Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #27, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Crystal structures for symmetric Grothendieck polynomials

Cara Monical*1, Oliver Pechenik†2, and Travis Scrimshaw‡3

[1] *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA*
[2] *Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA*
[3] *School of Mathematics and Physics, The University of Queensland, St. Lucia, QLD 4072, Australia*

Abstract. We construct a type A_n crystal structure on semistandard set-valued tableaux, which yields a new formula and proof for the Schur positivity of symmetric Grothendieck polynomials. For single rows and columns, we construct a K-theoretic analog of crystals and new interpretation of Lascoux polynomials. We relate our crystal structures to the 5-vertex model using Gelfand–Tsetlin patterns.

Keywords: Grothendieck polynomial, crystal, Lascoux polynomial, set-valued tableau

1 Introduction

The set of k-dimensional linear subspaces in $\mathbb{C}^n$ is the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ and has been well studied. One approach is through the cohomology ring, where a basis is given by the classes of the Schubert varieties. These correspond to Schur polynomials s_λ whose defining partition λ fits inside a $k \times (n-k)$ rectangle. In modern Schubert calculus, an alternative is to use the (connective) K-theory ring, where the Schubert classes are given by symmetric Grothendieck polynomials $\mathfrak{G}_\lambda$. By work of A. Buch [2], $\mathfrak{G}_\lambda$ is a generating function for semistandard set-valued tableaux of shape λ with entries at most n. Remarkably, the product $\mathfrak{G}_\mu \mathfrak{G}_\lambda$ (even in an infinite number of variables) can be written as a unique *finite* sum of $\mathfrak{G}_\nu$. There are many known positive combinatorial rules to compute the K-theoretic Schubert structure coefficients $C_{\lambda\mu}^{\nu}$, analogs of Littlewood–Richardson coefficients.

Our aim is to apply crystal theory [7] to the study of symmetric Grothendieck polynomials. Our main result is semistandard set-valued tableaux have a $U_q(\mathfrak{sl}_n)$-crystal

*caramonicalmath@gmail.com.

†pechenik@umich.edu. Oliver Pechenik was partially supported by a NSF Graduate Research Fellowship and the NSF Mathematical Sciences Postdoctoral Research Fellowship #1703696.

‡tcscrims@gmail.com. Travis Scrimshaw was partially supported by the NSF RTG grant DMS-1148634 and the ARC DP170102648.

structure that is isomorphic to a direct sum of crystals for irreducible highest weight representations. An immediate corollary is that $\mathfrak{G}_\lambda$ is a positive sum of Schur functions s_μ, where the multiplicities M_λ^μ are given by counting highest weight (*i.e.* Yamanouchi) semistandard set-valued tableaux. C. Lenart [10, Thm. 2.2] has given a different combinatorial formula for the multiplicities M_λ^μ in terms of certain flagged increasing tableaux. We recover his formula using the "uncrowding" operation of [2].

The larger goal of this project is to construct a K-theoretic analog of (combinatorial) crystal theory. Indeed, we believe there are additional K-theory crystal operators that extend our $U_q(\mathfrak{sl}_n)$-crystal structure to obtain a single connected component whose the K-theory analogs of Demazure subcrystal characters are the so-called Lascoux polynomials (see, *e.g.*, [9, 13] and references therein), the K-theory analog of key polynomials (or A_n Demazure characters) (see, *e.g.*, [1, 8] and references therein). Towards this goal, when λ is a single row or column, we construct such a K-theoretic Demazure crystal, yielding a new interpretation of the associated Lascoux polynomials.

Furthermore, we also expect a suitable notion of a tensor product such that the connected components are uniquely determined by what we call minimal highest weight elements, and so that the multiplicity of $\mathrm{SV}^n(\nu)$ in $\mathrm{SV}^n(\mu) \otimes \mathrm{SV}^n(\lambda)$ gives $C_{\lambda\mu}^\nu$. We expect this structure to connect with the column insertion given in [2] and to provide a K-theory analog of jeu de taquin (K-jdt) on semistandard set-valued tableaux. We make some progress in this direction by introducing a K-jdt for semistandard set-valued tableaux and showing that it is a $U_q(\mathfrak{sl}_n)$-crystal isomorphism.

Another approach to construct $\mathfrak{G}_\lambda$ is by using the 5-vertex model (see, *e.g.*, [14]). Configurations of the 5-vertex model with certain boundary conditions are in natural bijection with Gelfand–Tsetlin (GT) patterns, but this (and the corresponding crystal structure via the bijection with SSYT; see [5]) is a "coarse" structure, yielding a formula in analogy to the Tokuyama formula. We refine this by adding a marking to the GT pattern, so we can write a $\mathfrak{G}_\lambda$ as a sum over these marked GT patterns.

This extended abstract is organized as follows. In Section 2, we recall necessary background. In Section 3, we prove our main result: a $U_q(\mathfrak{sl}_n)$-crystal structure on semistandard set-valued tableaux and discuss the corollaries. We relate our crystal structure to (marked) Gelfand–Tsetlin patterns in Section 4. In Section 5, we define a jeu de taquin on semistandard set-valued tableaux. In Section 6, we construct a K-theory analog of crystals in some special cases and discuss extensions to the general case.

This is an extended abstract of our paper [12], which also has further connections.

2 Background

Let $\mathfrak{sl}_n$ be the type A_{n-1} Lie algebra with index set $I = \{1, 2, \ldots, n-1\}$, fundamental weights $\{\Lambda_i \mid i \in I\}$, and (Drinfeld–Jimbo) quantum group $U_q(\mathfrak{sl}_n)$. We use the usual identification of dominant weights, denoted P^+, with partitions, where a Λ_i corresponds to a column of height i. Let S_n denote the symmetric group on $\{1, \ldots, n\}$ with simple transpositions $\{s_i \mid 1 \leq i < n\}$ and longest element w_0.

A *$U_q(\mathfrak{sl}_n)$-crystal* is a nonempty set B together with maps $e_i, f_i \colon B \to B \sqcup \{0\}$ and $\mathrm{wt} \colon B \to P$ that satisfy certain compatibility properties and corresponds to a crystal basis of a $U_q(\mathfrak{sl}_n)$ representation. For more details, we refer the reader to [3]. We depict $f_i b = b'$ as $b \xrightarrow{i} b'$. An element $u \in B$ is called *highest weight* if $e_i u = 0$ for all $i \in I$.

The highest weight $U_q(\mathfrak{sl}_n)$-module $V(\lambda)$ for $\lambda \in P^+$ has a crystal basis [7] and is denoted by $B(\lambda)$ with highest weight element u_λ. This is given by the set $\mathrm{SST}^n(\lambda)$ of all semistandard tableaux of shape λ with all entries at most n. The crystal operators e_i and f_i act on $T \in \mathrm{SST}^n(\lambda)$ as follows: Write $+$ above each column of T containing i but not $i+1$, and write $-$ above each column containing $i+1$ but not i. Cancel ordered pairs $-+$. If every $+$ (resp. $-$) thereby cancels, then $f_i(T) = 0$ (resp. $e_i(T) = 0$. Otherwise $f_i T$ (resp. $e_i T$) is given by replacing the i (resp. $i+1$) corresponding to the rightmost uncanceled $+$ (resp. leftmost uncanceled $-$) with an $i+1$ (resp. i). The *weight* $\mathrm{wt}(T)$ of $T \in \mathrm{SST}^n(\lambda)$ is the *weak composition* $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, where a_i records the number of i's in the tableau T. Note that the highest weight element u_λ is the semistandard tableau given by filling the i-th row of λ from the top with i's.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be indeterminates. For a partition λ, let $s_\lambda(\mathbf{x})$ denote the corresponding *Schur function*. A *semistandard set-valued tableau of shape λ* is a filling T of the boxes of λ by nonempty finite sets of integers $\{1, \ldots, n\}$ so that for a set A to the left of a set B in the same row, we have $\max A \leq \min B$, and for C below A in the same column, we have $\max A < \min C$. Let $\mathrm{SV}^n(\lambda)$ denote the set of all semistandard set-valued tableaux of shape λ with all entries at most n. Following A. Buch [2], for a partition λ, we define the *symmetric (or stable) Grothendieck polynomial* by

$$\mathfrak{G}_\lambda(\mathbf{x}; \beta) := \sum_{T \in \mathrm{SV}^n(\lambda)} \beta^{|\mathrm{wt}(T)| - |\lambda|} \mathbf{x}^{\mathrm{wt}(T)}.$$

3 Crystal structure of semistandard set-valued tableaux

Definition 3.1. The action of f_i on $\mathrm{SV}^n(\lambda)$ is defined exactly as for the usual semistandard tableaux unless $i \in \mathsf{b}^{\rightarrow}$, where $\mathsf{b}^{\rightarrow}$ is the box immediately to the right of the box b that corresponds to the rightmost uncanceled $+$. In this case, $f_i(T)$ is given by removing i from $\mathsf{b}^{\rightarrow}$ and adding $i+1$ to b. If $i \notin \mathsf{b}^{\rightarrow}$, then we instead replace i with $i+1$ in b. The action of e_i is the reverse: We define the action of e_i exactly as for the usual semistandard tableaux unless $i+1 \in \mathsf{b}^{\leftarrow}$, where $\mathsf{b}^{\leftarrow}$ is the box immediately to the left of the box b that corresponds to the leftmost uncanceled $-$. In this case, $e_i(T)$ is given by removing $i+1$ from $\mathsf{b}^{\leftarrow}$ and adding i to b. If $i+1 \notin \mathsf{b}^{\leftarrow}$, we instead replace $i+1$ with i in b.

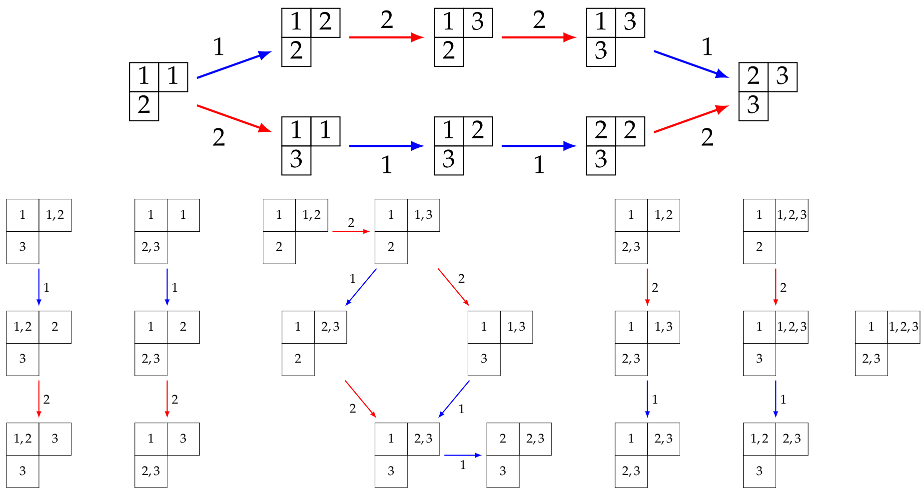


Figure 1: The $U_q(\mathfrak{sl}_3)$-crystal graph for $\mathrm{SV}^3(\lambda)$ with $\lambda = (2,1)$.

Proposition 3.2. *Fix some $s \in \mathbb{Z}_{>0}$, $k \in \{1,\dots,n\}$, and $\mu = \Lambda_k + (s-1)\Lambda_1$. We have $\mathrm{SV}^n_\mu(s\Lambda_1) \cong B(\mu)$, where $\mathrm{SV}^n_\mu(s\Lambda_1) \subseteq \mathrm{SV}^n(s\Lambda_1)$ is the closure under e_i and f_i, for all $i \in I$, of the $1 \times s$ tableau $U_\mu = \boxed{1}\boxed{1}\boxed{\cdots}\boxed{1}\boxed{1,\dots,k}$.*

One proof of Proposition 3.2 is to use the bijection Ψ_μ to the hook shape $B(\mu)$ following [2, Sec. 6], where we take the minimal element of each entry of a semistandard set-valued tableau T and place the remaining entries down the column.

The highest weight elements in $\mathrm{SV}^n(\lambda)$ are characterized as the tableaux whose Far-Eastern reading word, where we read a set as a column in decreasing order, is a *Yamanouchi word*. Hence, we call the highest weight elements *Yamanouchi set-valued tableaux*.

Theorem 3.3. *We have $\mathrm{SV}^n(\lambda) \cong \bigoplus_\mu B(\mu)^{\oplus M^\mu_\lambda}$, where M^μ_λ denotes the number of Yamanouchi set-valued tableaux of shape λ and weight μ.*

Corollary 3.4. *We have $\mathfrak{G}_\lambda = \sum_\mu \beta^{|\mu|-|\lambda|} M^\mu_\lambda s_\mu$ with M^λ_μ as in Theorem 3.3.*

C. Lenart [10, Theorem 2.2] gives a different combinatorial formula for the coefficients M^μ_λ of Corollary 3.4 using *flagged increasing tableaux*. Although these objects are superficially very different, applying the *uncrowding* bijection ψ given in [2, Sec. 6], our crystal operators define the coplactic classes in analogy to the Robinson–Schensted–Knuth (RSK) bijection with semistandard tableaux.

Example 3.5. Let $T = \begin{array}{ccc} 1,2 & 3 & 3,4 \\ 3,5 & 5,6 \end{array}$. We have $f_3 T = \begin{array}{ccc} 1,2 & 3,4 & 4 \\ 3,5 & 5,6 \end{array}$ and

$$\psi(T) = \left(\begin{array}{ccc} 1 & 3 & 3 \\ 2 & 4 \\ 3 & 5 \\ 5 \\ 6 \end{array}, \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot \\ 1 & 2 \\ 2 \\ 4 \end{array} \right), \qquad \psi(f_3 T) = f_3 \psi(T) = \left(\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 \\ 3 & 5 \\ 5 \\ 6 \end{array}, \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot \\ 1 & 2 \\ 2 \\ 4 \end{array} \right).$$

Example 3.6. The *K-Bender–Knuth moves* of [6] do not commute with uncrowding ψ:

$$\psi\left(K_2\left(\begin{array}{cc} 1 & 1,2 \\ 3 \end{array}\right)\right) = \psi\left(\begin{array}{cc} 1 & 1,3 \\ 2 \end{array}\right) = \left(\begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array}, \begin{array}{cc} \cdot & \cdot \\ \cdot & 1 \end{array}\right), \quad t_2\left(\psi\left(\begin{array}{cc} 1 & 1,2 \\ 3 \end{array}\right)\right) = \left(\begin{array}{cc} 1 & 1 \\ 2 \\ 3 \end{array}, \begin{array}{cc} \cdot & \cdot \\ \cdot \\ 2 \end{array}\right).$$

There is a natural definition of *K-evacuation* on $\mathrm{SV}^n(\lambda)$, given by applying K-Bender–Knuth moves in the analog of the Bender–Knuth version (see [4]) of the Schützenberger involution $K_1(K_2K_1)\cdots(K_{n-1}K_{n-2}\cdots K_2K_1)T$. However, K-evacuation does not generally coincide with the *Lusztig involution* (which is equal to the evacuation or Schützenberger involution on semistandard tableaux [11]) on $\mathrm{SV}^n(\lambda)$.

Example 3.7. Consider $\mathrm{SV}^3(\lambda)$ for $\lambda = (2,1)$. Then we have

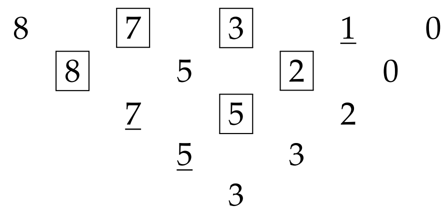
$$K_1K_2K_1\left(\begin{array}{cc} 1 & 2,3 \\ 2 \end{array}\right) = K_1K_2\left(\begin{array}{cc} 1 & 1,3 \\ 2 \end{array}\right) = K_1\left(\begin{array}{cc} 1 & 1,2 \\ 3 \end{array}\right) = \begin{array}{cc} 1,2 & 2 \\ 3 \end{array}, \qquad \left(\begin{array}{cc} 1 & 2,3 \\ 2 \end{array}\right)^* = \begin{array}{cc} 1 & 2,3 \\ 2 \end{array}.$$

4 Gelfand-Tsetlin patterns

Define a *horizontal strip* to be a skew partition that does not contain a vertical domino. Recall that a *Gelfand–Tsetlin (GT) pattern* with top row λ is a sequence of partitions $\Lambda = (\lambda^{(j)})_{j=0}^n$ such that $\lambda^{(0)} = \emptyset$, $\lambda^{(n)} = \lambda$, and $\lambda^{(j)}/\lambda^{(j-1)}$ is a horizontal strip. The weight of a GT pattern is $\mathrm{wt}(\Lambda) = \left(|\lambda^{(j)}| - |\lambda^{(j-1)}|\right)_{j=1}^n$.

Definition 4.1. A *marked Gelfand–Tsetlin (GT) pattern* is a GT pattern Λ together with a set M of entries that are "marked," where the entry (i,j), for $1 \leq i < \ell(\lambda^{(j)})$ and $2 \leq j \leq n$, is allowed to be marked if and only if $\lambda_{i+1}^{(j)} < \lambda_i^{(j-1)}$. If (Λ, M) is a marked GT pattern and $M^{(j)} = \{i \mid (i,j) \in M\}$, then the weight of (Λ, M) is $\mathrm{wt}(\Lambda, M) = \mathrm{wt}(\Lambda) + \left(|M^{(j)}|\right)_{j=1}^n$.

Example 4.2. For a GT pattern, we depict the marked entries with boxes and we underline those entries that are not allowed to be marked. Note that we can never mark the rightmost entry in any row and that positions in row j that cannot be marked correspond to those boxes of the tableau where we cannot add j. On the left is a marked GT pattern with top row $\lambda = (8,7,3,1)$ and the corresponding semistandard set-valued tableau is on the right:



1	1	1	2	2	3	3,4	4
2	2	2,3	3	3,5	5	5	
3	3,4,5	5					
5							

Proposition 4.3. *Summing over all (marked) GT patterns $\Lambda = (\lambda^{(j)})_{j=1}^{n}$ with top row λ:*

$$\mathfrak{G}_\lambda(x_1,\dots,x_n;\beta) = \sum_{\Lambda}\prod_{j=1}^{n}\mathfrak{G}_{\lambda^{(j)}/\lambda^{(j-1)}}(x_j;\beta) = \sum_{(\Lambda,M)} x^{\mathrm{wt}(\Lambda,M)}.$$

With this interpretation and [14, Prop. 3.4], we can compute $\mathfrak{G}_\lambda$ in similar fashion to the Tokuyama formula for Whittaker functions:

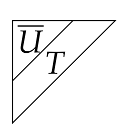
$$\mathfrak{G}_\lambda(x_1,\dots,x_n;\beta) = \sum_{\Lambda} x^{\mathrm{wt}(\Lambda)}\prod_{j=1}^{n}(1+\beta x_j)^{m_j(\Lambda)},$$

where we sum over all GT patterns with top row λ and $m_j(\Lambda)$ denotes the number of markable entries in row j of Λ. We can connect this to the 5-vertex model (with natural boundary conditions depending on λ) by a natural bijection with GT patterns, similar to a translation of [14, Cor. 3.6]. Note that GT patterns, and hence 5-vertex configurations (see also [5]), have a natural crystal structure coming from the bijection with semistandard tableaux. However, this crystal structure is a "coarse" version of the crystals on semistandard set-valued tableaux obtained by grouping monomials.

5 K-jeu de taquin

We propose a K-theoretic analog of jeu de taquin (K-jdt) for semistandard set-valued skew tableaux. Recall that a skew tableau is a filling of a skew partition λ/μ, where μ is contained within λ. Let $S * T$ denote the skew tableaux formed by placing the lower-left corner of S against the upper-right corner of T.

Consider $T \in \mathrm{SST}^n(\lambda/\mu)$ to rectify to some $R \in \mathrm{SST}^n(\nu)$ for some partition ν. For any $U \in \mathrm{SST}^m(\mu)$, let $\overline{U}$ be the tableau where we replace $k \mapsto \overline{k}$. Next, we define

$U \sqcup T :=$ 

of shape λ. Note that $U \sqcup T$ is a semistandard tableau of shape λ in the totally ordered alphabet $\overline{1} < \overline{2} < \cdots < \overline{m} < 1 < 2 < \cdots < n$. We define operators b_i on tableaux whose entries are in a totally ordered alphabet $\mathcal{A}$. Let j be the letter of $\mathcal{A}$ immediately greater than i. Then, b_i acts by first applying the Bender–Knuth operator t_i to the labels i and j, and then switching all instances of i and j.

The *rectification* $\mathrm{rect}_U(T)$ of T (with respect to the rectification order U) is the semistandard tableau obtained by restricting $b_{\overline{1}}^{\circ n} \circ \cdots \circ b_{\overline{m-1}}^{\circ n} \circ b_{\overline{m}}^{\circ n}(U \sqcup T)$ to the unbarred alphabet. Intuitively, we have pushed T and U through each other and $\mathrm{rect}_U(T)$ is the result of this pushing on T. Rectification is known to be independent of the choice of U, a property known as confluence.

Theorem 5.1. *The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ counts ordered pairs of semistandard tableaux $T \in \mathrm{SST}^n(\lambda)$ and $S \in \mathrm{SST}^n(\mu)$ such that $u_\nu \in \mathrm{SST}^n(\nu)$ is the rectification of $T * S$.*

Since the application of $b_{\overline{i}}^{n}$ during rectification preserves the semistandardness of the unbarred subtableau, rectification can be considered as a crystal isomorphism. Moreover, this implies that $\mathrm{rect}(T * S) = u_\nu$ if and only if $T * S$ is a highest weight element (*i.e.* Yamanouchi) of weight ν.

There is currently no known rule that directly extends Theorem 5.1 to K-theory. Towards this goal, we propose rectifying a set-valued tableau by the algorithm described above, except replacing everywhere the Bender–Knuth operators t_i with the K-Bender–Knuth operators K_i of T. Ikeda–T. Shimazaki [6]. Note that U can be set-valued as well.

Example 5.2. We have

$$U \sqcup T = \begin{array}{|c|c|c|c|}\hline \overline{1} & \overline{1},\overline{3} & 2 & 2 \\ \hline \overline{2},\overline{3} & 1,2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \xmapsto{b_{\overline{3}}} \begin{array}{|c|c|c|c|}\hline \overline{1} & \overline{1},1 & 2 & 2 \\ \hline \overline{2},1 & \overline{3},2 \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array} \xmapsto{b_{\overline{3}}} \begin{array}{|c|c|c|c|}\hline \overline{1} & \overline{1},1 & 2 & 2 \\ \hline \overline{2},1 & 2,\overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array} \xmapsto{b_{\overline{2}}} \begin{array}{|c|c|c|c|}\hline \overline{1} & \overline{1},1 & 2 & 2 \\ \hline 1,\overline{2} & 2,\overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array}$$

$$\xmapsto{b_{\overline{2}}} \begin{array}{|c|c|c|c|}\hline \overline{1} & \overline{1},1 & 2 & 2 \\ \hline 1,2 & \overline{2},\overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array} \xmapsto{b_{\overline{1}}} \begin{array}{|c|c|c|c|}\hline 1 & 1,\overline{1} & 2 & 2 \\ \hline \overline{1},2 & \overline{2},\overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array} \xmapsto{b_{\overline{1}}} \begin{array}{|c|c|c|c|}\hline 1 & 1,2 & 2 & \overline{1} \\ \hline 2,\overline{1} & \overline{2},\overline{3} \\ \cline{1-2} \overline{3} \\ \cline{1-1}\end{array} \Longrightarrow \mathrm{rect}_U(T) = \begin{array}{|c|c|c|}\hline 1 & 1,2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Example 5.3. Our proposed K-jdt for set-valued tableaux is not generally confluent. Consider $T = \begin{array}{|c|}\hline \\ \hline 1,2 \\ \hline\end{array}$, $\overline{U} = \begin{array}{|c|}\hline \overline{1} \\ \hline\end{array}$, $\overline{V} = \begin{array}{|c|}\hline \overline{1},\overline{2} \\ \hline\end{array}$. Then, we have $\mathrm{rect}_U(T) = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$, $\mathrm{rect}_V(T) = \begin{array}{|c|}\hline 1,2 \\ \hline\end{array}$.

Proposition 5.4. *K-jdt with respect to any fixed rectification order is a crystal isomorphism.*

Ideally, we want the Grothendieck product expansion $\mathfrak{G}_\lambda \cdot \mathfrak{G}_\mu = \sum_\nu C^{\nu}_{\lambda,\mu} \mathfrak{G}_\nu$, in parallel to Theorem 5.1, by taking ordered pairs of tableaux $T \in \mathrm{SV}(\lambda)$, $S \in \mathrm{SV}(\mu)$, rectifying with respect to some rectification orders, and obtaining exactly the tableau $R \in \mathrm{SV}(\nu)$ with the correct multiplicity. We do not know a general choice of a rectification order.

Open Problem 5.5. Determine a K-jdt rule on semistandard set-valued tableaux such that the K-rectification of $T * S$ equals the *Buch insertion* $S \overset{B}{\leftarrow} T$ from [2, Def. 4.1].

6 K-theory crystal arrows

Our approach to finding a K-analog of crystals is to construct an additional set of operators, which we call *K-crystal operators*, $e_i^K, f_i^K \colon \mathrm{SV}^n(\lambda) \to \mathrm{SV}^n(\lambda) \sqcup \{0\}$ with $f_i^K T$ formed by adding an $i+1$ to some cell of T and $e_i^K T' = T$ if and only if $T' = f_i^K T$. Such K-crystal operators should satisfy the following properties.

(K.1) The set $\mathrm{SV}^n(\lambda)$ is connected with the *minimal highest weight element* u_λ, the highest weight element of $B(\lambda)$, being the only highest weight element such that $e_i^K u_\lambda = 0$.

(K.2) The *K-Demazure crystal*

$$\mathrm{SV}_w^n(\lambda) := \{ b \in \mathrm{SV}^n(\lambda) \mid (e_{i_\ell}^K)^{\max} e_{i_\ell}^{\max} \cdots (e_{i_1}^K)^{\max} e_{i_1}^{\max} b = u_\lambda \}$$

does not depend on the choice of reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

(K.3) The character of $\mathrm{SV}_w^n(\lambda)$ is the *Lascoux polynomial* $L_{w\lambda}(\mathbf{x};\beta) = \varpi_w \mathbf{x}^\lambda$, where ϖ_i is the *Demazure–Lascoux operator* $\varpi_i f = \pi_i f + \beta \pi_i (x_{i+1} \cdot f)$, where $\pi_i f = \frac{x_i \cdot f - x_{i+1} \cdot s_i f}{x_i - x_{i+1}}$, and $\varpi_w = \varpi_{i_1} \cdots \varpi_{i_k}$ for some reduced expression $w = s_{i_1} \cdots s_{i_k}$.[1]

We can think of the Lascoux polynomials, named in honor of A. Lascoux who essentially introduced them in [9], as K-theoretic analogs of the key polynomials (also known as Demazure characters or standard bases), although no such interpretation using geometry or representation theory is currently known.

Definition 6.1. If the above properties (K.1)–(K.3) hold for some K-crystal operators, we say they give $\mathrm{SV}^n(\lambda)$ the structure of a *K-crystal*.

Note that in a K-crystal, we have $\mathrm{SV}_{w_0}^n(\lambda) = \mathrm{SV}^n(\lambda)$. We anticipate that there is a unique K-crystal structure on $\mathrm{SV}^n(\lambda)$ extending our $U_q(\mathfrak{sl}_n)$-crystal structure. Moreover, we expect our K-crystal operators to further satisfy:

(H.1) for all $T \in \mathrm{SV}^n(\lambda)$, we have $e_i^K e_i^K T = 0$ and $f_i^K f_i^K T = 0$;

(H.2) if $e_i T \neq 0$ or $f_i T = 0$, then $f_i^K T = 0$.

We expect (H.1) and (H.2) from the definition of ϖ_i and use them as heuristics for the K-crystal operators, where they should make natural axioms for a general K-crystal theory.

We first consider a K-crystal structure for single rows.

Definition 6.2. Let $\lambda \in \{k\Lambda_1, \Lambda_i\}$. The K-crystal operator f_i^K acts on $T \in \mathrm{SV}^n(\lambda)$ as follows: If $i \notin T$ or $i+1 \in T$, then $f_i^K T = 0$; otherwise $f_i^K T$ is given by adding $i+1$ to the rightmost box containing i in T. The K-crystal operator e_i^K acts on $T \in \mathrm{SV}^n(\lambda)$ as follows: If there is no box in T containing both i and $i+1$, then $e_i^K T = 0$; otherwise $e_i^K T$ is given by deleting $i+1$ from that (necessarily unique) box.

[1] Recall that both ϖ_i and the Demazure operator π_i satisfy the braid relations, so ϖ_w is well-defined.

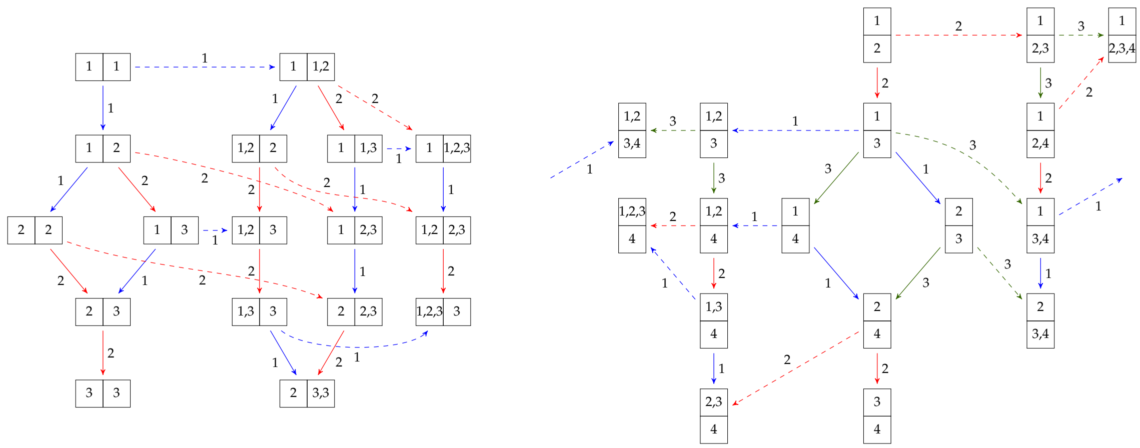
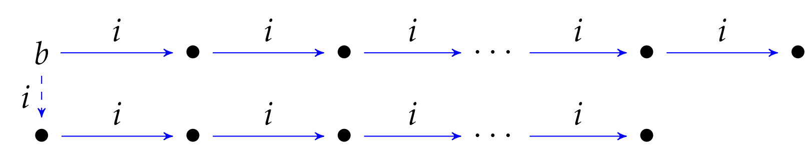


Figure 2: The K-crystal structure on $\mathrm{SV}^3(2\Lambda_1)$ (left) and $\mathrm{SV}^4(\Lambda_2)$ (right).

We need the following K-analog of [8, Prop. 3.3.4]. We propose the following K-analog of an i-string based on the Demazure–Lascoux operator. Define an *i-K-string* to be the subcrystal of the form



where the i-strings are as long as possible and the dashed arrow represents the f_i^K action. We say the i-K-string has *length* ℓ if $\varphi_i(b) = \ell$. (This is equivalent to saying that the i-string starting at b has length ℓ.) Note that $\varphi_i(f_i^K b) = \ell - 1$.

We have $\mathrm{SV}^n(\lambda)$ decomposing as a direct sum of i-K-strings when considering only i-(K-)arrows. Note that this is branching down to type $\mathfrak{sl}_2$.

Proposition 6.3. *Let S be an i-K-string of $\mathrm{SV}^n(\lambda)$ and let b be the highest weight element of S. Then, the set $\mathrm{SV}^n_w(\Lambda_k) \cap S$ is either empty, S, or $\{b\}$.*

Theorem 6.4. *Let $\lambda \in \{k\Lambda_1, \Lambda_i\}$. The $U_q(\mathfrak{sl}_n)$-crystal $\mathrm{SV}^n(\lambda)$ with the K-crystal operators given above is a K-crystal and satisfies (H.1) and (H.2).*

For $\lambda = k\Lambda_1$, we prove this using properties of the minimal length coset representatives of W/W_{Λ_1}, where $W_{\Lambda_1} = \langle s_2, \ldots, s_{n-1} \rangle$ is the stabilizer of Λ_1. For $\lambda = \Lambda_i$, we prove this by analyzing the local behavior at each element.

Next, we try to construct a K-crystal structure on $\mathrm{SV}^n(\lambda)$ for general λ. For this, the K-crystal operators in Definition 6.2 do not seem to give a K-crystal structure on $\mathrm{SV}^n(\lambda)$.

Example 6.5. If we attempt to naturally extend the K-crystal operators, for $\mathrm{SV}^3_{s_2s_1}(\Lambda_2 + \Lambda_1)$ and the 1-K-string S starting at the tableau $T = \begin{array}{|c|c|}\hline 1 & 1,3 \\ \hline 3 \\ \cline{1-1}\end{array}$, then we have $S \cap \mathrm{SV}^3_{s_2s_1}(\Lambda_2 + \Lambda_1) = \{T, f_1T, f_1^K T\} \neq \emptyset, S, \{T\}$. Hence, the naïve generalization of Proposition 6.3 is not true in this case. Moreover, if we attempt to extend this structure to $\mathrm{SV}^3_{s_1s_2s_1}(\Lambda_2 + \Lambda_1)$, then we fail to obtain one of the following two tableaux: $\begin{array}{|c|c|}\hline 1,2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2,3 \\ \cline{1-1}\end{array}$.

Proposition 6.6. *There does not exist a K-crystal structure on* $\mathrm{SV}^3(\Lambda_2 + \Lambda_1)$ *that satisfies (H.2).*

Open Problem 6.7. Does there exist a K-crystal structure satisfying (H.2) for $\mathrm{SV}^n(k\Lambda_i)$?

We believe a more natural condition would be to enforce Proposition 6.3. In this case, we require a weakening of the K-crystal structure, where the K-crystal operators depend on the choice of reduced expression for w_0. For $\lambda = \Lambda_2 + \Lambda_1$ and $n = 3$, we give an example of this weak K-crystal structure for $w_0 = s_1s_2s_1$ by Figure 3. Note that we do not have $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2,3 \\ \cline{1-1}\end{array}$ in the K-Demazure subcrystal $\mathrm{SV}^3_{s_2s_1s_2}(\lambda)$.

Let us focus on the weak K-crystal given in Figure 3. There are a few K-crystal operators that require more care than in the single row and single column cases:

$$f_1^K \begin{array}{|c|c|}\hline 1 & 1,3 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 1,3 \\ \hline 2,3 \\ \cline{1-1}\end{array}, \quad f_1^K \begin{array}{|c|c|}\hline 1 & X \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & X \\ \hline 2,3 \\ \cline{1-1}\end{array}, \qquad f_2^K \begin{array}{|c|c|}\hline 1 & 1,2 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & X \\ \hline 2 \\ \cline{1-1}\end{array}, \quad f_2^K \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2,3 \\ \cline{1-1}\end{array}.$$

A possible construction of a K-crystal on $\mathrm{SV}^n(\lambda)$ is to define a notion of tensor products of K-crystals. Such a tensor product rule should have connected components whose characters are Grothendieck polynomials. Then one could take a connected component of $\mathrm{SV}^n(\Lambda_1)^{\otimes|\lambda|}$ containing a minimal highest weight element of weight λ.

One approach to showing such the tensor product rule would be use the Pieri rule from [10] and construct a bijection $\mathrm{SV}^n(\Lambda_1) \otimes \mathrm{SV}^n(\lambda) \cong \bigoplus_\nu \mathrm{SV}^n(\nu)$, where ν/λ is a single box and $\ell(\nu) \leq n$, that we consider as a K-crystal isomorphism. By using [2, Thm. 5.4], the minimal highest weight elements should have a reading word that is highest weight except we read a box in *increasing* order, as opposed to decreasing order in how we construct the crystal operators. Therefore, the minimal highest weight elements should be of the form $\boxed{i_1 < \cdots < i_k} \otimes T_\lambda$ where $i_1, \dots, i_k$ are rows with addable corners in λ. We expect such an isomorphism to be given by Buch insertion $S \otimes T \mapsto (S \xleftarrow{B} T)$.

Open Problem 6.8. Construct a tensor product rule for (weak) K-crystals.

Recall that the insertion given in [2] does not give an associative product (unlike RSK); see [2, Ex. 4.3]. Thus, we do not expect the tensor product to be associative. Solving Open Problem 5.5 should help as we will want K-jdt to be a K-crystal isomorphism.

Acknowledgements

The authors thank A. Buch, Z. Hamaker, R. Patrias, K. Motegi, H. Nakajima, J. Pan, W. Poh, V. Reiner, K. Sakai, A. Schilling, A. Yong, and P. Zinn-Justin for useful discussions. TS thanks the University of Illinois at Urbana–Champaign for its hospitality in December 2015. This work benefited from computations using SageMath [15].

References

[1] S. Assaf and A. Schilling. "A Demazure crystal construction for Schubert polynomials". *Algebraic Combinatorics* **1**.2 (2018), pp. 225–247. Link.

[2] A. S. Buch. "A Littlewood-Richardson rule for the K-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78. Link.

[3] D. Bump and A. Schilling. *Crystal Bases*. World Scientific Publishing, Hackensack, NJ, 2017.

[4] M. Chmutov, M. Glick, and P. Pylyavskyy. "The Berenstein-Kirillov group and cactus groups". To appear in *J. Comb. Algebra*. 2016. arXiv:1609.02046.

[5] J. L. Espiro and L. Volk. "Crystals from 5-vertex ice models". *J. Lie Theory* **28**.4 (2018), pp. 1119–1136.

[6] T. Ikeda and T. Shimazaki. "A proof of K-theoretic Littlewood-Richardson rules by Bender-Knuth-type involutions". *Math. Res. Lett.* **21**.2 (2014), pp. 333–339. Link.

[7] M. Kashiwara. "On crystal bases of the q-analogue of universal enveloping algebras". *Duke Math. J.* **63**.2 (1991), pp. 465–516. Link.

[8] M. Kashiwara. "The crystal base and Littelmann's refined Demazure character formula". *Duke Math. J.* **71**.3 (1993), pp. 839–858. Link.

[9] A. Lascoux. "Transition on Grothendieck polynomials". *Physics and Combinatorics, 2000 (Nagoya)*. World Sci. Publ., River Edge, NJ, 2001, pp. 164–179. Link.

[10] C. Lenart. "Combinatorial aspects of the K-theory of Grassmannians". *Ann. Comb.* **4**.1 (2000), pp. 67–82. Link.

[11] C. Lenart. "On the combinatorics of crystal graphs. I. Lusztig's involution". *Adv. Math.* **211**.1 (2007), pp. 204–243. Link.

[12] C. Monical, O. Pechenik, and T. Scrimshaw. "Crystal structures for symmetric Grothendieck polynomials". 2018. arXiv:1807.03294.

[13] C. Monical, O. Pechenik, and D. Searles. "Polynomials from combinatorial K-theory". 2018. arXiv:1806.03802.

[14] K. Motegi and K. Sakai. "K-theoretic boson-fermion correspondence and melting crystals". *J. Phys. A: Math. Theor.* **47**.44 (2014), p. 445202. Link.

[15] *Sage Mathematics Software (Version 8.2)*. The Sage Development Team. 2018. Link.

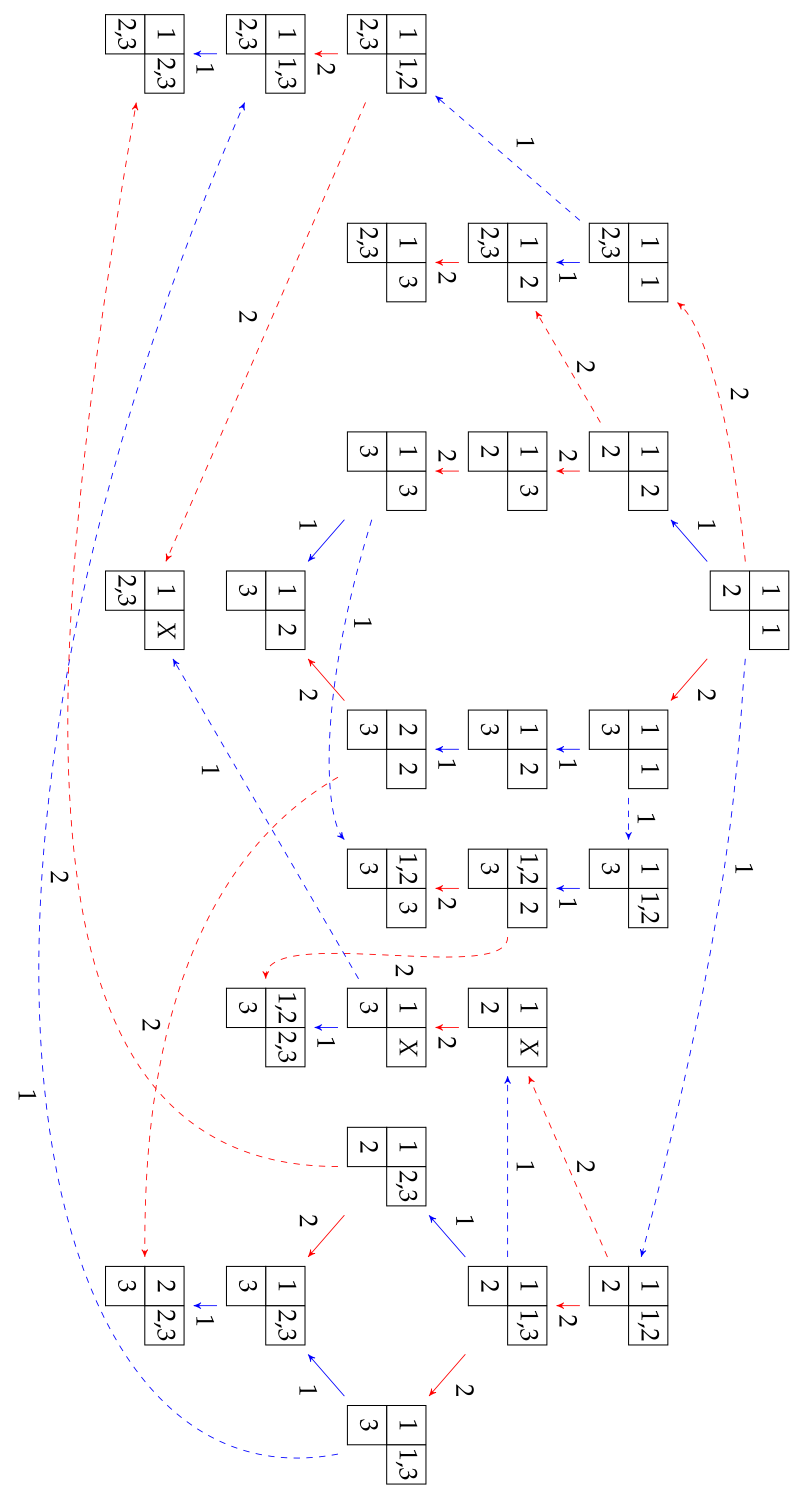


Figure 3: A weak K-crystal structure on $\mathcal{SV}^3(\Lambda_2 + \Lambda_1)$, where $X = \{1, 2, 3\}$, for $w_0 = s_1 s_2 s_1$.