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A Sundaram type bijection for $\mathrm{SO}(2k+1)$: vacillating tableaux and pairs consisting of a standard Young tableau and an orthogonal Littlewood-Richardson tableau

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Abstract. We present a bijection between vacillating tableaux and pairs consisting of a standard Young tableau and an orthogonal Littlewood-Richardson tableau for the special orthogonal group $\mathrm{SO}(2k+1)$. This bijection is motivated by the direct-sum decomposition of the rth tensor power of the defining representation of $\mathrm{SO}(2k+1)$. To formulate it, we present an explicit formulation of Kwon's Littlewood-Richardson tableaux and find alternative tableaux with which they are in bijection.

Moreover we use a suitably defined descent set for vacillating tableaux to determine the quasi-symmetric expansion of the Frobenius characters of the isotypic components.

Keywords: Special Orthogonal Groups, Vacillating Tableaux, Branching Rules

1 Introduction

In this extended abstract of [2], we present a bijection for $\mathrm{SO}(2k+1)$ between vacillating tableaux and pairs consisting of a standard Young tableau and an orthogonal Littlewood-Richardson tableau (introduced by Kwon in [4]).

This bijection explains the direct-sum-decomposition of a tensor power $V^{\otimes r}$ of the defining representation V of $\mathrm{SO}(2k+1)$ combinatorially. In particular we consider

$$V^{\otimes r} = \bigoplus_{\mu} V(\mu) \otimes U(r,\mu) = \bigoplus_{\mu} V(\mu) \otimes \bigoplus_{\lambda} c_{\lambda}^{\mu} S(\lambda),$$

as an $\mathrm{SO}(2k+1) \times \mathfrak{S}_r$ representation, where $V(\mu)$ is an irreducible representation of $\mathrm{SO}(2k+1)$ and $S(\lambda)$ is a Specht module. We concentrate on $U(r,\mu)$. A basis of $U(r,\mu)$ can be indexed by vacillating tableaux. The multiplicities c_{λ}^{μ} can be obtained by counting

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orthogonal Littlewood-Richardson tableaux. A basis of $S(\lambda)$ is indexed by standard Young tableaux.

To formulate our bijection, we introduce an explicit combinatorial description of Kwon's orthogonal Littlewood-Richardson tableaux. These are defined in a very general way, but too abstract for our purposes. Using our new description we find an alternative set of orthogonal Littlewood-Richardson tableaux, which is in bijection with Kwon's set. This alternative set reduces the problem to finding a bijection between vacillating tableaux and standard Young tableaux with $2k+1$ rows, all of them with lengths of the same parity. We solve this reduced problem with Algorithm 1.

The question of finding such a bijection was posed by Sundaram in her 1986 thesis [10] and has been attacked several times since Sundaram's thesis; in particular by Sundaram [11] and Proctor [6]. Okada [5] recently obtained the decomposition of $U(r,\mu)$ for multiplicity free cases implicitly using representation theoretic computations. We obtain parts of these results as a special case, which are themselves special cases of Okada's work. In fact, Okada asks for bijective proofs of his results.

One might assume that Fomin's machinery of growth diagrams could be employed to find such a bijection, as for the symplectic group this was done by Roby [7] and Krattenthaler [3]. However, for the orthogonal group the situation appears to be quite different. In particular, at least a naive application of Fomin's ideas does not even yield the desired bijection between vacillating tableaux and the set of standard Young tableaux in question, not even for dimension 3.

For SO(3) a bijection was provided in [1]. In dimension 3, vacillating tableaux are Riordan paths. This special combinatorial structure led to the stronger results for SO(3).

An advantage of our combinatorial, bijective approach is that we obtain additional properties and consequences such as the following.

We define a suitable notion of descents for vacillating tableaux and use the classical descent set for standard Young tableaux introduced by Schützenberger. We can show that our bijection is descent preserving. Thus we obtain the quasi-symmetric expansion of the Frobenius character of the isotypic space $U(r,\mu)$

$$\operatorname{ch} U(r,\mu) = \sum F_{\mathrm{Des}(w)},$$

where F_D denotes a fundamental quasi-symmetric function and the sum runs over all vacillating tableaux w of length r and shape μ.

Among others, this property justifies our bijection to be called "Sundaram-like", as she described a similar bijection for the defining representation of the symplectic group in her thesis [10]. There exists a similar (but less complicated) definition for descents in oscillating tableaux, which are used in the symplectic case instead of vacillating tableaux, and which Sundaram's bijection preserves. Thus there also exists a similar quasi-symmetric expansion of the Frobenius character, obtained by Rubey, Sagan and Westbury in [8].

2 Background

2.1 Schur-Weyl duality

We start with the general linear group and the classical Schur-Weyl duality,

$$V^{\otimes r} \cong \bigoplus_{\lambda \vdash r,\, \ell(\lambda) \leq n} V^{\mathrm{GL}}(\lambda) \otimes S(\lambda),$$

where V is a complex vector space of dimension n, $\mathrm{GL}(V)$ acts diagonally (and on each position by matrix multiplication) and $\mathfrak{S}_r$ permutes tensor positions. This is a decomposition as a $\mathrm{GL}(V) \times \mathfrak{S}_r$ representation. $V^{\mathrm{GL}}(\lambda)$ is an irreducible representation of $\mathrm{GL}(V)$ and $S(\lambda)$ is a Specht module. Now we consider for odd dimension $n = 2k+1$, the restriction $V(\lambda)^{\mathrm{GL}} \downarrow^{\mathrm{GL}(V)}_{\mathrm{SO}(V)} \cong \bigoplus c^{\mu}_{\lambda}(\mathfrak{d}) V^{SO}(\mu)$ from $\mathrm{GL}(V)$ to $\mathrm{SO}(V)$ to obtain a decomposition for $\mathrm{SO}(V)$.

$$V^{\otimes r} \cong \bigoplus_{\substack{\mu \text{ a partition} \\ \ell(\mu) \leq k}} V^{SO}(\mu) \otimes \bigoplus_{\substack{\lambda \vdash r \\ \ell(\lambda) \leq n}} \mathrm{c}^{\mu}_{\lambda}(\mathfrak{d}) S(\lambda) = \bigoplus_{\substack{\mu \text{ a partition} \\ \ell(\mu) \leq k}} V^{SO}(\mu) \otimes U(r, \mu),$$

where $\mathrm{c}^{\mu}_{\lambda}(\mathfrak{d})$ is the multiplicity of the irreducible representation $V^{\mathrm{SO}}(\mu)$ of $\mathrm{SO}(V)$ in $V^{\mathrm{GL}}(\lambda)$. For $\ell(\lambda) \leq k$ this simplifies to the classical branching rule due to Littlewood.

Kwon defined orthogonal Littlewood-Richardson tableaux as a set that is counted by $\mathrm{c}^{\mu}_{\lambda}(\mathfrak{d})$. A basis of $S(\lambda)$ can be indexed with standard Young tableaux. Combining Schur-Weyl duality and certain branching rules implies that a basis of $U(r, \mu)$ can be indexed by so called vacillating tableaux of shape μ. Therefore, we are interested in a bijection between vacillating tableaux and pairs that consist of a standard Young tableau and an orthogonal Littlewood-Richardson tableau.

Moreover we introduce descent sets for vacillating tableaux, which our bijection preserves, and follow the approach taken by Rubey, Sagan and Westbury [8] for the symplectic group. Therefore we get a quasi-symmetric expansion of the Frobenius character (see the text book by Stanley [9]), which can be defined by requiring that it be an isometry and

$$\mathrm{ch}\, S(\lambda) = s_{\lambda} = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(Q)},$$

where s_{λ} is a Schur function, $\mathrm{Des}(Q) \subseteq \{1, 2, \ldots, r\}$ denotes the descent set of Q and F_D denotes the *fundamental quasi-symmetric function* $\sum_{i_1 \leq i_2 \leq \cdots \leq i_r,\, j \in D \Rightarrow i_j < i_{j+1}} x_{i_1} x_{i_2} \ldots x_{i_r}$.

Theorem 1.

$$\mathrm{ch}\, U(r, \mu) = \sum F_{\mathrm{Des}(w)},$$

where the sum runs over all vacillating tableaux w of length r and shape μ and $\mathrm{Des}(w)$ *is the descent set of w.*

2.2 Standard Young Tableaux

We call $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of shape λ.

Definition 2. For a standard Young tableau $Q \in \mathrm{SYT}(\lambda)$ we call an entry j a *descent* if $j+1$ is in a row below j. We define the *descent set* of Q as: $\mathrm{Des}(Q) = \{j : j \text{ is a descent of } Q\}$.

Definition 3. The *concatenation* Q of two standard Young tableaux Q_1 and Q_2 is obtained as follows. First add the largest entry of Q_1 to each entry of Q_2 to obtain a tableau $\widetilde{Q_2}$. Then append row i of $\widetilde{Q_2}$ to row i of Q_1 to obtain Q.

In Figure 1 there are several standard Young tableaux. The third one has descent set: $\{5, 10, 12, 14, 15, 16, 18, 19, 20\}$. The fourth one is obtained by concatenating the one-column tableau filled with $1, 2, \ldots, 7$ with the third standard Young tableau in Figure 1.

2.3 Vacillating Tableaux

We define vacillating tableaux (as defined by Sundaram in [11, Def. 4.1]) in three different ways. By abuse of terminology we refer to all three objects as *vacillating tableaux*.

Definition 4.
1. A ($(2k+1)$-*orthogonal*) *vacillating tableau* of length r is a sequence of Young diagrams $\emptyset = \mu^0, \mu^1, \ldots, \mu^r = \mu$ each of at most k parts, such that:
   - μ^i and μ^{i+1} differ in at most one cell,
   - $\mu^i = \mu^{i+1}$ only occurs if the k^{th} row of cells is non-empty.

   We call the partition belonging to the final Young diagram μ the *shape*.

2. A ($(2k+1)$-*orthogonal*) *highest weight word* is a word w of length r with letters in $\{\pm 1, \pm 2, \ldots, \pm k, 0\}$ such that for every initial segment s of w the following holds (we write $\#i$ for the number of i's in s):
   - $\#i - \#(-i) \geq 0$,
   - $\#i - \#(-i) \geq \#(i+1) - \#(-i-1)$,
   - if the last letter is 0 then $\#k - \#(-k) > 0$.

   We call the partition $(\#1 - \#(-1), \#2 - \#(-2), \ldots, \#k - \#(-k))$ the *weight*. The vacillating tableau corresponding to w is the sequence of weights of the initial segments.

3. A k-tuple of Riordan paths of length r (Motzkin paths without horizontal steps on the x-axis) is a vacillating tableau of length r if it meets the following conditions:
   - The first path is a Riordan path of length r.

- Path i has steps where path $i-1$ has horizontal steps. Path i is never higher than path $i-1$.

The corresponding highest weight word is described as follows: A value i is an up step in path i and a horizontal step in paths 1 up to $i-1$. Similarly a value $-i$ is a down step in path i and a horizontal step in paths 1 up to $i-1$ and a value 0 is a horizontal step in every path, including path k.

Definition 5. The concatenation of vacillating tableaux of shape $\emptyset$ is obtained by writing them side by side.

In Figure 1 we see three vacillating tableaux written as tuples of paths. The first one is the concatenation of two vacillating tableaux.

Definition 6. We define descents for vacillating tableaux using highest weight words. We call a letter w_i of w a *descent* if there exists a directed path from w_i to w_{i+1} in the crystal graph for the defining representation of SO$(2k+1)$

$$1 \to 2 \to \cdots \to k \to 0 \to -k \to \cdots \to -1$$

and $w_i w_{i+1} \neq j(-j)$ if for the initial segment $w_1, w_2, \ldots, w_{i-1}$ holds $\#j - \#(-j) = 0$.

We define the *descent set* of w as $\mathrm{Des}(w) = \{j : j \text{ is a descent of } w\}$.

In our tuple of paths a descent is a convex edge of consecutive steps, but not an up step followed by a down step on the bottom. The last vacillating tableau in Figure 1 has descent set $\{5, 10, 12, 14, 15, 16\}$. Note that 6 is not a descent because the 6th position is a 2 immediately followed by a -2 on bottom level.

3 Orthogonal Littlewood-Richardson Tableaux

First we give an explicit description of Kwon's orthogonal Littlewood-Richardson tableaux, which are defined via crystal graphs and a system of inequalities in [4]. Then we use this explicit description to find a new set of tableaux that are in bijection with Kwon's tableaux. In order to do this, we introduce some notation.

Definition 7. Let T be a two column skew semistandard tableau of shape $(2^b, 1^m)/(1^a)$, with $b \geq a \geq 0$ and $m > 0$.

The *tail* of T is the part where only the first column exists, that is, the lower m entries of the first column. The topmost tail position is the *tail root* and the tail without the tail root is the *lower tail*. The *fin* of T is the largest entry in the second column.

The *residuum* of T is the number of positions the second column can be shifted down while maintaining semistandardness.

Definition 8. Let T be a semistandard tableau. We call a position $j > 1$ of T a *gap* (respectively *slot*) if $j-1$ (respectively $j+1$) is not in the same column as j.

Definition 9. For a partition μ with $\ell(\mu) \leq k$, the crystal graph $B^{\mathfrak{d}}(\mu)$ is the subgraph of the tensor product of $n = 2k+1$ one column crystal graphs, whose vertices are tuples $(T_1, T_2, \ldots, T_{\ell(\mu)}, S)$ of skew semistandard tableaux. Each T_j has shape $(2^{b_j}, 1^{\mu_j})/(1^{a_j})$, with $b_j \geq a_j \geq 0$, b_j, a_j even and residuum at most 1. S is of rectangular outer shape and has $n - 2\ell(\mu)$ (possibly empty) columns, all of whose lengths have the same parity. We say S is even if its columns have even length, and S is odd otherwise.

Theorem 10. *Let $\lambda \vdash r$, $\ell(\lambda) \leq n(= 2k+1)$, $\ell(\mu) \leq k$. Let $L = (T_1, T_2, \ldots, T_{\ell(\mu)}, S)$ be a vertex in $B^{\mathfrak{d}}(\mu)$. Then L is an orthogonal Littlewood-Richardson tableau in $\mathrm{LR}^{\mu}_{\lambda}(\mathfrak{d})$ for $\mathrm{SO}(n)$ if and only if for all i there are λ'_i i's in L and the following conditions are met:*

(H) $b_j \leq b_{j+1} - a_{j+1} + 2r_j r_{j+1}$ *for* $1 \leq j \leq \ell(\mu) - 1$.

(H′) $b_{\ell(\mu)} \leq ht(S^L)$ *if S is even and* $b_{\ell(\mu)} \leq ht(S^L) - 1 + 2r_{\ell(\mu)}$ *if S is odd, where $ht(S^L)$ denotes the length of the leftmost column of S.*

(C) *S contains no gap.*

(T1) *Tableaux $T_1, T_2, \ldots, T_{\ell(\mu)}$ are of one of the following three types: Type 1 tableaux have residuum 0. Gaps can be only in the tail. Type 2 tableaux have residuum 1. Gaps can be only in the lower tail. Type 3 tableaux have residuum 1. The fin is a gap. Other gaps can be only in the lower tail.*

If T_i is of type 3, $i < \ell(\mu)$, T_{i+1} has residuum 1 and the fin of T_i is not larger than the fin of T_{i+1}. If $T_{\ell(\mu)}$ is of type 3, S is odd. If $T_{\ell(\mu)}$ is of type 1, the tail root is smaller than or equal to the bottommost position in the leftmost column of S.

(T2) *The tails shifted together such that they share the top line form a semistandard tableau.*

(G) For each gap j there is a slot $j-1$ in a column to the right. This can be in the same tableau T_i or in another one that is right of T_i in L including S. More precisely, if there are m gaps j there are m slots $j-1$ such that we can build pairs of a gap and a slot such that each slot is to the right of its gap.

Definition 11. We define $\mathrm{aLR}^{\mu}_{\lambda}$, the set of alternative orthogonal Littlewood-Richardson tableaux as follows. A tableau $L \in \mathrm{aLR}^{\mu}_{\lambda}$ is a reverse skew semistandard tableau of inner shape λ and type μ (thus the filling consists of μ_j j's, for all j). The outer shape has $2k+1$ possibly empty rows, whose lengths have all the same parity. The reading word (rows concatenated from bottom to top) is a lattice permutation or Yamanouchi word.

Now we go through the reading word of L from right to left. Let p be the current position. We define a sequence v_p of positions of the reading word. Let p be the first entry of v_p. If $m-1$ entries of v_p are defined, let e be entry number $m-1$. We search now for entry number m. For that we consider entries whose letter is larger than the letter of e and which are in exactly $m-1$ sequences of positions left of p (thus sequences already defined). If this set is nonempty we search for the smallest letter in it and take the leftmost position with this letter as entry m. If it is empty v_p has no more entries.

We call r_p the row where p appears. Now we define the value o_p to be the number of entries in v_p with the following properties. It is the rightmost occurrence of its letter and, if it is number m in v_p, all $v_{\tilde{p}}$ with $\tilde{p} \neq p$ in the same row as p, have at most $m-1$ entries.

We require $r_p \geq 2|v_p| - o_p$.

Theorem 12. *Kwon's orthogonal Littlewood-Richardson tableaux are in bijection with our new alternative Littlewood-Richardson tableaux.*

In Figure 1 we have an orthogonal Littlewood-Richardson tableau of Kwon, and the corresponding alternative orthogonal Littlewood-Richardson tableau. The latter has reading word $(1_1, 1_2, 2_3, 3_4, 2_5, 1_6)$ where l is the letter and p counts the position in l_p. The sequences v are: (1_6), (2_5), (3_4); $(2_3, 3_4)$; $(1_2, 2_5, 3_4)$, $(1_1, 2_3)$.

4 The Bijection

Theorem 13. *Let $\lambda \vdash r$, $\ell(\lambda) \leq n (= 2k+1)$, $\ell(\mu) \leq k$. The map defined in this section maps a pair (Q, L) consisting of a standard Young tableau Q in* SYT(λ) *and an orthogonal Littlewood-Richardson tableau L in* $\mathrm{LR}^{\mu}_{\lambda}(\mathfrak{o})$ *to a vacillating tableau of length r and shape μ. Moreover it is bijective and descent-preserving.*

Definition 14 (The bijection for SO$(2k+1)$)**.** We start with a pair (Q, L) consisting of a standard Young tableau Q in SYT(λ) and an orthogonal Littlewood-Richardson tableau L in $\mathrm{LR}^{\mu}_{\lambda}(\mathfrak{o})$. First we map L to its corresponding alternative orthogonal Littlewood-Richardson tableau $\tilde{L} \in \mathrm{aLR}^{\mu}_{\lambda}$.

If e is the biggest entry in Q we add cells labeled $e + (\mu_{j+1} + \cdots + \mu_{\ell(\mu)}) + 1$, $e + (\mu_{j+1} + \cdots + \mu_{\ell(\mu)}) + 2, \ldots, e + (\mu_{j+1} + \cdots + \mu_{\ell(\mu)}) + \mu_j$ to the spots where cells labeled j are in $\tilde{L}$, such that the numbers in the horizontal strip belonging to j are increasing from left to right. We obtain a new standard Young tableau $\tilde{Q}$ with the same shape as $\tilde{L}$. Thus the row lengths of $\tilde{Q}$ have all the same parity.

Now we distinguish two cases: If our resulting tableau $\tilde{Q}$ consists of even length rows this is the tableau we will use in Algorithm 1. Otherwise, when $\tilde{Q}$ consists of n rows of odd length, we concatenate the one column tableau filled with $1, 2, \ldots, n$ from left to $\tilde{Q}$. We obtain an even-rowed standard Young tableau that we will use in Algorithm 1.

Next we apply Algorithm 1 to obtain a vacillating tableau $\tilde{V}$ with shape $\emptyset$ ending with $\mu_{\ell(\mu)}$ $(-\ell(\mu))$'s, $\ldots$, μ_2 (-2)'s and μ_1 (-1)'s.

Table 1: Notation for Algorithm 1

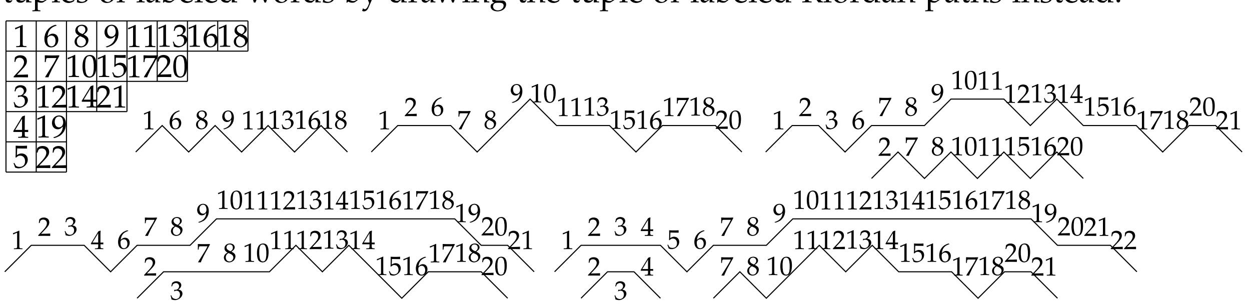
A *labeled word* w with letters in $\{\pm 1, \cdots \pm k, 0\}$.	A word, where each letter is labeled by an integer $1 \leq i \leq r$ strictly increasing from left to right. Each position consists of a label and an entry. We denote by $w(p)$ the entry of w labeled with p.
A position q is on l-level m.	The maximum of the following two sums over entries with absolute value l is $-l \cdot m$. For the first sum we consider entries strictly to the *right* of q. For the second one we consider entries to the *right* including q.
A position q is a height violation in l.	The l-level of q is smaller than the $(l+1)$-level of q. If $w(q) = \pm(l+1)$ we take the $(l+1)$-level plus one instead.
Insert q with l.	We insert a new position with entry l and label q such that the labels are still sorted.
Ignore q.	Act as if this position was not here, for example in level calculations.
A position p is a 3-row-position in j.	p is either the rightmost 0 of an odd sequence of 0's on j-level one or a 0 that is on j-level two or higher.
A position p is a 2-row-position in j.	p is either a j on j-level one or the leftmost 0 of a sequence of 0's.
A position p is in an j-even position	The number of positions q strictly to the left with $w(q) \in \{0, \pm j\}$ is even.

Once again we distinguish the two cases from before. If we did not concatenate with a column, we do not change $\tilde{V}$. If we concatenated a column to $\tilde{Q}$, we now delete the first n entries of $\tilde{V}$. In this case those are always $1, 2, \ldots, k, 0, -k, \ldots, -2, -1$.

Finally we delete the last $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_k$ entries to obtain a vacillating tableau V of shape μ and length $r = |\lambda|$.

In Figure 1 we illustrate our bijection in an odd case. In Table 1 we explain notation used in Algorithm 1.

Example 15. We run our algorithm on the following standard Young tableau. We draw tuples of labeled words by drawing the tuple of labeled Riordan paths instead.



Algorithm 1: Standard Young Tableaux to Vacillating Tableaux

input : $n = 2k+1$, standard Young tableau Q, at most n rows, all rows of even length
output: vacillating tableau V, dimension k, weight $\emptyset$, same number of entries as Q

```
let w be the word (1,−1,...,1,−1) with the same length as the first row of Q,
  labeled by the first row elements of Q;                          /* insert row 1 */
for i = 2,3,...,n do                                               /* insert row i */
  j := ⌊i/2⌋; unmark everything;
  if i even then change 0-entries of w into j,−j,...,j,−j ;        /* initialize j */
  for pairs of elements a,b in row i, start with the rightmost, go to left do
    a_1 := a, b_1 := b, a_l := b_l := 0 for l = 2,3,...,j+1;
    if b is biggest position so far then insert b_1 with −1;       /* b */
    let p be the rightmost position so far;
    let p̃ be the next position left of p with w(p̃) ∈ {0,±j};
    while a_{j+1} < p or w(p) ∉ {0,±j} do
      if p < b_l, p ≠ a_l, w(p) = −l for an l < j, a_{l+1} = 0 then
        if p not marked, b_{l+1} = 0 then w(p) := −l−1, b_{l+1} := p;  /* b_{l+1} */
        else if p < a_l, p < b_{l+1} then w(p) := −l−1, a_{l+1} := p;  /* a_{l+1} */
      if i is even, w(p) ∈ {0,±j} then                             /* i even */
        if b_j < p, w(p̃),w(p) = j,−j then               /* adjust separation point */
          for l < j change ±l on l-level 0 between p and p̃ into ±(l+1), if
            p < b_l, b_{l+1} = 0 ignore b_l, if p < a_l, a_{l+1} = 0 ignore a_l;
          mark changed positions;
          change −j,j between p and p̃ into 0,0;
        else if a_j < p, w(p̃),w(p) = j,−j then                    /* mark it + connect */
          w(p̃),w(p) := 0,0; for l < j mark ±l on l-level 0 between p and
            p̃, if p < a_l, a_{l+1} = 0 ignore a_l;
        else if p = a_j, w(p̃) = 0 on j-level 1 then               /* a_{j+1} 1 */
          w(p̃),w(p) := j,0, a_{j+1} := p̃;
        else if p < a_j, w(p) = −j, a_{j+1} = 0 then               /* a_{j+1} 2 */
          w(p) := j, a_{j+1} := p;
        if p < b_j, b_{j+1} = 0 then b_{j+1} := p;                 /* b_{j+1} */
```

if *i is odd,* $w(p) \in \{0, \pm j\}$ **then** `/*` i `odd */`
  if $b_{j+1} < p$, $w(p), w(\tilde{p}) = 0, 0$, *$p$ j-even position on j-level* 1 *if* $b_j < p$ *or* 2 *if* $p < b_j$ **then** `/* adjust separation point */`
    for $l < j$ change $\pm l$ on l-level 0 between p and $\tilde{p}$ into $\pm(l+1)$, if $p < b_l$, $b_{l+1} = 0$ ignore b_l, if $p < a_l$, $a_{l+1} = 0$ ignore a_l;
    mark changed positions;
  else if $a_{j+1} < p < b_{j+1}$, $w(p) = j$ *on j-level* 1 *for* $p < a_j$ *or* 0 *for* $a_j < p$ **then** $w(\tilde{p}), w(p) := 0, 0$; `/* connect */`
  else if $a_{j+1} < p < b_{j+1}$, $w(\tilde{p}), w(p) = 0, 0$, *$p$ j-even position on j-level* 2 *if* $p < a_j$ *or* 1 *if* $a_j < p$ **then** `/* mark it + separate */`
    $w(\tilde{p}), w(p) := -j, j$;
    for $l < j$ mark $\pm l$ on l-level 0 between p and $\tilde{p}$, if $p < a_l$, $a_{l+1} = 0$ ignore a_l
  else if $p < b_j$, $p \neq a_j$, $w(p) = -j$, $a_{j+1} = 0$ **then**
    if *p not marked,* $b_{j+1} = 0$ **then** $w(p) := 0$, $b_{j+1} := p$; `/*` b_{j+1} `*/`
    else if $p < a_j$, $p < b_{j+1}$ **then** $w(p) := 0$, $a_{j+1} := p$; `/*` a_{j+1} `*/`

if $p = a_l$ *on l-level* 0*, for an* $l < j$*, the* l *to the right is marked* **then**
  mark a_l; `/* mark` a_l `*/`

if *p height violation in l for an* $l < j$*, (*$p < a_l$ *or p not marked), if* $p < a_l$*,* $a_{l+1} = 0$ *ignore* a_l **then** `/* height violation */`
  $w(p) := l + 1$, **if** $a_{l+1} = 0$ **then** $b_{l+1} := 0$ **else** $a_{l+1} := 0$;
  if *i is even,* $a_{j+1} \neq 0$ **then** $w(a_{j+1}), w(p) := 0, 0$, $a_{j+1} := 0$;
  if *i is odd,* $w(\tilde{p}) = 0$ *on j-level* 0 **then** $w(\tilde{p}) := -j$, $b_{j+1} := 0$;

if *b between p and the position to the left* **then** `/* b */`
  insert b_1 with -1
else if *a between p and the position to the left* **then** `/* a */`
  insert a_1 with -1

let p be one position to the left in w, change $\tilde{p}$ according to it;

do one additional iteration of the inner for-loop with $a = b = 0$;

forget the labels of w, set $V = w$ and **return** V;

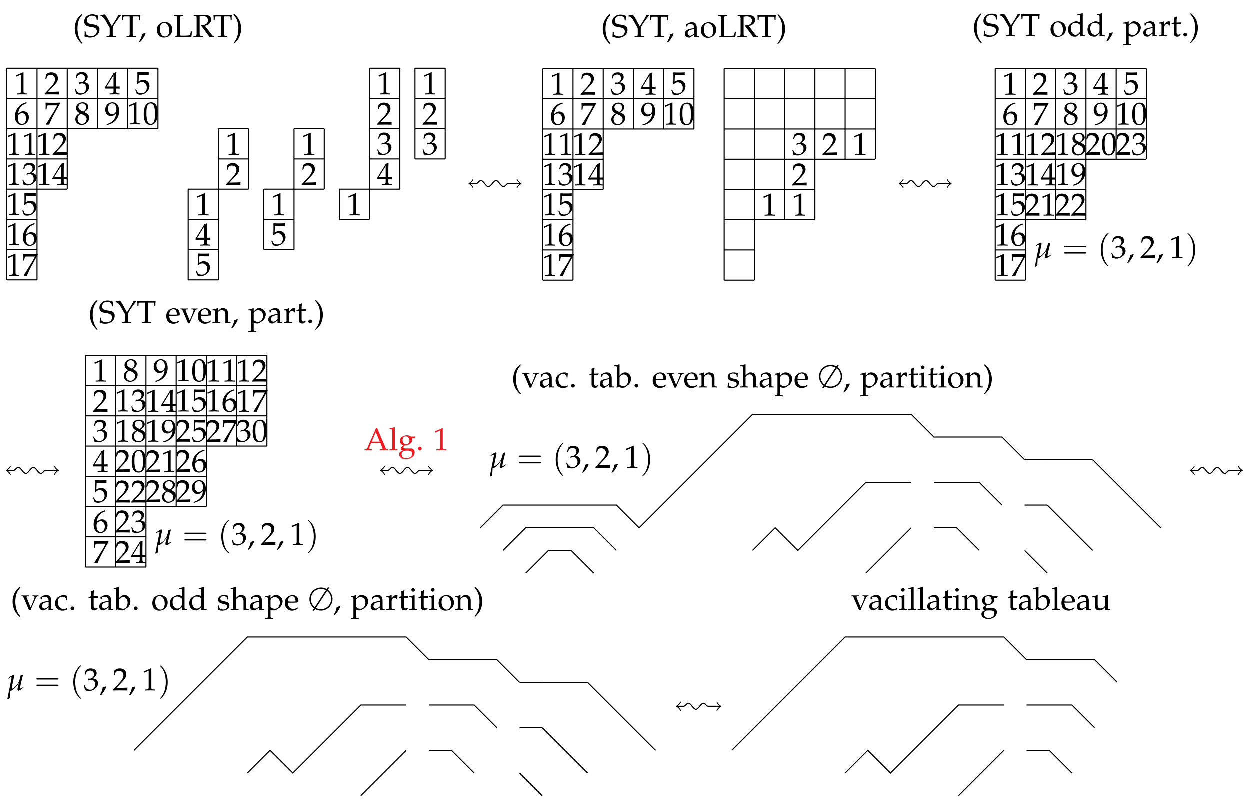


Figure 1: The strategy of our bijection outlined in an odd case.

We start with a path consisting only of up and down steps. Inserting the second and the third row works exactly as in dimension 3 in [1]. After that we initialize the second path and insert row four. There we see that our algorithm causes height violations. Those completely between an a and a b we just mark as "allowed height violations" in *separation point - mark it*. Those which used to be between an a and a b we change in *adjust separation point* and for all the others we use *height violation* to deal with them.

Conjecture 16. *Concatenation of standard Young tableaux, whose row lengths have all the same parity corresponds to concatenation of vacillating tableaux of shape* $\emptyset$*.*

This is proven for $k = 1$ in [1]. For standard Young tableaux with even row lengths it is easy to show in any odd dimension.

Conjecture 17. *Evacuation of standard Young tableaux corresponds to reversal of vacillating tableaux.*

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References

[1] J. Jagenteufel. "A Sundaram type bijection for SO(3): vacillating tableaux and pairs of standard Young tableaux and orthogonal Littlewood-Richardson tableaux". *Electron. J. Combin.* **25**.3 (2018), Paper 3.50, 44 pp. Link.

[2] J. Jagenteufel. "A Sundaram type bijection for SO (2k+1): vacillating tableaux and pairs consisting of a standard Young tableau and an orthogonal Littlewood-Richardson tableau". 2019. arXiv:1902.03843.

[3] C. Krattenthaler. "Bijections between oscillating tableaux and (semi)standard tableaux via growth diagrams". *J. Combin. Theory Ser. A* **144** (2016), pp. 277–291. Link.

[4] J.-H. Kwon. "Combinatorial extension of stable branching rules for classical groups". *Trans. Amer. Math. Soc.* **370**.9 (2018), pp. 6125–6152. Link.

[5] S. Okada. "Pieri rules for classical groups and equinumeration between generalized oscillating tableaux and semistandard tableaux". *Electron. J. Combin.* **23**.4 (2016), Paper 4.43, 27 pp. Link.

[6] R. A. Proctor. "A Schensted algorithm which models tensor representations of the orthogonal group". *Canad. J. Math.* **42**.1 (1990), pp. 28–49. Link.

[7] T. W. Roby. "Applications and extensions of Fomin's generalization of the Robinson-Schensted correspondence to differential posets". PhD thesis. Massachusetts Institute of Technology, 1991.

[8] M. Rubey, B. E. Sagan, and B. W. Westbury. "Descent sets for symplectic groups". *J. Algebraic Combin.* **40**.1 (2014), pp. 187–208. Link.

[9] R. P. Stanley. *Enumerative Combinatorics. Vol. 2*. Cambridge Studies in Advanced Mathematics 62. Cambridge University Press, Cambridge, 1999. Link.

[10] S. Sundaram. "On the combinatorics of representations of the symmetric group". PhD thesis. Massachusetts Institute of Technology, 1986.

[11] S. Sundaram. "Orthogonal tableaux and an insertion algorithm for SO$(2n+1)$". *J. Combin. Theory Ser. A* **53**.2 (1990), pp. 239–256. Link.