. Now we have the following immanant evaluation identity for star networks (cf. [7, Thm. 3.7]).

**Theorem 1.** *Assign weights to the edges of  $G = G_{J_1} \circ \cdots \circ G_{J_m}$  as above and let  $B$  be the resulting path matrix. Then for any linear function  $\theta_q : H_n(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  we have*

$$\theta_q(\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q)) = [z_G]\sigma_B(\text{Imm}_{\theta_q}(x)). \quad (2.10)$$

*Proof.* Omitted. □

To illustrate, we let  $n = 4$  and consider the element

$$\tilde{C}_{s_{[2,4]}}(q)\tilde{C}_{s_{[1,3]}}(q) = (1+q) \sum_{w \leq 3421} T_w \quad (2.11)$$

of  $H_4(q)$ . Its star network (2.8) has weighted path matrix

$$B = \begin{bmatrix} z_{1,1,1}z_{1,2,2} & z_{1,2,1}z_{2,2,2} & z_{1,2,1}z_{3,2,2} & 0 \\ z_{2,1,1}(z_D + z_U)z_{1,2,2} & z_{2,1,1}(z_D + z_U)z_{2,2,2} & z_{2,1,1}(z_D + z_U)z_{3,2,2} & z_{2,1,1}z_{4,1,2} \\ z_{3,1,1}(z_D + z_U)z_{1,2,2} & z_{3,1,1}(z_D + z_U)z_{2,2,2} & z_{3,1,1}(z_D + z_U)z_{3,2,2} & z_{3,1,1}z_{4,1,2} \\ z_{4,1,1}(z_D + z_U)z_{1,2,2} & z_{4,1,1}(z_D + z_U)z_{2,2,2} & z_{4,1,1}(z_D + z_U)z_{3,2,2} & z_{4,1,1}z_{4,1,2} \end{bmatrix}, \quad (2.12)$$

where  $z_D = z_{2,1,2}z_{2,2,1}$ ,  $z_U = z_{3,1,2}z_{3,2,1}$ . Now consider the linear function  $\theta_q : H_4(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  defined by  $\theta_q(T_{3412}) = 1$ ,  $\theta_q(T_{4312}) = -1$ , and  $\theta_q(T_w) = 0$  otherwise. Computing the left-hand side of (2.10) we have

$$\theta_q(\tilde{C}_{s_{[2,4]}}(q)\tilde{C}_{s_{[1,3]}}(q)) = (1+q)(1) + (0)(-1) = 1+q, \quad (2.13)$$

since  $T_{3412}$  appears in (2.11) with coefficient  $1+q$  and  $T_{4312}$  appears with coefficient 0. To compute the right-hand side of (2.10), we begin by writing

$$\text{Imm}_{\theta_q}(x) = q^{-2}x_{1,3}x_{2,4}x_{3,1}x_{4,2} - q^{-\frac{5}{2}}x_{1,4}x_{2,3}x_{3,1}x_{4,2}.$$

Substituting  $b_{i,j}$  for  $x_{i,j}$ , we have  $q^{\frac{5}{2}}b_{1,4}b_{2,3}b_{3,1}b_{4,2} = 0$  and

$$q^{-2}b_{1,3}b_{2,4}b_{3,1}b_{4,2} = q^{-2}z_{1,2,1}z_{3,2,2}z_{2,1,1}z_{4,1,2}z_{3,1,1}(z_D + z_U)z_{1,2,2}z_{4,1,1}(z_D + z_U)z_{2,2,2}. \quad (2.14)$$

Now since  $z_D^2$  and  $z_U^2$  are not square-free, we ignore terms in the expansion containing these. Since we have

$$\begin{aligned} z_{3,2,2}z_{4,1,2}z_D &= q^{\frac{1}{2}}z_Dz_{3,2,2}z_{4,1,2}, & z_{3,2,2}z_{1,2,2} &= q^{\frac{1}{2}}z_{1,2,2}z_{3,2,2}, & z_Uz_D &= qz_Dz_U, \\ z_{3,2,2}z_{4,1,2}z_U &= q^{\frac{1}{2}}z_Uz_{3,2,2}z_{4,1,2}, & z_{3,2,2}z_{2,2,2} &= q^{\frac{1}{2}}z_{2,2,2}z_{3,2,2}, \end{aligned}$$

we express the nonzero square-free monomials of (2.14) in lexicographic order to obtain

$$q^{-2}(q^2 + q^3)z_{1,2,1}z_{1,2,2}z_{2,1,1}z_Dz_{2,2,2}z_{3,1,1}z_Uz_{3,2,2}z_{4,1,1}z_{4,1,2} = (1 + q)z_G,$$

which matches (2.13).

An important property of the map  $\sigma_B$  is that its evaluation at natural basis elements of  $\mathcal{A}_{[n],[n]}$  is closely related to coefficients in the natural expansion of  $\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q)$ .

**Proposition 1.** *Let  $B$  be the weighted path matrix of star network  $G = G_{J_1} \circ \cdots \circ G_{J_m}$ , and fix  $w \in \mathfrak{S}_n$ . Then  $\sigma_B(x^{e,w})$  is equal to  $q^{\frac{\ell(w)}{2}}$  times the coefficient of  $T_w$  in the product  $\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q)$ .*

*Proof.* Omitted. □

### 3 $G$ -tableaux and evaluation of induced sign characters

**Theorem 1** provides half of the solution to the problem of evaluating the left-hand side of (2.4). The other half is a combinatorial interpretation of the right-hand-side of (2.10), which is a linear combination of expressions of the form  $\sigma_B(x^{u,w}) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ . To combinatorially interpret such evaluations, we arrange the paths of a path family  $\pi$  covering a star network  $G$  into a (French) Young diagram, using each path exactly once. We call the resulting structure a  $G$ -tableau, or more specifically a  $\pi$ -tableau. If  $\text{type}(\pi) = w$ , we say that the tableau has type  $w$ . For example, the following path family  $\pi$  covering the star network of  $\tilde{C}_{s_{[1,2]}}(q)\tilde{C}_{s_{[2,3]}}(q)\tilde{C}_{s_{[1,2]}}(q)$  yields six  $\pi$ -tableaux of shape 21 and type 213:

$$\begin{array}{c} \pi_3 \cdots \cdots \\ \pi_2 \cdots \cdots \\ \pi_1 \cdots \cdots \end{array} \begin{array}{c} \diagdown \\ \diagup \\ \diagdown \\ \diagup \end{array}, \quad \begin{array}{|c|c|} \hline \pi_3 \\ \hline \pi_1 | \pi_2 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \pi_2 \\ \hline \pi_1 | \pi_3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \pi_3 \\ \hline \pi_2 | \pi_1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \pi_1 \\ \hline \pi_2 | \pi_3 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \pi_2 \\ \hline \pi_3 | \pi_1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \pi_1 \\ \hline \pi_3 | \pi_2 \\ \hline \end{array}. \quad (3.1)$$

Given a  $\pi$ -tableau  $U$ , we define (integer) Young tableaux  $L(U)$ ,  $R(U)$  by replacing each path by its source index and sink index, respectively. For example, if  $U$  is the first  $\pi$ -tableau in (3.1), then we have

$$L(U) = \begin{array}{|c|c|} \hline 3 \\ \hline 1 | 2 \\ \hline \end{array}, \quad R(U) = \begin{array}{|c|c|} \hline 3 \\ \hline 2 | 1 \\ \hline \end{array}.$$

It is easy to see that given two Young tableaux  $P, Q$  of the same shape, there is at most one  $\pi$ -tableau  $U$  satisfying  $L(U) = P, R(U) = Q$ .

We also define several statistics on  $G$ -tableaux. Suppose that two paths  $\pi_a, \pi_b$  in a star network  $G = G_{J_1} \circ \cdots \circ G_{J_m}$  pass through the central vertex of some simple star network  $G_{J_p}$ . We call the triple  $(p, \pi_a, \pi_b)$  a *crossing* of  $\pi$  if the two paths cross there, and a *noncrossing* otherwise. Let  $U$  be any  $\pi$ -tableau. Define  $c(U) = c(\pi)$  to be the number of crossings of  $\pi$ . Define  $\text{INVNC}(U)$ , the number of *inverted noncrossings* of  $U$ , to be the number of noncrossings  $(p, \pi_a, \pi_b)$  of  $\pi$  such that  $\pi_a, \pi_b$  intersect at the central vertex of  $G_{J_p}$  with  $\pi_b$  above  $\pi_a$ ,



and  $\pi_b$  appearing in an earlier column of  $U$  than  $\pi_a$  (whether or not  $b > a$ ). For example, each tableau  $U$  in (3.1) satisfies  $c(U) = 1$  because  $c(\pi) = 1$ . The inverted noncrossings in these tableaux are triples  $(1, \pi_1, \pi_2)$  with  $\pi_2$  in an earlier column than  $\pi_1$ , or  $(2, \pi_2, \pi_3)$  with  $\pi_3$  in an earlier column than  $\pi_2$ . The numbers of inverted noncrossings for the six tableaux are 1, 0, 0, 0, 1, 1, respectively.

Combining the above tableau statistics, we may combinatorially interpret  $\sigma_B(x^{u,w})$  in terms of tableaux of shape  $(n)$  (i.e., consisting of a single row). A fixed path family  $\pi$  of type  $v$  and a permutation  $u \in \mathfrak{S}_n$  determine a path tableau  $U(\pi, u, uv) = \pi_{u_1} \cdots \pi_{u_n}$  which satisfies  $L(U(\pi, u, uv)) = u_1 \cdots u_n$  and  $R(U(\pi, u, uv)) = (uv)_1 \cdots (uv)_n$ . The inclusion of  $uv$  in our notation  $U(\pi, u, uv)$  is superfluous but makes clear the ordering of sinks as they appear in the tableau.

**Proposition 2.** *Let star network  $G$  have weighted path matrix  $B$ . For  $u, w \in \mathfrak{S}_n$  we have*

$$\sigma_B(x^{u,w}) = \sum_{\pi} q^{\frac{c(\pi)}{2}} q^{\text{INVNC}(U)}, \quad (3.3)$$

where the sum is over path families  $\pi$  of type  $u^{-1}w$  covering  $G$ , and  $U = U(\pi, u, w)$  is the unique  $\pi$ -tableau of shape  $(n)$  satisfying  $L(U) = u_1 \cdots u_n, R(U) = w_1 \cdots w_n$ .

*Proof.* Omitted. □

The special case  $u = e$  of Proposition 2 yields a proof of a generalization of Deodhar's defect formula [4, Prop. 3.5] for coefficients of the expression  $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$ . Let  $\pi = (\pi_1, \dots, \pi_n)$  be a path family covering a star network  $G = G_{J_1} \circ \cdots \circ G_{J_m}$ . If two paths  $\pi_i, \pi_j$  intersect at the central vertex of  $G_{J_p}$ , call the intersection *defective* if the paths have previously crossed an odd number of times (i.e., in  $G_{J_1}, \dots, G_{J_{p-1}}$ ). Define  $\mathfrak{D}(\pi)$ , the number of *defects* of  $\pi$ , to be the number of triples  $(p, \pi_i, \pi_j)$  such that  $\pi_i$  and  $\pi_j$  intersect defectively at the central vertex of  $G_{J_p}$ .



**Corollary 1.** *The coefficients in the expansion  $\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q) = \sum_w a_w T_w$  are*

$$a_w = \sum_{\pi} q^{D(\pi)},$$

where the sum is over all path families of type  $w$  which cover the star network  $G_{J_1} \circ \cdots \circ G_{J_m}$ .

*Proof.* Omitted. □

By **Theorem 1**, the map  $\sigma_B$  (2.9) can be used to evaluate  $\epsilon_q^\lambda(\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q))$  when one has a simple expression for the generating function  $\text{Imm}_{\epsilon_q^\lambda}(x)$ . Such an expression was given by Konvalinka and the second author in [9, Thm. 5.4]:

$$\text{Imm}_{\epsilon_q^\lambda}(x) = \sum_I \det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r}), \quad (3.4)$$

where  $\det_q$  and  $x_{L, M}$  are defined as in **Section 2**, and the sum is over all ordered set partitions  $I = (I_1, \dots, I_r)$  of  $[n]$  satisfying  $|I_j| = \lambda_j$ . We will say that such an ordered set partition has *type*  $\lambda$ .

To evaluate  $\sigma_B(\text{Imm}_{\epsilon_q^\lambda}(x))$ , we expand each term on the right-hand side of (3.4) in a monomial basis  $\{x^{u, v} \mid v \in \mathfrak{S}_n\}$  of  $\mathcal{A}_{[n], [n]}$ , where  $u = u(I)$  is the concatenation of the  $r$  strictly increasing subwords

$$u_1 \cdots u_{\lambda_1}, \quad u_{\lambda_1+1} \cdots u_{\lambda_1+\lambda_2}, \quad u_{\lambda_1+\lambda_2+1} \cdots u_{\lambda_1+\lambda_2+\lambda_3}, \quad \dots, \quad u_{n-\lambda_r+1} \cdots u_n \quad (3.5)$$

formed by listing the elements of each block  $I_1, \dots, I_r$  in increasing order. As  $I$  varies over all ordered set partitions of  $[n]$  of type  $\lambda$ , the permutations  $u(I)$  vary over the Bruhat-minimal representatives  $\mathfrak{S}_\lambda^-$  of cosets  $\mathfrak{S}_\lambda u$ , where  $\mathfrak{S}_\lambda$  is the *Young subgroup* of  $\mathfrak{S}_n$  generated by

$$\{s_1, \dots, s_{n-1}\} \setminus \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, s_{\lambda_1+\lambda_2+\lambda_3}, \dots, s_{n-\lambda_r}\}.$$

Expanding each term on the right-hand side of (3.4) and applying  $\sigma_B$  we have

$$\sigma_B(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) = \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{-\frac{\ell(y)}{2}} \sigma_B(x^{u(I), yu(I)}). \quad (3.6)$$

To combinatorially interpret the sum in (3.6) we may apply **Proposition 2** and compute certain statistics for tableaux belonging to the set

$$\mathcal{U}_I = \mathcal{U}_I(G) \stackrel{\text{def}}{=} \{U(\pi, u, yu) \mid \pi \text{ covers } G, u = u(I), y \in \mathfrak{S}_\lambda\}.$$

Note that our restriction on  $y$  forces the sink indices of paths in components

$$(\lambda_1 + \cdots + \lambda_{k-1} + 1), \dots, (\lambda_1 + \cdots + \lambda_k) \quad (3.7)$$

of  $U(\pi, u, yu)$  to be a permutation of the source indices of the same paths.

On the other hand, the sum in (3.6) has both positive and negative signs. We obtain a subtraction-free expression for the sum by applying a sign-reversing involution to the tableaux in each set  $\mathcal{U}_I$ . Tableaux which remain after cancellation are in bijection with  $G$ -tableaux in which paths in a single column have increasing indices and do not intersect. We call such tableaux *column-strict*.

**Theorem 2.** *Let  $G = G_{J_1} \circ \cdots \circ G_{J_m}$ . Then for  $\lambda \vdash n$  we have*

$$\epsilon_q^\lambda(\tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q)) = \sum_W q^{\text{INVNC}(W) + c(W)/2}, \quad (3.8)$$

where the sum is over all column-strict  $G$ -tableaux of type  $e$  and shape  $\lambda^\top$ .

*Proof.* (Idea) Let  $B$  be the path matrix of  $G$ . Combining the Theorems 1 and [9, Thm. 5.4] (i.e., (3.4)) with the identity (3.6), we see that the left-hand side of (3.8) is

$$\begin{aligned} \sigma_B(\text{Imm}_{\epsilon_q^\lambda}(x)) &= \sum_I \sigma_B(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) \\ &= \sum_I \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{-\frac{\ell(y)}{2}} \sigma_B(x^{u(I), yu(I)}), \end{aligned} \quad (3.9)$$

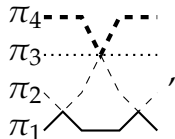
where the first two sums are over ordered set partitions  $I = (I_1, \dots, I_r)$  of  $[n]$  of type  $\lambda$ . Fixing one such partition  $I$  and writing  $u = u(I)$ , we may use Proposition 2 and other lemmas to express the sum over elements of  $\mathfrak{S}_\lambda$  as

$$\sum_{y \in \mathfrak{S}_\lambda} \sum_{\pi} (-1)^{\ell(y)} q^{-\frac{\ell(y)}{2}} q^{\frac{c(\pi)}{2}} q^{\text{INVNC}(U(\pi, u, yu))} = \sum_{y \in \mathfrak{S}_\lambda} \sum_{\pi} (-1)^{\ell(y)} q^{-\frac{\ell(y)}{2}} q^{\frac{c(\pi)}{2}} q^{\text{INVNC}(W) + \text{CDNC}(W)}, \quad (3.10)$$

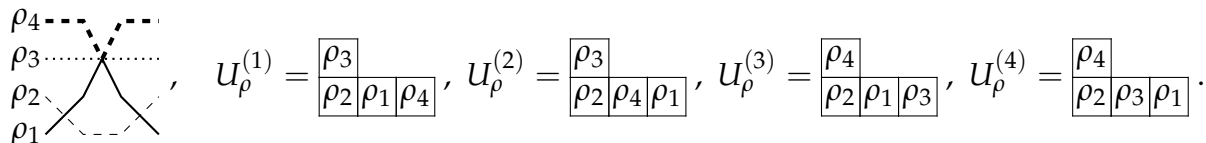
where the inner sums are over path families  $\pi$  of type  $u^{-1}yu$  which cover  $G$ , and where  $W = W(\pi, u, yu)$  is a related tableau of shape  $\lambda^\top$ , and  $\text{CDNC}$  is a statistic related to defects and noncrossings.

A sign reversing involution eliminates those tableaux  $W$  which are not column-strict, and another lemma allows us to interpret the given powers of  $q$  in terms of crossings and inverted noncrossings in the remaining column-strict tableaux. Thus the three expressions in (3.9) are equal to the right-hand side of (3.8).  $\square$

To illustrate the theorem, we compute  $\epsilon_q^{211}(\tilde{C}_{s_{[1,2]}}(q)\tilde{C}_{s_{[2,4]}}(q)\tilde{C}_{s_{[1,2]}}(q))$  using the star network  $G = G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$  pictured in (2.5). There are two path families of type  $e$  which cover  $G$ , and four column-strict  $G$ -tableau of shape  $211^\top = 31$  for each:



$$U_\pi^{(1)} = \begin{array}{|c|c|c|} \hline \pi_3 & & \\ \hline \pi_1 & \pi_2 & \pi_4 \\ \hline \end{array}, \quad U_\pi^{(2)} = \begin{array}{|c|c|c|} \hline \pi_3 & & \\ \hline \pi_1 & \pi_4 & \pi_2 \\ \hline \end{array}, \quad U_\pi^{(3)} = \begin{array}{|c|c|c|} \hline \pi_4 & & \\ \hline \pi_1 & \pi_2 & \pi_3 \\ \hline \end{array}, \quad U_\pi^{(4)} = \begin{array}{|c|c|c|} \hline \pi_4 & & \\ \hline \pi_1 & \pi_3 & \pi_2 \\ \hline \end{array};$$



$$U_\rho^{(1)} = \begin{array}{|c|c|c|c|} \hline \rho_3 & & & \\ \hline \rho_2 & \rho_1 & \rho_4 & \\ \hline \end{array}, \quad U_\rho^{(2)} = \begin{array}{|c|c|c|c|} \hline \rho_3 & & & \\ \hline \rho_2 & \rho_4 & \rho_1 & \\ \hline \end{array}, \quad U_\rho^{(3)} = \begin{array}{|c|c|c|c|} \hline \rho_4 & & & \\ \hline \rho_2 & \rho_1 & \rho_3 & \\ \hline \end{array}, \quad U_\rho^{(4)} = \begin{array}{|c|c|c|c|} \hline \rho_4 & & & \\ \hline \rho_2 & \rho_3 & \rho_1 & \\ \hline \end{array}.$$

The path family  $\pi$  has no crossings, so tableau  $U_\pi^{(i)}$  contributes  $q^{\text{INVNC}(U_\pi^{(i)})} q^{c(U_\pi^{(i)})/2} = q^{\text{INVNC}(U_\pi^{(i)})}$  for all  $i$ . We have one noncrossing for each of the pairs  $(\pi_2, \pi_3)$ ,  $(\pi_2, \pi_4)$  and  $(\pi_3, \pi_4)$  and two for the pair  $(\pi_1, \pi_2)$ . Counting only the inverted noncrossings, such as  $\pi_2$  and  $\pi_3$  in  $U_\pi^{(1)}$ , we find the contributions from  $U_\pi^{(1)}, \dots, U_\pi^{(4)}$  are  $q, q^2, q^2, q^3$ , respectively. The tableaux for the path family  $\rho$  each have two crossings, and one noncrossing for each of the pairs  $(\rho_1, \rho_3)$ ,  $(\rho_1, \rho_4)$  and  $(\rho_3, \rho_4)$ . Adding the contributions together we find the contributions for  $U_\rho^{(1)}, \dots, U_\rho^{(4)}$  are  $q^1 q^{2/2} = q^2, q^2 q^{2/2} = q^3, q^2 q^{2/2} = q^3$  and  $q^3 q^{2/2} = q^4$  respectively. Hence we have  $\epsilon_q^{211}(\tilde{C}_{s_{[1,2]}}(q)\tilde{C}_{s_{[2,4]}}(q)\tilde{C}_{s_{[1,2]}}(q)) = q + 3q^2 + 3q^3 + q^4$ .

**Theorem 2** allows one to combinatorially interpret evaluations of  $\epsilon_q^\lambda$  at (multiples of) certain elements  $\tilde{C}_w(q)$  of the Kazhdan-Lusztig basis of  $H_n(q)$ . In particular, for some elements  $\tilde{C}_w(q)$  there exists a polynomial  $g_w(q)$  such that we have

$$g_w(q)\tilde{C}_w(q) = \tilde{C}_{s_{J_1}}(q) \cdots \tilde{C}_{s_{J_m}}(q) \tag{3.11}$$

for some sequence  $s_{J_1}, \dots, s_{J_m}$  of reversals. Such permutations include all 3412-avoiding, 4231-avoiding permutations, all of  $\mathfrak{S}_4$  (even 4231 and 3412), all of  $\mathfrak{S}_5$  except 45312, and all 321-hexagon-avoiding permutations. (See [1].)

**Corollary 2.** *Suppose that  $\tilde{C}_w(q)$  satisfies a factorization of the form (3.11) and define  $G = G_{J_1} \circ \cdots \circ G_{J_m}$ . Then we have*

$$\epsilon_q^\lambda(\tilde{C}_w(q)) = \frac{1}{g_w(q)} \sum_U q^{\text{INVNC}(U) + c(U)/2}, \tag{3.12}$$

where the sum is over all column-strict  $G$ -tableaux of type  $e$  and shape  $\lambda^\top$ .

It would be interesting to characterize the factorizations (3.11) [12, Quest. 4.5].

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