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Skew polynomials and extended Schur functions

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Abstract. Using Kohnert's algorithm, we associate a polynomial to any cell diagram in the positive quadrant, simultaneously generalizing Schubert polynomials and GL_n Demazure characters. We survey properties of these Kohnert polynomials and their stable limits, which are quasisymmetric functions. As a first application, we introduce and study two new bases of Kohnert polynomials, one of which stabilizes to the skew-Schur functions and is conjecturally Schubert-positive, the other stabilizes to a new basis of quasisymmetric functions that contains the Schur functions.

Keywords: Schubert polynomials, Demazure characters, monomial slide polynomials

1 Introduction

Certain bases of polynomials play a central role in representation theory and geometry. Principal examples are the Schubert polynomials [13], which are characters of Kraśkiewicz-Pragacz modules [11, 12] and represent Schubert basis classes in the cohomology of the complete flag variety; and the Demazure characters [8] (also known as key polynomials), which are the characters of GL_n Demazure modules. We seek a general combinatorial framework to construct and understand bases of polynomials that share key properties with these bases. This framework may be used to study Schubert polynomials and Demazure characters, and the new bases introduced may moreover be of independent representation-theoretic or geometric interest.

Kohnert [10] introduced a combinatorial model for the monomial expansion of a Demazure character. Given the diagram $\mathbb{D}(\mathbf{a})$ of a weak composition $\mathbf{a}$, the cell diagram in $\mathbb{N} \times \mathbb{N}$ which has $\mathbf{a}_i$ left-justified cells in row i, Kohnert defined an algorithmic process that moves the rightmost cell of a row down to the first empty position below. The Kohnert diagrams for $\mathbf{a}$ are the cell diagrams that may be obtained by a (possibly empty) sequence of these Kohnert moves on $\mathbb{D}(\mathbf{a})$. Kohnert proved that the Demazure character for $\mathbf{a}$ is the generating function of the Kohnert diagrams of $\mathbb{D}(\mathbf{a})$. Kohnert also conjectured that Schubert polynomials are the generating function of the Kohnert diagrams of different initial cell diagrams; two proofs appear in the literature by Winkel [18, 19], though are not widely accepted due to the intricate nature of the arguments. A

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direct, bijective proof by Assaf [1] uses the Demazure character expansion of Schubert polynomials.

We consider the polynomials arising from application of Kohnert's algorithm to *any* cell diagram in $\mathbb{N} \times \mathbb{N}$. Given a weak composition $\mathbf{a}$, there are several different (though finitely many) Kohnert polynomials for $\mathbf{a}$: when creating an initial cell diagram one must place $\mathbf{a}_i$ cells in row i, but one may choose the columns in which the cells are placed. If one Kohnert polynomial is chosen for each weak composition $\mathbf{a}$, the resulting set of polynomials is a basis of the polynomial ring; we call this a *Kohnert basis*. Kohnert bases thus comprise a vast collection of combinatorially-defined bases of polynomials, including the Schubert and Demazure character bases.

We prove that every Kohnert polynomial expands positively in the monomial slide basis introduced by the authors in [3]. As a consequence, the stable limit of a Kohnert polynomial exists and is quasisymmetric; we call these stable limits *Kohnert quasisymmetric functions*. In fact, every Kohnert quasisymmetric function expands positively in Gessel's [9] fundamental basis of quasisymmetric functions. We conjecture a simple condition on diagrams characterizing when a Kohnert polynomial expands positively in Demazure characters; the stable limits of such Kohnert polynomials are Schur-positive symmetric functions. This conjecture has been extensively computer-checked. The condition turns out to be the same as the *northwest* condition of Reiner and Shimozono [16, 17] in their study of Specht modules associated to diagrams, which suggests a potential connection between Kohnert polynomials and flagged Weyl modules.

As a first application, we introduce and study two new Kohnert bases. The *skew polynomials* are the Kohnert polynomials associated to diagrams arising from certain rightward shifts of contiguous rows of cells. Skew polynomials expand positively into Demazure characters, and we conjecture, based on extensive computations, that they also expand positively into Schubert polynomials. This is suggestive of a connection with geometry. The stable limits of skew polynomials in fact are the well-studied skew-Schur functions. This Kohnert perspective yields a new canonical expansion of the product of skew-Schur functions in skew-Schur functions, via taking stable limits of the (unique) skew expansion of the product of skew polynomials.

The *lock polynomials* are the Kohnert polynomials associated to right-justified diagrams. Their stable limits form a new basis of quasisymmetric functions that contains the Schur functions, thus is a lifting of the Schur basis from symmetric to quasisymmetric functions. We call this basis the *extended Schur functions*. The theory of Kohnert bases implies the extended Schur functions expand positively in the fundamental basis. We give a tableau description for the extended Schur functions, and use this to give an explicit formula for the expansion of an extended Schur function in terms of fundamental quasisymmetric functions and to extract further interesting properties of this basis.

2 Kohnert polynomials

A *diagram* is an array of finitely many cells in $\mathbb{N} \times \mathbb{N}$. The *weight* of a diagram D, denoted by $\mathrm{wt}(D)$, is the weak composition whose ith part is the number of cells in row i. For example, four diagrams with weight $(0,2,1,2)$ are shown in Figure 1.

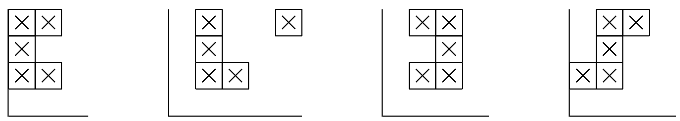

Figure 1: Four diagrams of weight $(0,2,1,2)$.

A diagram is called a *key diagram* if the rows are left justified. For each weak composition $\mathbf{a}$, the unique key diagram of weight $\mathbf{a}$ is called the *key diagram for* $\mathbf{a}$ and denoted by $\mathbb{D}(\mathbf{a})$. For example, the key diagram for $(0,2,1,2)$ is the first diagram in Figure 1.

In his thesis, Kohnert [10] described an algorithm for generating a Demazure character from a key diagram by iteratively applying certain moves to the diagram.

Definition 2.1 ([10])**.** A *Kohnert move* on a diagram selects the rightmost cell of a given row and moves the cell to the first available position below, jumping over other cells in its way as needed. Given a diagram D, let $\mathrm{KD}(D)$ denote the set of all diagrams that can be obtained by applying a series of Kohnert moves to D.

For example, Figure 2 shows all 16 Kohnert diagrams for the key diagram $\mathbb{D}(0,2,1,2)$. For comparison, the third diagram gives rise to 9 Kohnert diagrams shown in Figure 6.

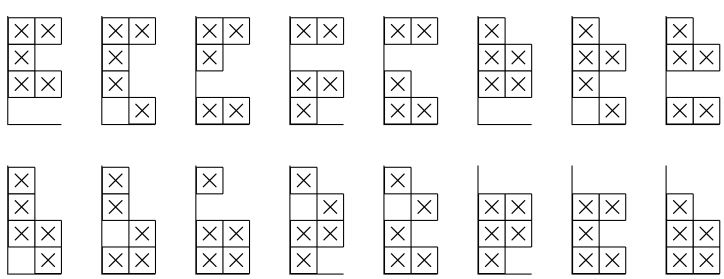

Figure 2: Kohnert diagrams for $\mathbb{D}(0,2,1,2)$.

Definition 2.2. The *Kohnert polynomial indexed by D* is

$$\mathfrak{K}_D = \sum_{T \in \mathrm{KD}(D)} x^{\mathrm{wt}(T)},$$

where $x^{\mathbf{a}}$ denotes the monomial $x_1^{a_1} \cdots x_n^{a_n}$.

For example, from Figure 2, we see that

$$\begin{aligned}\mathfrak{K}_{\mathbb{D}(0,2,1,2)} &= x^{2210} + x^{2201} + x^{2120} + 2x^{2111} + x^{2102} + x^{2021} + x^{2012} + x^{1220} \\ &\quad + 2x^{1211} + x^{1202} + x^{1121} + x^{1112} + x^{0221} + x^{0212}.\end{aligned}$$

Theorem 2.3. *Given any set of diagrams $\{D_{\mathbf{a}}\}$, one for every weak composition, such that $\mathrm{wt}(D_{\mathbf{a}}) = \mathbf{a}$, the corresponding Kohnert polynomials $\{\mathfrak{K}_{D_{\mathbf{a}}}\}$ form a basis of the polynomial ring.*

Definition 2.4. A basis $\{\mathfrak{B}_{\mathbf{a}}\}$ for polynomials is a *Kohnert basis* if each element $\mathfrak{B}_{\mathbf{a}}$ can be realized as a Kohnert polynomial for some diagram D with $\mathrm{wt}(D) = \mathbf{a}$.

Kohnert's motivation for studying key diagrams arose from characters of Demazure modules for the general linear group [7], which may be regarded as truncations of irreducible characters [8]. These polynomials were studied combinatorially by Lascoux and Schützenberger [14], who called them *key polynomials*. For a nice survey of the combinatorial aspects, see [15]; for a recent treatment from Kohnert's perspective, see [4].

Theorem 2.5 ([10])**.** *The Demazure character $\kappa_{\mathbf{a}}$ is equal to the Kohnert polynomial $\mathfrak{K}_{\mathbb{D}(\mathbf{a})}$, where $\mathbb{D}(\mathbf{a})$ is the key diagram for the indexing composition $\mathbf{a}$.*

Schubert polynomials were introduced by Lascoux and Schützenberger [13] as polynomial representatives for Schubert classes in the cohomology ring of the flag manifold for the general linear group. The *Rothe diagram* of a permutation w, denoted by $\mathbb{D}(w)$, is

$$\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\}.$$

For example, the second diagram in Figure 1 is the Rothe diagram for 143625.

Kohnert asserted that his rule, when applied to a Rothe diagram, yielded the Schubert polynomial for the corresponding permutation.

Theorem 2.6 ([18, 19, 1])**.** *The Schubert polynomial $\mathfrak{S}_w$ is equal to the Kohnert polynomial $\mathfrak{K}_{\mathbb{D}(w)}$, where $\mathbb{D}(w)$ is the Rothe diagram for the indexing permutation w.*

In this way, Kohnert polynomials generalise both Schubert polynomials and Demazure characters.

3 Positive expansions and quasisymmetric stable limits

Assaf and Searles [3] introduced a new basis for the polynomial ring called *monomial slide polynomials*. These give a natural polynomial generalization of the *monomial quasisymmetric functions* of Gessel [9]. We use this basis to study Kohnert polynomials, and to prove in particular that stable limits of Kohnert polynomials exist and are in fact quasisymmetric functions.

For weak compositions $\mathbf{a}, \mathbf{b}$ of length n, let $\mathrm{flat}(\mathbf{a})$ denote the (strong) composition obtained by removing all 0 terms from $\mathbf{a}$, and let $\mathbf{b} \geqslant \mathbf{a}$ mean $b_1 + \cdots + b_k \geqslant a_1 + \cdots + a_k$ for all $k = 1, \ldots, n$.

Definition 3.1 ([3])**.** Given a weak composition $\mathbf{a}$, the *monomial slide polynomial* $\mathfrak{M}_{\mathbf{a}}$ is

$$\mathfrak{M}_{\mathbf{a}} = \sum_{\substack{\mathbf{b} \geqslant \mathbf{a} \\ \mathrm{flat}(\mathbf{b}) = \mathrm{flat}(\mathbf{a})}} x^{\mathbf{b}}$$

For example, $\mathfrak{M}_{(2,0,1,2)} = x^{2120} + x^{2102} + x^{2012}$.

Theorem 3.2. *Kohnert polynomials expand non-negatively into monomial slide polynomials.*

Let $0^m \times \mathbf{a}$ denote the weak composition obtained by prepending m 0's to $\mathbf{a}$. The *stable limit* of a polynomial $P_{\mathbf{a}}$ associated to $\mathbf{a}$ is the limit $\lim_{m \to \infty} P_{0^m \times \mathbf{a}}(x_1, x_2, \ldots)$, if this exists. Assaf and Searles [3] showed that the monomial slide polynomials stabilize and that their stable limits are precisely the monomial quasisymmetric functions M_α of [9].

Theorem 3.3 ([3])**.** *For a weak composition* $\mathbf{a}$*, we have*

$$\lim_{m \to \infty} \mathfrak{M}_{0^m \times \mathbf{a}} = M_{\mathrm{flat}(\mathbf{a})}(x_1, x_2, \ldots).$$

Let $0^m \times D$ denote the diagram of D shifted up vertically by m rows. For example, once again taking D to be the third diagram in Figure 1, we may compute

$$\begin{aligned} \mathfrak{K}_{0 \times D} \;&=\; \mathfrak{M}_{(0,0,2,1,2)} + \mathfrak{M}_{(0,1,1,1,2)} + \mathfrak{M}_{(1,1,1,0,2)} + \mathfrak{M}_{(0,2,1,1,1)} \\ &\quad + \mathfrak{M}_{(0,1,2,0,2)} + \mathfrak{M}_{(0,1,2,1,1)} + \mathfrak{M}_{(1,1,2,0,1)} + 2\mathfrak{M}_{(1,1,1,1,1)}. \end{aligned}$$

Moreover, for any $m \geqslant 0$, we have the following expansion,

$$\begin{aligned} \mathfrak{K}_{0^{m+2} \times D} \;&=\; \mathfrak{M}_{0^m \times (0,0,0,2,1,2)} + \mathfrak{M}_{0^m \times (0,0,1,1,1,2)} + \mathfrak{M}_{0^m \times (0,1,1,1,0,2)} \\ &\quad + \mathfrak{M}_{0^m \times (0,0,2,1,1,1)} + \mathfrak{M}_{0^m \times (0,0,1,2,0,2)} + \mathfrak{M}_{0^m \times (0,0,1,2,1,1)} \\ &\quad + \mathfrak{M}_{0^m \times (0,1,1,2,0,1)} + 2\mathfrak{M}_{0^m \times (0,1,1,1,1,1)} + \mathfrak{M}_{0^m \times (1,1,1,1,0,1)}. \end{aligned}$$

In particular, the monomial slide expansion of a Kohnert polynomial eventually stabilizes. Theorems 3.2 and 3.3 allow us to consider the stable limit of Kohnert polynomials.

Definition 3.4. The *Kohnert quasisymmetric function indexed by* D is

$$\mathcal{K}_D(X) = \lim_{m \to \infty} \mathfrak{K}_{0^m \times D},$$

where $0^m \times D$ denotes the diagram of D shifted up vertically by m rows.

For example, continuing with D the third diagram in Figure 1, we have

$$\mathcal{K}_D = M_{(2,1,2)} + 2M_{(1,1,1,2)} + M_{(2,1,1,1)} + M_{(1,2,2)} + M_{(1,2,1,1)} + M_{(1,1,2,1)} + 3M_{(1,1,1,1,1)}.$$

Justifying our nomenclature, by Theorems 3.2 and 3.3, we have

Theorem 3.5. *For any diagram D, $\mathcal{K}_D(X)$ is a well-defined quasisymmetric function that expands non-negatively into the monomial quasisymmetric functions.*

Assaf and Searles [3] also introduced another basis of polynomials called the *fundamental slide polynomials*, which stabilize to the fundamental quasisymmetric functions [9]. Kohnert polynomials do not, in general, expand positively in the fundamental slide basis, however, we can characterize the diagrams for which this expansion is positive [5]. Surprisingly, despite nonpositivity at the level of polynomials, positivity in the stable limit holds *in general*.

Theorem 3.6. *Kohnert quasisymmetric functions expand non-negatively into fundamental quasisymmetric functions.*

4 Demazure character positivity and skew polynomials

Lascoux and Schützenberger [14] proved that Schubert polynomials always expand as a nonnegative integral sum of Demazure characters. Thus it is natural to explore the question of when a general Kohnert polynomial expands as a nonnegative integral sum of Demazure characters.

Definition 4.1. Let (r_i, c_j) denote the cell in row i and column j. A diagram D is *southwest* if whenever a pair of cells (r_2, c_1) and (r_1, c_2) are in D, where $r_1 < r_2$ and $c_1 < c_2$, then the cell (r_1, c_1) is also in D.

For example, the diagram on the left of Figure 3 is not southwest since it contains cells in positions $(4, 2)$ and $(2, 4)$ but not the cell in position $(2, 2)$. In contrast, the diagram on the right of Figure 3 is southwest, precisely because this impediment has been removed.

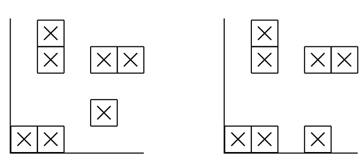

Figure 3: The left diagram is not southwest; the right is southwest.

Both Schubert polynomials and (trivially) Demazure characters expand positively in Demazure characters. Indeed, we note

Proposition 4.2. *Both composition diagrams and Rothe diagrams are southwest.*

This together with extensive computations supports the following conjecture.

Conjecture 4.3. *Given a southwest diagram D, the Kohnert polynomial $\mathfrak{K}_D$ expands non-negatively into Demazure characters.*

If true, Conjecture 4.3 gives an enormous class of Kohnert polynomials that have representation-theoretic and geometric significance. In further support of the conjecture, the southwest condition is exactly the same as the *northwest* condition of Reiner and Shimozono [16, 17] in their study of Specht modules associated to diagrams, expressed in English notation. This suggests a connection between flagged Weyl modules and Kohnert polynomials.

Since Demazure characters stabilize to Schur functions, Conjecture 4.3 gives an enormous class of Kohnert quasisymmetric functions that are *Schur-positive*. This would be striking, since Kohnert quasisymmetric functions are not generally even symmetric.

We define a new southwest Kohnert basis that further supports Conjecture 4.3.

Definition 4.4. For a weak composition $\mathbf{a}$, the *skew diagram* $\mathbb{S}(\mathbf{a})$ is constructed as follows:

(1) left justify $\mathbf{a}_i$ cells in row i,
(2) for j from 1 to n such that $\mathbf{a}_j > 0$, take $i < j$ maximal such that $\mathbf{a}_i > 0$, and if $\mathbf{a}_i > \mathbf{a}_j$, then shift rows $k \geqslant j$ rightward by $\mathbf{a}_i - \mathbf{a}_j$ columns,
(3) shift each row j rightward by $\#\{i < j \mid a_i = 0\}$ columns.

The *skew polynomial* $\mathfrak{K}_{\mathbb{S}(\mathbf{a})}$ is the Kohnert polynomial associated to the skew diagram $\mathbb{S}(\mathbf{a})$.

For example, Figure 4 shows the step-by-step construction of the skew diagram $\mathbb{S}(1,0,3,2,0,3)$ from the composition diagram $\mathbb{D}(1,0,3,2,0,3)$.

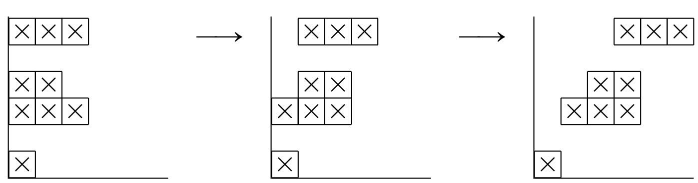

Figure 4: Illustration of construction of the skew diagram $\mathbb{S}(1,0,3,2,0,3)$.

Proposition 4.5. *Skew diagrams are southwest.*

Conjecture 4.3 therefore suggests that skew polynomials should expand nonnegatively in Demazure characters. Indeed, we have

Theorem 4.6. *Skew polynomials* $\{\mathfrak{K}_{\mathbb{S}(\mathbf{a})}\}$ *form a basis of* $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. *They expand nonnegatively in Demazure characters, and therefore stabilize to Schur-positive symmetric functions.*

Thanks to the stability results for Kohnert polynomials, we have the following.

Corollary 4.7. *Let* $\mathbf{a}$ *be a weak composition. Let* λ *be the partition given by reading the column index of the rightmost cell in each nonempty row of* $\mathbb{S}(\mathbf{a})$*, from top to bottom, and let* μ *be the partition given by* $\mu_i = \lambda_i - \mathrm{rev}(\mathrm{flat}(\mathbf{a}))_i$. *Then the stable limit* $\mathcal{K}_{\mathbb{S}(\mathbf{a})}$ *of the skew polynomial indexed by* $\mathbf{a}$ *is*

$$\mathcal{K}_{\mathbb{S}(\mathbf{a})} = \lim_{m \to \infty} \mathfrak{K}_{\mathbb{S}(\mathbf{a})} = s_{\lambda/\mu}.$$

In particular, skew polynomials stabilize to skew Schur functions.

For example, for $\mathbb{S}(1,0,3,2,0,3)$ we have $\lambda = (6,4,4,1)$ and $\mu = (3,2,1)$, therefore

$$\mathcal{K}_{\mathbb{S}(1,0,3,2,0,3)} = s_{(6,4,4,1)/(3,2,1)}.$$

One application of the skew basis is that it may be used to give a new, canonical expansion of the product of skew-Schur functions in skew-Schur functions. This is achieved by taking the limit of the skew polynomial expansion of a product of two skew polynomials. For example

$$\mathfrak{K}_{\mathbb{S}(2,0,2)} \cdot \mathfrak{K}_{\mathbb{S}(0,2,0)} = \mathfrak{K}_{\mathbb{S}(2,2,2)} + \mathfrak{K}_{\mathbb{S}(3,1,2)} + \mathfrak{K}_{\mathbb{S}(4,0,2)} + \mathfrak{K}_{\mathbb{S}(2,3,1)} + \mathfrak{K}_{\mathbb{S}(2,4,0)}.$$

In this case, Corollary 4.7 yields

$$s_{(3,2)/(1)} \cdot s_{(2)} = s_{(2,2,2)} + s_{(4,3,3)/(2,2)} + s_{(5,4)/(3)} + s_{(3,3,2)/(2)} + s_{(4,2)}.$$

The structure constants of skew polynomials, and thus the coefficients in this canonical expansion, are often, but not always, positive. However, signs appearing can be natural (e.g. see [2]), and the canonical nature of this expansion makes exploring structure constants of skew polynomials a worthwhile endeavor.

Shifting to a more positive direction, skew polynomials coincide with Demazure characters in the case when $\mathbf{a}$ is weakly increasing, in which case both are Schur polynomials. Skew polynomials also coincide with Schubert polynomials in certain cases, even outside of the above coincidence with Schur polynomials. For instance, we have

$$\mathfrak{K}_{\mathbb{S}(1,0,3,2,0,3)} = \mathfrak{S}_{216539478}.$$

This coincidence is non-obvious; Figure 5 shows the skew diagram $\mathbb{S}(1,0,3,2,0,3)$ and the somewhat-different Rothe diagram $\mathbb{D}(216539478)$.

We conjecture that Schubert polynomials are, in fact, nested between Demazure characters and skew polynomials in the following sense.

Conjecture 4.8. *Skew polynomials expand as nonnegative sums of Schubert polynomials.*

Conjecture 4.8 has been verified for degree up to 10 in up to 6 variables. If true, this conjecture is highly suggestive that skew polynomials are a combinatorial shadow of representation-theoretic and geometric objects yet to be discovered.

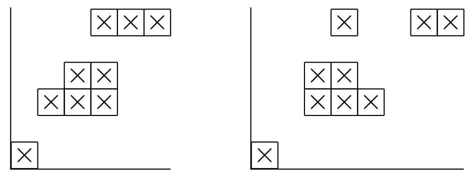

Figure 5: The skew diagram $\mathbb{S}(1,0,3,2,0,3)$ and Rothe diagram $\mathbb{D}(216539478)$.

5 Lock polynomials and extended Schur functions

We now introduce a second new Kohnert basis. For each weak composition $\mathbf{a}$, there is a unique right-justified diagram of weight $\mathbf{a}$ which we call the *lock diagram for* $\mathbf{a}$ and denote by $\mathbb{L}(\mathbf{a})$. For example, the third diagram in Figure 1 is the lock diagram for $(0,2,1,2)$. The Kohnert diagrams for this diagram are shown in Figure 6.

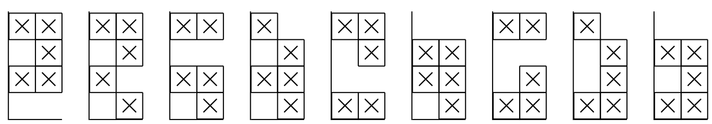

Figure 6: Kohnert diagrams for $\mathbb{L}(0,2,1,2)$.

Definition 5.1. The *lock polynomial indexed by* $\mathbf{a}$ is

$$\mathfrak{L}_a = \mathfrak{K}_{\mathbb{L}(\mathbf{a})}.$$

For example, from Figure 6, we see that

$$\mathfrak{L}_{(0,2,1,2)} = x^{2120} + x^{2111} + x^{2102} + x^{2012} + x^{1220} + x^{1211} + x^{1202} + x^{1112} + x^{0212}.$$

Since lock polynomials are a Kohnert basis, by Theorem 2.3 they are, in particular, a basis of polynomials.

Lock polynomials do not expand non-negatively into Demazure characters, as predicted by Conjecture 4.3 since lock diagrams are not, in general, southwest. In fact, the lock diagram for $\mathbf{a}$ is southwest if and only the entries of $\mathrm{flat}(\mathbf{a})$ weakly decrease. In this case, the lock polynomial coincides with the key polynomial:

Theorem 5.2. *Let* $\mathbf{a}$ *be a weak composition such that* $\mathrm{flat}(\mathbf{a})$ *weakly decreases. Then* $\mathfrak{L}_{\mathbf{a}} = \kappa_{\mathbf{a}}$.

By Theorem 3.5, we may consider the stable limits of lock polynomials, which we call *extended Schur functions*.

Definition 5.3. Given a (strong) composition α, the *extended Schur function* $\mathcal{E}_\alpha(X)$ is

$$\mathcal{E}_\alpha(X) = \lim_{m\to\infty} \mathfrak{L}_{0^m\times\alpha} = \mathcal{K}_{\mathbb{L}(\alpha)}.$$

Proposition 5.4. *Given a weak composition* $\mathbf{a}$*, we have*

$$\mathcal{K}_{\mathbb{L}(\mathbf{a})} = \mathcal{E}_{\mathrm{flat}(\mathbf{a})}.$$

In particular, Kohnert quasisymmetric functions for lock diagrams are extended Schur functions.

Justifying their name, the extended Schur functions contain the Schur functions:

Proposition 5.5. *For* λ *a partition, we have*

$$\mathcal{E}_\lambda(X) = s_\lambda(X).$$

Demazure characters do not expand non-negatively into lock polynomials. However, Proposition 5.5 shows that the stable limits of Demazure characters do expand nonnegatively into the stable limits of lock polynomials.

Campbell, Feldman, Light, Shuldiner and Xu [6] introduced the *shin functions*, which are a basis of non-commutative symmetric functions that generalize the Schur functions. They give the expansion of noncommutative homogeneous symmetric functions into shin functions, and as a result state the positive expansion of the dual basis into monomial quasisymmetric functions. The extended Schur functions are this dual basis; we develop further properties of this basis.

We give a more compact formula for extended Schur functions in terms of the fundamental quasisymmetric functions F_α:

Definition 5.6. A *standard extended tableau of shape* α is a filling of $\mathbb{L}(\alpha)$ that uses each of $1, 2, \ldots, n$ exactly once, such that rows decrease left to right and columns decrease top to bottom. Denote the set of standard extended tableaux of shape α by $\mathrm{SET}(\alpha)$.

For example, the standard extended tableaux of shape $(2, 1, 2)$ are shown in Figure 7. Let T be a standard extended tableau with entries $1, 2, \ldots, n$. The entry i is a *descent* of T

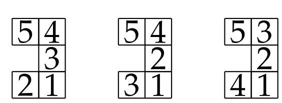

Figure 7: The set $\mathrm{SET}(2, 1, 2)$ of standard extended tableaux of shape $(2, 1, 2)$.

if it is weakly left of $i + 1$ in T. Define the *descent composition of* T, denoted by $\mathrm{Des}(T)$, to be the (strong) composition of n corresponding to the subset of $[n - 1]$ consisting of the descents of T. For example, the descent compositions for the standard extended tableaux in Figure 7 are $(2, 1, 2)$, $(1, 2, 2)$, and $(1, 1, 2, 1)$, respectively.

Theorem 5.7. *For α a (strong) composition, we have*

$$\mathcal{E}_\alpha(X) = \sum_{T \in \mathrm{SET}(\alpha)} F_{\mathrm{Des}(T)}(X).$$

The extended Schur basis exhibits many nice properties and should have interesting applications to symmetric and quasisymmetric functions. We close our introduction of this basis with two such properties.

Proposition 5.8. *The extended Schur function $\mathcal{E}_\alpha$ is equal to a single fundamental quasisymmetric function F_α if and only if $\mathbb{D}(\alpha)$ is a (reverse) hook, i.e. $\alpha = (1^k, \ell)$ for some k and ℓ.*

Proposition 5.9. *Let α be a (strong) composition and let β be obtained from α by exchanging two adjacent parts $\alpha_i < \alpha_{i+1}$. Then $\mathcal{E}_\beta - \mathcal{E}_\alpha$ is F-positive. In particular, the terms of the fundamental quasisymmetric expansion of $\mathcal{E}_\alpha$ are a sub(multi)set of those of s_λ where $\lambda = \mathrm{sort}(\alpha)$.*

For example, let $\alpha = (2,1,2)$. Exchanging α_2 and α_3 to get $\beta = (2,2,1)$, we have

$$\mathcal{E}_{(2,2,1)} - \mathcal{E}_{(2,1,2)} = F_{(2,2,1)} + F_{(1,2,1,1)}.$$

For further examples, compare entries for the extended Schur functions in Table 1.

$$\begin{aligned}
\mathcal{E}_{(1)} &= F_{(1)} \\
\mathcal{E}_{(2)} &= F_{(2)} \\
\mathcal{E}_{(11)} &= F_{(11)} \\
\mathcal{E}_{(3)} &= F_{(3)} \\
\mathcal{E}_{(21)} &= F_{(12)} + F_{(21)} \\
\mathcal{E}_{(12)} &= F_{(12)} \\
\mathcal{E}_{(111)} &= F_{(111)}
\end{aligned}
\qquad
\begin{aligned}
\mathcal{E}_{(4)} &= F_{(4)} \\
\mathcal{E}_{(31)} &= F_{(13)} + F_{(22)} + F_{(31)} \\
\mathcal{E}_{(13)} &= F_{(13)} \\
\mathcal{E}_{(22)} &= F_{(121)} + F_{(22)} \\
\mathcal{E}_{(211)} &= F_{(112)} + F_{(121)} + F_{(211)} \\
\mathcal{E}_{(121)} &= F_{(112)} + F_{(121)} \\
\mathcal{E}_{(112)} &= F_{(112)} \\
\mathcal{E}_{(1111)} &= F_{(1111)}
\end{aligned}$$

Table 1: A table of the fundamental expansion of the extended Schur functions.

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