Stanley symmetric functions for signed involutions

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Abstract. Involution words are variations of reduced words associated to twisted involutions in Coxeter groups. These words are saturated chains in a partial order first considered by Richardson and Springer in their study of symmetric varieties. In the symmetric group, involution words can be enumerated in terms of tableaux using appropriate analogues of the symmetric functions introduced by Stanley to count reduced words. We adapt this approach to the group of signed permutations. We show that the involution words for the longest element in the Coxeter group $C_n$ are in bijection with reduced words for the longest element in $A_n = S_{n+1}$, which are known to be in bijection with standard tableaux of shape $(n, n-1, \ldots, 2, 1)$.

Keywords: Signed permutations, Stanley symmetric functions, transition equations

1 Introduction

Let $W$ be a Coxeter group with simple generating set $S$. A reduced word for $w \in W$ is a minimal-length sequence $(r_1, r_2, \ldots, r_\ell)$ of simple generators $r_i \in S$ with $w = r_1 r_2 \cdots r_\ell$. Let $\mathcal{R}(w)$ be the set of reduced words for $w$.

Of primary interest are the finite Coxeter groups of classical types A and B/C. Fix an integer $n \geq 1$ and let $[n] = \{1, 2, \ldots, n\}$ and $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let $A_n = S_{n+1}$ be the group of permutations of $[n+1]$. Let $C_n$ be the group of permutations $w$ of $[\pm n]$ with $w(-i) = -w(i)$ for all $i$. Define $s_1, s_2, \ldots, s_n \in A_n$ and $t_0, t_1, \ldots, t_{n-1} \in C_n$ by

$$s_i = (i, i+1), \quad t_0 = (-1, 1), \quad \text{and} \quad t_i = (-i-1, -i)(i, i+1) \quad \text{for} \quad i \neq 0. \quad (1.1)$$

Then $A_n$ is a Coxeter group relative to the generating set $S = \{s_1, s_2, \ldots, s_n\}$ while $C_n$ is a Coxeter group relative to the generating set $S = \{t_0, t_1, \ldots, t_{n-1}\}$. We refer to elements of $C_n$ as signed permutations.

Each finite Coxeter group contains a unique element of maximal length, where the length of an element $w$ refers to the common length of any word in $\mathcal{R}(w)$. Let $w_n^A$ and $w_n^C$ denote the longest elements of $A_n$ and $C_n$. Then $w_n^A$ is the permutation $i \mapsto n+2-i$ while $w_n^C$ is the negation map $i \mapsto -i$. There are attractive product formulas for the number of reduced words for both of these permutations:

$$\left| \mathcal{R}(w_n^A) \right| = \frac{(n+1)!}{\prod_{i=1}^{n} (2i-1)^i} \quad \text{and} \quad \left| \mathcal{R}(w_n^C) \right| = \frac{(n^2)!}{n^n \prod_{i=1}^{n-1} [i(2n-i)]^i}. \quad (1.2)$$
Stanley proved the first of these identities [13, Corollary 4.3] and conjectured the second, which was later shown by Haiman [4, Theorem 5.12].

Let \( \mathcal{SYT}(\lambda) \) be the set of standard Young tableaux of shape \( \lambda \). Define \( \delta_n = (n, n-1, \ldots, 2, 1) \) and write \( (n^n) \) for the partition with \( n \) parts of size \( n \). The identities (1.2) are equivalent to \( \left| \mathcal{R}(w_n^A) \right| = \left| \mathcal{SYT}(\delta_n) \right| \) and \( \left| \mathcal{R}(w_n^C) \right| = \left| \mathcal{SYT}(n^n) \right| \) via the well-known hook-length formula [14, Corollary 7.21.6]. As one would expect from this formulation, there are natural bijective proofs of the identities (1.2), due to Edelman and Greene [3]

The main result of this paper is a product formula similar to (1.2) for the cardinality of a set of reduced-word-like objects associated to \( w \) in type A and to Haiman [4] and Kraska\'wicz [8] in type C.

Theorem 1.2. It holds that
\[
\hat{\mathcal{SYT}}(\delta_n) = \left( \binom{p+1}{2} + \binom{q+1}{2} \right) \left| \mathcal{SYT}(\delta_p) \right| \left| \mathcal{SYT}(\delta_q) \right|
\]

and conjectured the following theorem, which is our main new result.

Theorem 1.2. It holds that \( \left| \hat{\mathcal{R}}(w_n^A) \right| = \left| \mathcal{SYT}(\delta_n) \right| = \left| \mathcal{R}(w_n^A) \right| \).

There is an algebraic approach to enumerating \( \mathcal{R}(w_n^A), \mathcal{R}(w_n^C), \hat{\mathcal{R}}(w_n^A), \) and \( \hat{\mathcal{R}}(w_n^C) \) by means of certain generating functions called Stanley symmetric functions. We write \([x_1x_2 \cdots]f\) for the coefficient of a square-free monomial in a homogeneous symmetric function \( f \). The Stanley symmetric functions of interest have the following properties:

- Each (type A) Stanley symmetric function \( F_w \) has \([x_1x_2 \cdots]F_w = |\mathcal{R}(w)|\).
- Each (type C) Stanley symmetric function \( G_w \) has \([x_1x_2 \cdots]G_w = 2^\ell(w) |\mathcal{R}(w)|\).
Each (type A) involution Stanley symmetric function $\hat{F}_y$ is a multiplicity-free sum of certain instances of $F_\omega$, and it holds that $[x_1x_2\cdots]\hat{F}_y = |R(y)|$.

Each (type C) involution Stanley symmetric function $\hat{G}_y$ is a multiplicity-free sum of certain instances of $G_\omega$, and it holds that $[x_1x_2\cdots]\hat{G}_y = 2^\ell(y)|\hat{R}(y)|$.

There are expressions for $F_\omega^A$, $G_\omega^C$, and $\hat{F}_\omega^A$ as Schur functions $s_\lambda$, Schur Q-functions $Q_\lambda$, and Schur S-functions $S_\lambda$. For the definitions of these symmetric functions, see Section 2. The identities (1.2) and (1.4) are corollaries of the following formulas:

**Theorem 1.3** (Stanley [13]). $F_\omega^A = s_{\delta_n}$.

**Theorem 1.4** (Worley [15]; Billey and Haiman [2]). $G_\omega^C = Q(2n-1,2n-3,\ldots,3,1) = S(n^n)$.

**Theorem 1.5** (Hamaker, Marberg, and Pawlowski [6]). $\hat{F}_\omega^A = 2^{\ell_q}Q(n,n-2,n-4,\ldots) = s_{\delta_p}s_{\delta_q}$.

Theorem 1.2, in turn, is an immediate corollary of the following result, which adds an entry for $\hat{G}_\omega^C$ to the preceding sequence of identities.

**Theorem 1.6**. $\hat{G}_\omega^C = G_\omega^A = S_{\delta_n}$.

We relegate most technical arguments in this extended abstract to the full length article [12], but sketch the outline of some proofs. In particular, our strategy for proving Theorem 1.6 is as follows.

One can write $\hat{G}_\omega^C$ as a sum $\sum_{v \in A_n} G_v$ indexed by a certain set $A_n$ of signed permutations $v \in C_n$, the atoms of $\omega_n^C$. The transition equations of Lascoux-Schützenberger as adapted by Billey [1] generate various identities between sums of type C Stanley symmetric functions. Work of Lam [10] implies that $G_\omega^A = S_{\delta_n}$, and we show that one can apply a specific sequence of transition equations to rewrite $G_\omega^A$ as exactly the sum $\sum_{v \in A_n} G_v$. The fact that this is possible is somewhat miraculous. It is an intriguing open problem to find bijective or geometric proofs of our results.

## 2 Preliminaries

Fix a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$. The Young diagram of $\lambda$ is the set of pairs $D_\lambda = \{(i,j) : i \in [k] \text{ and } j \in [\lambda_i]\}$, which we envision as a collection of left-justified boxes oriented as in a matrix. A semistandard tableau of shape $\lambda$ is a filling of the boxes of the Young diagram $D_\lambda$ by positive integers, such that each row is weakly increasing from left to right and each column is (strictly) increasing from top to bottom. Such a tableau is standard if its boxes contain exactly the numbers $1,2,\ldots,|\lambda|$.

Similarly, a marked semistandard tableau of shape $\lambda$ is a filling of the Young diagram of $\lambda$ by numbers from the alphabet of primed and unprimed positive integers $\{1,2,3,\ldots\}$ \sqcup \{1',2',\ldots\}$.
\{1', 2', 3', \ldots \} such that (i) the rows and columns are weakly increasing under the order 1' < 1 < 2' < 2 < \cdots , (ii) no unprimed letter \( i \) appears twice in the same column, and (iii) no primed letter \( i' \) appears twice in the same row.

Assume \( \lambda \) is a strict partition, i.e., has all distinct parts. A marked semistandard shifted tableau of shape \( \lambda \) is a filling of the shifted Young diagram \( \{(i, i + j - 1) : (i, j) \in D_\lambda \} \) with primed and unprimed positive integers satisfying properties (i)-(iii) from the previous paragraph. A semistandard marked (shifted) tableau \( T \) of shape \( \lambda \) is standard if exactly one of \( i \) or \( i' \) appears in \( T \) for each \( i = 1, 2, \ldots, |\lambda| \).

Given a tableau \( T \), write \( x^T \) for the monomial formed by replacing the boxes in \( T \) containing \( i \) or \( i' \) by \( x_i \) and then multiplying the resulting variables.

**Example 2.1.** If \( T, U, \) and \( V \) are the tableaux of shape \( \lambda = (4, 3, 1) \) given by

\[
T = \begin{array}{cccc}
2 & 2 & 2 & 3 \\
3 & 3 & 4 & 5 \\
\end{array} \quad \text{and} \quad U = \begin{array}{cccc}
1' & 1 & 1 & 3 \\
1' & 3 & 4' & 5 \\
\end{array} \quad \text{and} \quad V = \begin{array}{cccc}
1 & 2' & 3 & 3 \\
2' & 4 & 6 & 5 \\
\end{array}
\]

then \( T \) is semistandard, \( U \) is marked and semistandard, and \( V \) is marked, semistandard, and shifted. We have \( x^T = x_2^3x_3^2x_5 \) and \( x^U = x_1^4x_2^2x_4x_5 \) and \( x^V = x_1x_2^2x_3^2x_4x_5 \).

**Definition 2.2.** Let \( \lambda \) be a partition and let \( \mu \) be a strict partition. The Schur function of \( \lambda \), the Schur \( S \)-function of \( \lambda \), and the Schur \( Q \)-function of \( \mu \) are then the respective sums

\[
s_\lambda = \sum_T x^T, \quad S_\lambda = \sum_U x^U, \quad \text{and} \quad Q_\mu = \sum_V x^V
\]

where \( T \) runs over semistandard tableaux of shape \( \lambda \), \( U \) runs over semistandard marked tableaux of shape \( \lambda \), and \( V \) runs over marked semistandard shifted tableaux of shape \( \mu \).

It is well-known that the Schur functions \( s_\lambda \), with \( \lambda \) ranging over all partitions, form a basis for the algebra \( \Lambda \) of symmetric functions. Similarly, the Schur \( Q \)-functions \( Q_\mu \), with \( \mu \) ranging over all strict partitions, form a basis for the subalgebra \( \Gamma \subset \Lambda \) generated by the odd-indexed power sum symmetric functions. Each Schur \( Q \)-function is itself Schur-positive, i.e., an \( \mathbb{N} \)-linear combination of Schur functions.

The set of Schur \( S \)-functions, with \( \lambda \) ranging over all partitions, is not linearly independent, but also spans the subalgebra \( \Gamma \). The set \( \{S_\lambda : \lambda \text{ is a strict partition} \} \) is a basis for \( \Gamma \). For more properties of these functions, see [11, Chap. I, §3] (for \( s_\lambda \)), [11, Chap. III, §8] (for \( Q_\lambda \)), and [11, Chap. III, §8, Ex. 7] (for \( S_\lambda \)).

Next, we review the definitions of \( F_w, G_w, \) and \( \tilde{F}_y \) from [1, 2, 7, 13].

**Definition 2.3.** The type \( A \) Stanley symmetric function associated to \( w \in A_n = S_{n+1} \) is

\[
F_w = \sum_{a \in R(w)} \sum_{(i_1 \leq i_2 \leq \cdots \leq i_l) \in \mathcal{C}(a)} x_{i_1}x_{i_2} \cdots x_{i_l}
\]

where if \( a = (s_{a_1}, s_{a_2}, \cdots, s_{a_l}) \), then \( \mathcal{C}(a) \) is the set of all weakly increasing sequences of positive integers \( i_1 \leq i_2 \leq \cdots \leq i_l \) such that if \( a_j > a_{j+1} \) for \( 1 \leq j < l \) then \( i_j < i_{j+1} \).
Definition 2.4. The type C Stanley symmetric function associated to $w \in C_n$ is
\[ G_w = \sum_{a \in R(w)} \sum_{i_1 \leq i_2 \leq \cdots \leq i_l \in D(a)} 2^{\{i_1,i_2,\ldots,i_l\}} x_{i_1} x_{i_2} \cdots x_{i_l} \]
where if $a = (t_{a_1}, t_{a_2}, \cdots, t_{a_l})$, then $D(a)$ is the set of all weakly increasing sequences of positive integers $i_1 \leq i_2 \leq \cdots \leq i_l$ such that if $a_{j-1} < a_j > a_{j+1}$ for $1 < j < l$ then either $i_{j-1} < i_j \leq i_{j+1}$ or $i_{j-1} \leq i_j < i_{j+1}$.

Let $(W, S)$ be any Coxeter system. Recall the definition of $\circ$ from the introduction.

Definition 2.5. For each $y \in \mathcal{I}(W) = \{ z \in W : z = z^{-1} \}$ let $A(y)$ be the set of elements $w \in W$ of minimal length with $w^{-1} \circ w = y$. We refer to elements of this set as atoms.

Definition 2.6. The type A and type C involution Stanley symmetric functions associated to $y \in \mathcal{I}(A_n)$ and $z \in \mathcal{I}(C_n)$ are $\hat{F}_y = \sum_{w \in A(y)} F_w$ and $\hat{G}_z = \sum_{w \in A(z)} G_w$, respectively.

Each $F_w$ is an $\mathbb{N}$-linear combination of Schur functions [3]. Each $G_w$ is an $\mathbb{N}$-linear combination of Schur Q-functions [10, Theorem 3.12]. It follows that $\hat{F}_y$ and $\hat{G}_z$ are likewise Schur-positive and Schur-Q-positive. For $\hat{F}_y$, a stronger statement holds: if $\kappa(y)$ is the number of 2-cycles in $y \in \mathcal{I}(A_n)$, then $2^{\kappa(y)} \hat{F}_y$ is also Schur-Q-positive [6, Corollary 4.62]. We do not know if $\hat{G}_z$ has any stronger positivity property along these lines.

Let $t : A_{n-1} \rightarrow C_n$ be the unique group homomorphism with $t(s_i) = i$ for $i \in [n-1]$, and define $G_w = G_{t(w)}$ for $w \in A_{n-1}$. To relate $F_w$ and $G_w$ for $w \in A_{n-1}$, recall that $\Lambda = Q\text{-span}\{s_\lambda\}$ and $\Gamma = Q\text{-span}\{S_\lambda\}$ where both spans are over all partitions $\lambda$. The second space is also given by $\Gamma = \{Q_\mu : \mu \text{ is a strict partition}\}$. The superposition map is the linear map $\phi : \Lambda \rightarrow \Gamma$ with $\phi(s_\lambda) = S_\lambda$ for all partitions $\lambda$.

Theorem 2.7 (Lam [9]). If $w \in A_{n-1}$ then $G_w = \phi(F_w)$.

A reflection in a Coxeter group is an element conjugate to a simple generator. With our notation as in [1, §3], the reflections in $C_n$ are the elements $s_{ii} = (i, i)$ for $i \in [n]$ along with $s_{ij} = s_{ji} = (i, j)(i, j)$ and $t_{ij} = t_{ji} = (i, j)(i, j)$ for $i, j \in [n]$ with $i < j$. If $t \in C_n$ is a reflection and $u, v \in C_n$ are such that $v = ut$ and $\ell(v) = \ell(u) + 1$, then we write $u < v$; then $\prec$ is the covering relation of the Bruhat order of $C_n$. For $w \in C_n$ and $j \in [n]$, define
\[ S_j(w) = \{ ws_{ij} : i \in [n], w < ws_{ij} \} \subseteq C_n, \]
\[ T_j^-(w) = \{ wt_{ij} : 1 \leq i < j, w < wt_{ij} \} \subseteq C_n, \]
\[ T_j^+(w) = \{ wt_{jk} : j < k \leq n + 1, w < wt_{jk} \} \subseteq C_{n+1}. \]
We refer to the following technical result as the transition formula for $G_w$.

Theorem 2.8 (Billey [1]). If $w \in C_n$ and $j \in [n]$ then
\[ \sum_{v \in T_j^+(w)} G_v = \sum_{v \in S_j(w)} G_v + \sum_{v \in T_j^-(w)} G_v. \]
3 Atoms and quasi-atoms

Let $S \subset \mathbb{Z}$ be a set of integers. A perfect matching on a set $S$ is a set of pairwise disjoint 2-element subsets $\{i,j\}$, referred to as blocks, whose union is $S$. A perfect matching $M$ is symmetric if $\{i,j\} \in M$ implies $-\{i,j\} \in M$, and noncrossing if it does not occur that $i < a < j < b$ for any blocks $\{i,j\}, \{a,b\} \in M$. Let NCSP$(n)$ denote the set of noncrossing, symmetric, perfect matchings on the set $[\pm n]$. The three elements of NCSP$(3)$ are $\{\{\pm 1\}, \{\pm 2\}, \{\pm 3\}\}$, $\{\{\pm 1, 2\}, \{\pm 3\}\}$, and $\{\{\pm 1\}, \{\pm 2, 3\}\}$.

For us, a word is a finite sequence of nonzero integers. The one-line representation of $w \in \mathbb{C}_n$ is the word $w_1 w_2 \cdots w_n$ where $w_i = w(i)$. We usually write $\overline{w}$ in place of $-w$ so that, for example, the elements of $C_2$ are $12, \overline{12}, \overline{1} \overline{2}, \overline{2} 1, \overline{2} \overline{1}$, and $\overline{2} 1$. If $w = w_1 w_2 \cdots w_n$ is a word then we write $[[w]]$ for the subword formed by omitting each repeated letter after its first appearance. For example, $[[312311243]] = 3124$. Suppose $M$ is a symmetric, noncrossing, perfect matching on a subset of $[\pm n]$. Define

$$\text{Pair}(M) = \{(a, b) : (a, b) \in M, 0 < a < b\} \uplus \{(-a, -a) : \{-a, a\} \in M, 0 < a\}.$$ 

Let $(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)$ and $(c_1, d_1), (c_2, d_2), \ldots, (c_l, d_l)$ be the elements of $\text{Pair}(M)$ listed in order such that $b_1 < b_2 < \cdots < b_l$ and $c_1 < c_2 < \cdots < c_l$, and define

$$\alpha_{\text{min}}(M) = [[a_1 b_1 a_2 b_2 \cdots a_l b_l]] \quad \text{and} \quad \alpha_{\text{max}}(M) = [[c_1 d_1 c_2 d_2 \cdots c_l d_l]].$$

If $M \in \text{NCSP}(n)$, then $\alpha_{\text{min}}(M)$ and $\alpha_{\text{max}}(M)$ contain exactly one letter from $\{\pm i\}$ for each $i \in [n]$, so are the one-line representations of elements of $C_n$. If $u$ and $v$ are words, both with $n \geq i + 2$ letters, then we write $u \prec_i v$ to mean that

$$u_i u_{i+1} u_{i+2} = cab, \quad v_j v_{j+1} v_{j+2} = bca, \quad \text{and} \quad u_j = v_j \text{ for } j \notin \{i, i+1, i+2\}$$

(3.1)

for some $a < b < c$. Let $<_A$ be the transitive closure of the relations $\prec_i$ for all $i \geq 1$. We apply $<_A$ to signed permutations in one-line notation. Finally, define $A_n = A(w_n^C) \subset C_n$ and $A_M = \{w \in C_n : \alpha_{\text{min}}(M) \preceq_A w \preceq_A \alpha_{\text{max}}(M)\}$ for $M \in \text{NCSP}(n)$.

**Example 3.1.** If $M = \{\{\pm 1\}, \{\pm 2, 3\}, \{\pm 4, 5\}\} \in \text{NCSP}(5)$ then the interval $A_M$ is

```
\begin{array}{cccccc}
\text{12345} & \text{12453} & \text{23145} \\
\downarrow & & \downarrow \\
\text{12453} & \text{23145} \\
\downarrow & & \downarrow \\
\text{T4523} & \text{23451} \\
\downarrow & & \downarrow \\
\text{45123} & \text{24531} \\
\downarrow & & \downarrow \\
\text{45231} \\
\end{array}
```
Theorem 3.2 (Hamaker and Marberg [5]). It holds that $\mathcal{A}_n = \bigsqcup_{M \in \text{NCSP}(n)} \mathcal{A}_M$.

For example, we have $\mathcal{A}_4 = \{\overline{123}, \overline{231}\}$, $\mathcal{A}_5 = \{\overline{312}\}$, and $\mathcal{A}_6 = \{\overline{321}\}$ for the matchings $L = \{\{\pm 1\}, \{\pm 2, 3\}\}$, $M = \{\{\pm 1, 2\}, \{\pm 3\}\}$, and $N = \{\{\pm 1\}, \{\pm 2\}, \{\pm 3\}\}$, and $\mathcal{A}_3 = \mathcal{A}_4 \sqcup \mathcal{A}_5 \sqcup \mathcal{A}_6$. For an atom $w \in \mathcal{A}_n$, define $M(w)$ to be the unique noncrossing, symmetric, perfect matching in NCSP$(n)$ with $w \in \mathcal{A}_{M(w)}$.

Given a word $w = w_1 w_2 \cdots w_n$ such that $|w_1|, |w_2|, \ldots, |w_n|$ are distinct and nonzero, define $\text{fl}_\pm (w) \in C_n$ to be the signed permutation whose one-line representation is formed by replacing each letter of $w$ by its image under the order-preserving bijection $\{\pm w_1, \pm w_2, \ldots, \pm w_n\} \to [\pm n]$. For example, $\text{fl}_\pm (3\overline{257}) = 2\overline{134} \in C_4$. If $M$ is a partition of a symmetric $2n$-element subset $X = -X \subset [\pm n]$, then define $\text{fl}_\pm (M)$ to be the partition of $[\pm n]$ formed by replacing each element of each block of $M$ by its image under the order-preserving bijection $X \to [\pm n]$.

Suppose $w \in C_n$ and $v = \text{fl}_\pm (w_2 w_3 \cdots w_n) \in \mathcal{A}_{n-1}$. Define $M'(w)$ to be the unique perfect matching on $[n] \setminus \{\pm w_1\}$ with $\text{fl}_\pm (M'(w)) = M(v)$. Since $M(v)$ is symmetric and noncrossing, $M'(w)$ is symmetric and noncrossing.

The matching $M'(w)$ may be read off directly from the one-line representation of $w$ by the following procedure. Let $w^0, w^1, w^2, \ldots, w^l$ be any sequence of words whose first term is $w^0 = w_2 w_3 \cdots w_n$ (note the deliberate omission of $w_1$) and whose final term is strictly increasing, in which $w^i$ for $i > 0$ is formed from $w^{i-1}$ by removing a consecutive subword $q_i p_i$ with $p_i < q_i$. Let $\{c_1, c_2, \ldots, c_k\}$ be the set of letters in $w^l$. Then $M'(w)$ is the matching whose blocks consist of $\{p, -q\}$, $\{-p, q\}$, and $\{c\}$ for each $(q, p) \in \{(q_1, p_1), (q_2, p_2), \ldots, (q_l, p_l)\}$ and each $c \in \{c_1, c_2, \ldots, c_k\}$. It is a nontrivial fact that this construction is independent of the choices of descents [5]. One can read off $M(v)$ from the one-line representation of $v \in \mathcal{A}_{n-1}$ by a similar procedure.

Example 3.3. Let $w = 316\overline{745}\overline{2}$. One sequence of words $w^0, w^1, \ldots, w^l$ as above is

\[ w^0 = 16\overline{745}\overline{2}, \quad w^1 = 14\overline{52}, \quad w^2 = 1\overline{2}, \quad w^3 = \emptyset, \]

so $M'(w) = \{\pm\{1, 2\}, \pm\{4, 5\}, \pm\{6, 7\}\}$. Setting $v = \text{fl}_\pm (w_2 w_3 \cdots w_n)$, we have

\[ \alpha_{\min}(M) = \overline{563412} <_3 \overline{561342} <_1 \overline{156342} = v \]

for $M = \{\pm\{1, 2\}, \pm\{3, 4\}, \pm\{5, 6\}\} = \text{fl}_\pm (M'(w))$, so $v \in \mathcal{A}_M$.

Definition 3.4. An element $w \in C_n$ is a quasi-atom if the following conditions hold:

(a) One has $w_1 > 0$ and $\text{fl}_\pm (w_2 w_3 \cdots w_n) \in \mathcal{A}_{n-1}$, where $\mathcal{A}_0 = \{\emptyset\}$.

(b) At most one block $\{a, b\} \in M'(w)$ has $0 < a < w_1 < b$.

(c) No symmetric block $\{\pm c\} \in M'(w)$ has $0 < w_1 < c$. 

A quasi-atom \( w \) is odd if no block \( \{a, b\} \in M'(w) \) exists with \( 0 < a < w_1 < b \); otherwise, \( w \) is even. We write \( Q_{n}^+ \) and \( Q_{n}^- \) for the sets of even and odd quasi-atoms in \( C_n \), and define \( Q_{n} = Q_{n}^+ \sqcup Q_{n}^- \). By convention \( A_0 = \{\emptyset\} \), \( Q_{1}^+ = \emptyset \), and \( Q_{1}^- = \{1\} \). In rank two we have \( A_1 = \{1\} \), \( Q_{2}^+ = \emptyset \), and \( Q_{2}^- = \{21\} \). In rank three we have \( A_2 = \{21, 12\} \), \( Q_{3}^+ = \{213\} \), and \( Q_{3}^- = \{321, 312, 123\} \). The following is not obvious:

**Proposition 3.5.** The sets \( A_{n} \) and \( Q_{n} \) are disjoint.

### 4 Transition graphs

In this section, we construct a directed bipartite graph \( \overrightarrow{L}_{n} \) with vertex set \( A_{n} \sqcup Q_{n} \). We use the letter \( L \) to denote this graph since it will later serve as one “layer” in a larger graph of interest.

Explicitly, define \( \overrightarrow{L}_{n} \) to be the smallest directed graph containing each element of \( A_{n} \sqcup Q_{n} \) as a vertex and containing each of the following directed edges:

- If \( v \in Q_{n}^+ \) and \( b = v_1 > 0 \), and \( \{a, c\} \in M'(v) \) is the unique block with \( 0 < a < b < c \), then \( \overrightarrow{L}_{n} \) has an incoming edge \( t_{bc}v \to v \) and an outgoing edge \( v \to t_{ab}v \).

- If \( v \in A_{n} \) then \( v \) has a single incoming edge \( u \to v \) in \( \overrightarrow{L}_{n} \), where if \( v_1 < 0 \) then \( u = vt_0 \), and if \( v_1 > 0 \) then \( u = t_{bc}v \) where \( b = v_1 \) and \( c \) is such that \( \{b, -c\} \in M(v) \).

See Figure 1 for an example. It is not obvious from this description of \( \overrightarrow{L}_{n} \) that each vertex incident to \( v \in A_{n} \sqcup Q_{n}^+ \) is contained in \( A_{n} \sqcup Q_{n} \), but this turns out to be the case.

![Figure 1](image-url)

**Figure 1:** The directed bipartite graph \( \overrightarrow{L}_{4} \). The vertices in the top row are all odd.

**Theorem 4.1.** The vertices in \( \overrightarrow{L}_{n} \) are precisely the signed permutations \( A_{n} \sqcup Q_{n} \). Each edge in \( \overrightarrow{L}_{n} \) goes from an even quasi-atom to an odd quasi-atom, from an odd quasi-atom to an even quasi-atom, or from an odd quasi-atom to an atom.
Corollary 4.2. The directed graph $\vec{L}_n$ is bipartite.

Recall the definitions of $S_j(w) \subset C_n$ and $T_j^+(w) \subset C_{n+1}$ for $w \in C_n$ from Theorem 2.8. Our next theorem relates the edges of $\vec{L}_n$ to Billey’s transition formula for $G_w$.

Theorem 4.3. Suppose $u, w \in Q_n$ and $w_1 < n$. Then $v \in S_1(u)$ if and only if $u \to v$ is an edge in $\vec{L}_n$, and $v \in T_1^+(w)$ if and only if $v \to w$ is an edge in $\vec{L}_n$. Hence $T_1^+(w) \subset C_n$.

A sink/source in a directed graph is a vertex with no outgoing/incoming edges.

Theorem 4.4. The set of sinks in $\vec{L}_n$ is precisely $A_n$. The set of sources in $\vec{L}_n$ is precisely the set of odd quasi-atoms of the form $nv_1v_2 \cdots v_{n-1} \in Q_n$ where $v_1v_2 \cdots v_{n-1} \in A_{n-1}$.

The preceding result indicates a natural way of packaging the graphs $\vec{L}_n$ together for successive values of $n$. For $0 < m < n$, write $\uparrow^m_n : C_m \to C_n$ for the transformation

$$\uparrow^m_n(v_1v_2 \cdots v_m) = n(n-1) \cdots (m+3)(m+2)v_1v_2 \cdots v_m(m+1) \in C_n.$$  

Let $C_0$ be the set consisting of just the empty word $\emptyset$ and define $\uparrow^0_n : C_0 \to C_n$ to be the map $\emptyset \mapsto n \cdots 321$. View $\vec{L}_0$ as the edgeless graph with vertex set $C_0$. Define $\vec{L}_{m,n}$ for $0 \leq m < n$ to be the directed graph given by replacing each vertex in $\vec{L}_m$ by its image under $\uparrow^m_n$. Interpret $\uparrow^{m+1}_n$ as the identity map $C_n \to C_n \mapsto C_{n+1}$ and identify $\vec{L}_{n,n+1}$ with $\vec{L}_n$. Finally, define $\vec{G}_n$ to be the graph given by the disjoint union

$$\vec{L}_{0,n+1} \sqcup \vec{L}_{1,n+1} \sqcup \vec{L}_{2,n+1} \cdots \sqcup \vec{L}_{n,n+1}$$

with these additional edges: for $m \in [n]$ and $w \in A_{m-1}$, include an edge from the sink $$\uparrow^{m+1}_{m-1}(w_1w_2 \cdots w_{m-1}) = (n+1)n \cdots (m+3)(m+2)w_1w_2 \cdots w_{m-1}m$$  

in $\vec{L}_{m-1,n+1}$ to the source $$\uparrow^m_m(mw_1w_2 \cdots w_{m-1}) = (n+1)n \cdots (m+3)(m+2)mw_1w_2 \cdots w_m(m+1)$$  

in $\vec{L}_{m,n+1}$. A vertex in $\vec{L}_{m,n+1}$ is odd it is the image under $\uparrow^{m+1}_m$ of an odd quasi-atom in $\vec{L}_m$. All other vertices in $\vec{L}_{m,n+1}$ or $\vec{G}_n$ are even; in particular, the unique vertex $\uparrow^{n+1}_0(\emptyset)$ in $\vec{L}_{0,n+1}$ is even. Figure 2 shows the graph $\vec{G}_4$. The resulting division into even and odd vertices affords a bipartition of $\vec{G}_n$. The following key properties of $\vec{G}_n$ are straightforward corollaries of Theorems 2.8, 4.3 and 4.4.

Corollary 4.5. The unique source in $\vec{G}_n$ is $\uparrow^{n+1}_0(\emptyset) = w_n^A$. The set of sinks in $\vec{G}_n$ is $\uparrow^{n+1}_n(A_n) = A_n$. If $v$ is any odd vertex in $\vec{G}_n$, then $\sum_{u \to v} G_u = \sum_{v \to w} G_w$.
Finally, using these results we can prove Theorem 1.6.

Proof of Theorem 1.6. For each vertex \( v \in \overrightarrow{G}_n \), define \( f(v) \) to be 0 if \( v \) is odd and \( G_v \) if \( v \) is even. The last property in Corollary 4.5 implies \( \sum_{\{u \rightarrow v\} \in \overrightarrow{G}_n} (f(v) - f(u)) = 0 \). On the other hand, \( \sum_{\{u \rightarrow v\} \in \overrightarrow{G}_n} (f(v) - f(u)) = \sum_{v \in \overrightarrow{G}_n} \text{sdeg}(v)f(v) \) where \( \text{sdeg}(v) \) is the outdegree of \( v \) minus its indegree. The unique source in \( \overrightarrow{G}_n \) has outdegree 1; every sink in \( \overrightarrow{G}_n \) has indegree 1; and every even vertex that is not a source or a sink has indegree 1 and outdegree 1. Hence \( 0 = \sum_{u \in \text{Source}(\overrightarrow{G}_n)} G_u - \sum_{v \in \text{Sink}(\overrightarrow{G}_n)} G_v = G_{w^n} - \sum_{v \in A_\theta} G_v \). The last term is \( S_{(n,n-1,...,3,2,1)} - \hat{G}_{w^n} \) by Theorems 1.3 and 2.7 and the definition of \( \hat{G}_w \). \( \square \)

5 Conjectures in type D

Assume \( n \geq 3 \) and let \( D_n \) be the subgroup of signed permutations in \( C_n \) whose one-line representations have an even number of negative letters. This subgroup is the finite Coxeter group of classical type D relative to the generating set \( S = \{ t'_1, t_1, t_2, \ldots, t_{n-1} \} \) where \( t'_1 = t_0 t_1 t_0 \). To conclude this article, we describe a conjecture that gives a type D analogue of Theorem 1.2.

For \( w \in D_n \) and \( a \in \mathcal{R}(w) \), let \( \overline{a} \) be the word obtained from \( a \) by replacing each \( t'_1 \) with \( t_1 \), and define \( \mathcal{R}(w) = \{ a : a \in \mathcal{R}(w) \} \). For instance, \( \mathcal{R}(T^32) = \{ (t_1, t'_1, t_2), (t'_1, t_1, t_2) \} \) while \( \mathcal{R}(T32) = \{ (t_1, t_1, t_2) \} \). In type D it is the sets \( \mathcal{R}(w) \) that have simple tableau enumerations. Let \( w^n_D \) be the longest element of \( D_n \).

**Theorem 5.1** (Billey and Haiman [2]). It holds that \( |\mathcal{R}(w^n_D)| = |\text{SYT}((n-1)^n)| \), which is also the number of unmarked shifted standard tableaux of shape \((2n-2,2n-4,\ldots,2)\). \( \lambda \)

Let \( (W,S) \) be a Coxeter system with a group automorphism \( \theta : W \rightarrow W \) such that \( \theta(S) = S \) and \( \theta = \theta^{-1} \). Let \( \mathcal{I}_\theta(W) = \{ w \in W : \theta(w) = w^{-1} \} \). The set of (twisted) atoms \( A_\theta(y) \) of \( y \in \mathcal{I}_\theta(W) \) consists of the minimal-length elements \( w \in W \) with \( \theta(w)^{-1} \circ w = y \). The set of (twisted) involution words for \( y \in \mathcal{I}_\theta(W) \) is \( \mathcal{R}_\theta(y) = \bigcup_{w \in A_\theta(y)} \mathcal{R}(w) \).

Assume \( W \) is finite with longest element \( w_0 \). If \( W \) is \( A_n \), \( C_n \), or \( D_{2n+1} \) for \( n > 1 \), then the only possibilities for \( \theta \) are the identity map and \( w \mapsto w_0 w w_0 \), and it holds that \( |\mathcal{R}_\theta(w_0)| = |\mathcal{R}(w_0)| \) and (in type D) \( |\mathcal{R}_\theta(w_0)| = |\mathcal{R}(w_0)| \). Define \( * \) to be the automorphism of \( D_n \) that interchanges \( t_1 \) and \( t'_1 \) and fixes \( t_i \) for \( 1 < i < n \). When \( n \) is odd, \( * \) is the inner automorphism \( w \mapsto w_0 w w_0 \). Computations support the following:

**Conjecture 5.2.** It holds that \( |\mathcal{R}(w^n_D)| = |\text{SYT}(\lambda)| \) and \( |\mathcal{R}_*(w^n_D)| = |\text{SYT}(\mu)| \) for \( \lambda = (n-1,n-2,\ldots,\lfloor \frac{n}{2} \rfloor,\lfloor \frac{n}{2} \rfloor,\ldots,2,1) \) and \( \mu = (n-1,n-2,\ldots,\lfloor \frac{n}{2} \rfloor-1,\lfloor \frac{n}{2} \rfloor-1,\ldots,2,1) \).
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References

Figure 2: The directed bipartite graph $\overrightarrow{G}_4$. The dashed arrows correspond to edges between vertices of the form (4.1) and (4.2). We have omitted the terminal 5 from all vertices in the final layer $\overrightarrow{L}_{4,5} \subset \overrightarrow{G}_4$. In contrast to what we see in this example, the graph $\overrightarrow{G}_n$ is not a directed tree for $n \geq 5$. 