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Semistable reduction in characteristic 0

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Abstract. We address the semistable reduction conjecture of Abramovich and Karu: we prove that every surjective morphism of complex projective varieties can be modified to a semistable one. The key ingredient is a combinatorial result on triangulating lattice Cayley polytopes.

Keywords: semistable reduction, Cayley polytopes, mixed subdivisions

1 Introduction

Motivated by the desire to replace unwieldy morphisms of varieties with sufficiently nice ones, Abramovich and Karu [1] proposed a best possible conjecture.

Conjecture 1.1. *Let $X \to B$ denote a surjective morphism of complex projective varieties with geometrically integral generic fiber. Then there is a projective alteration $B_1 \to B$, and a projective modification $Y \to X \times_B B_1$ such that $Y \to B_1$ is semistable.*

Abramovich and Karu [1] themselves proved a weak version of this conjecture, namely the existence to a weak semistable morphism. These are defined as follows:

A morphism $f : X \to B$ is called *weakly semistable* if

1. the varieties X and B admit toroidal structures $U_X \subset X$ and $U_B \subset B$, with $U_X = f^{-1}U_B$;
2. with this structure, the morphism f is toroidal;
3. the morphism f is equidimensional;
4. all the fibers of the morphism f are reduced; and
5. B is nonsingular.

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However, the case when X is additionally nonsingular, and in which the morphism $f : X \to B$ is called *semistable*, was left as an open conjecture. We build on the combinatorial methods of [2] to resolve Conjecture 1.1 positively. The present manuscript is an extended abstract of the full paper, which can be found here [3].

History

In addition to the work of Abramovich and Karu, a few other cases are known; see [1] for a more detailed history. In particular, we know the cases $\dim B = 1$ and $\dim X = 2$ (see [6]), $\dim B = 1$ (see [10]), and the case of codimension one [1].

De Jong proved the case $\dim X = \dim B + 1$ [8]. Using work of Alexeev, Kollár and Shepherd-Barron (see [4], [5]), one obtains a version of the case $\dim X = \dim B + 2$.

Abramovich and Karu were the first to provide substantial progress to the case $\dim X > \dim B + 2$, explaining their dissatisfaction with the quote "since we do not have a semistable reduction result over a base of higher dimension, we will work around it in the following technical manner...".

Finally, Karu [9] succeeded in another breakthrough: he proved semistable reduction in the case $\dim X - \dim B \leq 3$ based on reduction to combinatorics.

A breakthrough was obtained in the recent work of the three authors and Pak, who proved a local version in [2] here using recent constructions of Haase, Paffenholz, Piechnik and Santos [7]. We expand further on this here, and obtain a complete solution to the conjecture.

2 Proof

2.1 Statement of the conjecture

Following the usual approach, Abramovich and Karu translated Conjecture 1.1 to a combinatorial one. We shall not repeat that translation here, and instead refer to [1] and review the combinatorial statement of the conjecture. We follow the notation and terminology on rational cones, polyhedral complexes, etc., from this paper. Recall that a *rational conical polyhedral complex*, or *conical complex*, is an (abstract) polyhedral complex formed by gluing together finitely many polyhedral cells σ, each of which is equipped with a lattice $N_\sigma \cong \mathbf{Z}^{d_\sigma}$ such that σ is a full-dimensional, rational, strictly convex polyhedral cone in $N_\sigma \otimes \mathbf{R}$, and such that if τ is a face of σ, then $N_\tau = N_\sigma|_{\mathrm{Span}(\tau)}$. We will denote such a conical complex by $\{(\sigma, N_\sigma)\}$. A *map* $f : X \to Y$ of conical complexes $X = \{(\sigma, N_\sigma)\}$ and $Y = \{(\rho, N_\rho)\}$ is a collection of group homomorphisms $f_\sigma : N_\sigma \to N_\rho$, one for each $\sigma \in X$ (and with ρ chosen for each σ), such that the extension $f_\sigma : N_\sigma \otimes \mathbf{R} \to N_\rho \otimes \mathbf{R}$ maps σ into ρ, and such that if τ is a face of σ, then $f_\tau = f_\sigma|_\tau$. A conical complex is *nonsingular* if each of its cells σ is generated by a lattice basis of N_σ.

Definition 2.1. Let X, B be conical complexes and $f : X \to B$ a map of conical complexes. We say that f is *semistable* if the following conditions hold.

1. $f^{-1}(0) = \{0\}$.
2. For every $\sigma \in X$, we have $f(\sigma) \in B$.
3. For every $\sigma \in X$, we have $f(N_\sigma) = N_{f(\sigma)}$.
4. X and B are nonsingular.

Definition 2.2. Let X be a conical complex. A *lattice alteration* is a map $X_1 \to X$ where X_1 is a conical complex of the form $\{(\sigma, N^1_\sigma) : \sigma \in X\}$ and each N^1_σ is a sublattice of N_σ. An *alteration* is a composition $X_1 \to X' \to X$ of a lattice alteration $X_1 \to X'$ with a subdivision $X' \to X$. The alteration is *projective* if the subdivision $X' \to X$ is projective.

Fix a map $f : X \to Y$ of conical complexes. For any alteration $g : Y_1 \to Y$, there is an alteration $X_1 \to X$ with

$$X_1 = \{(\sigma \cap f^{-1}(g(\tau)), N_\sigma \cap f^{-1}(g(N_\tau)) : \sigma \in X, \tau \in Y_1\}$$

which is the unique minimal alteration admitting a map $X_1 \to Y_1$. We call $X_1 \to X$ the alteration *induced* by g (with respect to f). If g is projective, then $X_1 \to X$ is as well.

We can now state the conjecture of Abramovich and Karu.

Conjecture 2.3. *Let $f : X \to B$ be a map of conical complexes with $f^{-1}(0) = \{0\}$. Then there exists a projective alteration $B_1 \to B$, with induced alteration $X_1 \to X$, and a projective subdivision $Y \to X_1$ such that $Y \to B_1$ is semistable.*

2.2 Polytopal complexes

Our next step is to translate the problem from conical complexes to polytopal complexes, which will be more natural for our constructions. A *lattice polytope* is a pair (P, N_P) (usually denoted as just P) where $N_P \cong \mathbf{Z}^{d_P}$ is an affine lattice and P is a polytope in $N_P \otimes \mathbf{R}$ with vertices in N_P. A *face* of a lattice polytope (P, N_P) is a lattice polytope (F, N_F) where F is a face of P and $N_F = N_P|_{\mathrm{AffSpan}(F)}$. A *lattice polytopal complex*, or *polytopal complex*, is a polyhedral complex formed by gluing together finitely many lattice polytopes (with the usual gluing conditions of polyhedral complexes) such that lattices agree on common faces and each polytope P is full-dimensional in N_P. Maps between polytopal complexes are defined analogously to the conical case.

Let (P, N_P) be a lattice polytope. We let L_P denote the affine sublattice of N_P spanned by the vertices of P. We say that the *index* of P is the index

$$[N_P \cap \mathrm{AffSpan}(P) : L_P].$$

A *unimodular simplex* is a lattice simplex of index 1. A *unimodular triangulation* is a polytopal complex all of whose elements are unimodular simplices. If the vertices of P are contained in some sublattice N' of N_P, then we say that the *index of P with respect to* N' is the index of (P, N'). We define unimodularity with respect to N' accordingly.

Usually, we will specify an origin in N_P, which allows us to define dilations of lattice polytopes and Minkowski sums of polytopes with the same lattice. Since all of our results will be invariant under translation, the choice of origin will not matter. If $X = \{(P, N_P)\}$ is a polytopal complex, then we can define a polytopal complex $cX = \{(cP, N_P)\}$. Given a map $f : X \to Y$ of polytopal complexes, there is the obvious induced map $cX \to cY$; we denote this map by cf.

We define lattice alterations and alterations of polytopal complexes analogously to the conical case. For induced alterations, we need to proceed more carefully. Let $f : X \to Y$ be a map of polytopal complexes and $g : Y_1 \to Y$ an alteration. Consider the rational subdivision

$$X_1 := \{(P \cap f^{-1}(g(Q)), N_P \cap f^{-1}(g(N_Q))) : P \in X, Q \in Y_1\}. \tag{2.1}$$

The vertices of X_1 are not necessarily lattice points. However, if they are, then X_1 is a lattice polytopal complex, and $X_1 \to X$ is the unique minimal alteration admitting a map $X_1 \to Y_1$. In this case we say that $X_1 \to X$ is *induced* by g (with respect to f).

We say a map $f : X \to Y$ of polytopal complexes is *good* if for every $P \in X$, we have $f(P) \in Y$. We have the following.

Proposition 2.4. *Let $f : X \to Y$ be a map of polytopal complexes. Then there exists a positive integer c and a projective subdivision $Y' \to cY$ which induces with respect to cf an alteration $X' \to cX$ such that $X' \to Y'$ is good.*

Proof. It is easy to see that there is a rational projective subdivision $\tilde{Y}$ of Y such that for every $P \in X$, we have that $f(X)$ is a union of cells of Y. This induces as in (2.1) a rational subdivision $\tilde{X}$ of X. For some c, we have that $c\tilde{X}$ and $c\tilde{Y}$ are lattice subdivisions of cX and cY, and the map $c\tilde{X} \to c\tilde{Y}$ gives the result. □

Proposition 2.5. *Let $f : X \to Y$ be a good map of polytopal complexes and $Y_1 \to Y$ an alteration. Then there exists a positive integer c such that if $Y_1 \to cY_1$ is the lattice alteration given by $(P, N_P) \cong (cP, cN_P) \mapsto (cP, N_P)$ and g is the alteration $Y_1 \to cY_1 \to cY$, then g induces with respect to cf an alteration $X_1 \to cX$.*

Proof. As in the previous proof, there is c such that $cY_1 \to cY$ induces a lattice subdivision $X' \to cX$. Then the lattice alteration $Y_1 \to cY_1$ induces an alteration $X_1 \to X'$, and $X_1 \to X' \to cX$ is the desired alteration. □

Our goal now is to prove the following theorem:

Theorem 2.6. *Let $f : X \to B$ be a map of polytopal complexes. Then there exists a positive integer c, a projective alteration $B_1 \to cB$ which induces with respect to cf an alteration $X_1 \to cX$, and a projective subdivision $Y \to X_1$ such that Y and B_1 are both unimodular triangulations.*

Proposition 2.7. *Theorem 2.6 implies Conjecture 2.3.*

Proof. Let $\hat{f} : \hat{X} \to \hat{B}$ be a map of conical complexes with $\hat{f}^{-1}(0) = \{0\}$. Then we can construct a map $f : X \to B$ of polytopal complexes such that there are bijections $\hat{X} \to X$ and $\hat{B} \to B$ which preserve the combinatorial structure and such that each cone σ is mapped to the intersection of σ and some hyperplane H_σ. Applying Theorem 2.6, we have a positive integer c, a projective alteration $B_1 \to cB$ with induced alteration $X_1 \to cX$, and a projective subdivision $Y \to X_1$ such that Y and B_1 are unimodular triangulations. Note that Y and B_1 are alterations of the polytopal complexes formed by intersecting each σ in $\hat{X}$ and $\hat{B}$ with the hyperplane cH_σ. Hence, by coning Y and B_1, we have alterations $\hat{Y} \to \hat{X}$ and $\hat{B}_1 \to \hat{B}$, respectively. Since B_1 is unimodular, the map $\hat{Y} \to \hat{B}_1$ satisfies (2) and (3) of Definition 2.1. Since Y is also unimodular, $\hat{Y}$ and $\hat{B}_1$ are nonsingular. □

2.3 Canonical subdivisions

As we will see, much of our proof relies on being able to construct "canonical" subdivisions for polytopes. We now formalize his notion.

An *ordered polytope* is a polytope along with a total order on its vertices. A *face* of an ordered polytope is a face of the underlying polytope along with the induced ordering. Let $\mathcal{P}$ to be the category whose objects are ordered lattice polytopes and whose morphisms are $F \to P$ where F is a face of P. Let $\mathcal{S}$ be the category whose objects are subdivisions of ordered lattice polytopes and whose morphisms are $F' \to P'$ where F' is the subdivision induced on a face of the underlying ordered polytope of P'.

Let $\Gamma : \mathcal{F} \to \mathcal{P}$ be a full and faithful functor for some category $\mathcal{F}$. A *canonical subdivision* of Γ is a functor $\Sigma : \mathcal{F} \to \mathcal{S}$ such that $\Sigma(P)$ is a subdivision of $\Gamma(P)$ for all $P \in \mathrm{Ob}(\mathcal{F})$. If $\Sigma(P)$ is projective for all P, then we call Σ *projective*. If $\Sigma(P)$ is a triangulation for all P, then we call Σ a *canonical triangulation*.

2.3.1 Canonical triangulations of dilated simplices

Let Δ be the category defined as follows. The objects are ordered pairs (P, c) where P is an ordered lattice simplex and c is a positive integer. The morphisms are $(P', c) \to (P, c)$ where P' is a face of P. We have a full and faithful functor $\mu : \Delta \to \mathcal{P}$ defined by $\mu(P, c) = cP$. We will assume the origin is in L_P, so that cP has vertices in L_P.

The following is a key result from Haase et al. [7].

Theorem 2.8. *There is a projective canonical triangulation Σ of μ such that for all $(P, c) \in \mathrm{Ob}(\Delta)$, we have that $\Sigma(P, c)$ is unimodular with respect to L_P.*

Later we will prove a generalization of this to polysimplices, Lemma 2.10. For now, we will state a modified version of this Theorem. Let Δ' be the full subcategory of Δ whose objects are $(P, c) \in \mathrm{Ob}(\Delta)$ with $c \geq \dim(P) + 1$. Let μ' be the restriction of μ to Δ'. Then we have the following.

Lemma 2.9. *There is a projective canonical triangulation Σ' of μ' such that for all $(P, c) \in \mathrm{Ob}(\Delta')$ and all full-dimensional simplices Q of $\Sigma'(P, c)$, we have the following:*

1. *$L_P = L_Q$. (That is, $\Sigma'(P, c)$ is unimodular with respect to L_P.)*
2. *If the vertices of Q are ordered $v_1, v_2, \ldots, v_{\dim(P)+1}$, then for all $i = 1, \ldots, \dim(P)$, every face of P which contains v_i also contains v_{i+1}.*

Proof. Note: This proof uses ideas and notation from the next section. We have put the proof in this section for the sake of organization.

We construct the triangulation $\Sigma'(P, c)$ of cP as follows. For each face F of P, let O_F be the barycenter of F. If F has dimension k and F' is a face of F, then we note that

$$\phi(F', F) := (c - k - 1)F' + (k + 1)O_F$$

is a lattice polytope contained in cF.

Let $d = \dim(P)$. Let $F_r, F_{r+1}, \ldots, F_d$ be a sequence of nonempty faces of P with $F_r < \cdots < F_d$ and $\dim(F_i) = i$ for all $r \leq i \leq d$. We define

$$(cP)_{F_r,\ldots,F_d} := \mathrm{conv} \bigcup_{i=r}^{d} \phi(F_r, F_i).$$

Then the collection of all such $(cP)_{F_r,\ldots,F_d}$ are the full-dimensional cells of a subdivision Σ of cP.

The final step is to refine Σ to a triangulation. When viewed as lattice polytopes in L_P, each $(cP)_{F_r,\ldots,F_d}$ is lattice equivalent to the Cayley polytope

$$\mathcal{C}((c - r - 1)F_r, (c - r - 2)F_r, \ldots, (c - d - 1)F_r).$$

Thus, by Lemma 2.10, there are canonical triangulations of each $(cP)_{F_r,\ldots,F_d}$ which are unimodular with respect to L_P. These extend to a triangulation of Σ which is unimodular in L_P. The fact that this triangulation satisfies property (2) is easy to check, as is canonicity. $\square$

2.4 Cayley polytopes

Let $(P_1, \mathbf{Z}^d)$, $(P_2, \mathbf{Z}^d)$ $\ldots$, $(P_n, \mathbf{Z}^d)$ be lattice polytopes. To simplify some statements later, we will assume that L_{P_j} contains the origin for all j. (Note that we can always translate the P_j so that this holds.) We define the *Cayley polytope* $\mathcal{C}(P_1, \ldots, P_n)$ to be the polytope

$$\mathcal{C}(P_1, \ldots, P_n) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times e_i\right) \subset \mathbf{R}^d \times \mathbf{R}^n$$

where conv denotes convex hull and e_i is the i-th standard basis vector of $\mathbf{R}^n$. We make $\mathcal{C}(P_1, \ldots, P_n)$ a lattice polytope by equipping it with the affine lattice $\mathbf{Z}^d \times \Lambda^{n-1}$, where $\Lambda^{n-1} := \{x \in \mathbf{Z}^n : x_1 + \cdots + x_n = 1\} \cong \mathbf{Z}^{n-1}$.

If $P_1, \ldots, P_n$ are ordered polytopes, then we make $\mathcal{C}(P_1. \ldots, P_n)$ and ordered polytope with the following ordering: First the vertices of $P_1 \times e_1$ in the order given by P_1, then the vertices of $P_2 \times e_2$ in the order given by P_2, and so on. Note that these are precisely the vertices of $\mathcal{C}(P)$.

A *polysimplex*, or *product of simplices*, is a polytope of the form $\sum_i P_i$, where $\{P_i\}$ is an affinely independent set of simplices and $\sum$ denotes Minkowski sum. If $\{P_i\}$ is an ordered set of such simplices and each P_i is an ordered polytope, then we make $\sum_i P_i$ an ordered polytope by lexicographic ordering on its vertices.

Let A be an $m \times n$ matrix with nonnegative integer entries. We define

$$\begin{aligned}\mathcal{C}(P_1, \ldots, P_n, A) &:= \mathcal{C}\left((P_1, \cdots, P_n)A^T\right) \\ &:= \mathcal{C}\left(\sum_{j=1}^{n} A_{1j}P_j, \sum_{j=1}^{n} A_{2j}P_j, \ldots, \sum_{j=1}^{n} A_{mj}P_j\right).\end{aligned}$$

If $P_1, \ldots, P_n$ are affinely independent ordered simplices, then $\mathcal{C}(P_1, \ldots, P_n, A)$ is an ordered polytope as described in the previous two paragraphs. In addition, with the assumption that L_{P_j} contains the origin for all j, the vertices of $\mathcal{C}(P_1, \ldots, P_n, A)$ are contained in $L_{\tilde{P}} \times \Lambda^{m-1}$, where $\tilde{P} := P_1 + \cdots + P_n$.

Let A be as above. For each row A_i of A, let supp A_i denote the set of column indices at which A_i is nonzero. We say that A is *full* if

$$[n] = \operatorname{supp} A_1 \supseteq \operatorname{supp} A_2 \supseteq \cdots \supseteq \operatorname{supp} A_m.$$

2.4.1 Canonical triangulations of polysimplices

We now turn to subdividing Cayley polytopes of polysimplices. Let $\mathcal{F}$ be the category whose objects are tuples $(P_1, \ldots, P_n, A)$ satisfying the following.

1. $P_1, \ldots, P_n$ are ordered lattice simplices with the same lattice.

2. $P_1, \ \ldots, P_n$ are affinely independent.

3. A is an $m \times n$ matrix with nonnegative integer entries.

The morphisms in $\mathcal{F}$ are $(F_1, \ldots, F_n, A') \to (P_1, \ldots, P_n, A)$, where F_i is a face of P_i for all i, and A' is obtained from A by taking a subset of the rows of A. We have a full and faithful functor $\mathcal{C} : \mathcal{F} \to \mathcal{P}$ given by $(P_1, \ldots, P_n, A) \mapsto \mathcal{C}(P_1, \ldots, P_n, A)$.

We generalize Theorem 2.8 as follows.

Lemma 2.10. *There is a projective canonical subdivision Σ of $\mathcal{C}$ such that for all $(P_1, \ldots, P_n, A) \in \mathrm{Ob}(\mathcal{F})$, we have that $\Sigma(P_1, \ldots, P_n, A)$ is unimodular with respect to $L_{\tilde{P}} \times \Lambda^{m-1}$, where $\tilde{P} := P_1 + \cdots + P_n$.*

Proof. See [3]. □

A *box point* of a lattice polytope P is a nonzero element of N_P/L_P. If F is a face of P, then there is a natural inclusion $N_F/L_F \hookrightarrow N_P/L_P$, and so any box point of F can be regarded as a box point of P. Moreover, if we have two polytopes P, Q with $L_P = L_Q$, then we identify the box points of P with the box points of Q.

Let $\mathcal{F}^*$ denote the full subcategory of $\mathcal{F}$ whose objects are $(P_1, \ldots, P_n, A) \in \mathrm{Ob}(\mathcal{F})$ such that

1. For each entry a of A, either $a = 0$ or $a \geq \max_j \dim P_j$.

2. A is full.

Let $\mathcal{C}^*$ be the restriction of $\mathcal{C}$ to $\mathcal{F}^*$.

We now state the main result of this section.

Lemma 2.11. *Let $\mathbf{m}$ be a box point for some polytope. Then there is a projective canonical triangulation $\Sigma^{\mathbf{m}}$ of $\mathcal{C}^*$ such that for all $(P_1, \ldots, P_n, A) \in \mathrm{Ob}(\mathcal{F}^*)$ and all full-dimensional simplices Q in $\Sigma^{\mathbf{m}}(P_1, \ldots, P_n, A)$, we have the following.*

- *If $\mathbf{m}$ is a box point of $\tilde{P} := P_1 + \cdots + P_n$, then*
$$\mathrm{index}(Q) < \mathrm{index}(\tilde{P}).$$

- *If $\mathbf{m}$ is not a box point of $\tilde{P}$, then $L_Q = L_{\tilde{P}} \times \Lambda^{m-1}$.*

Proof. See [3]. □

2.5 Proof of Theorem 2.6

We are now ready to prove Theorem 2.6. By Proposition 2.4, we may assume $f : X \to Y$ is a good map.

2.5.1 Reducing the base

The first step is to alter B so that it is a unimodular triangulation. By the KKMS theorem [10], there is a positive integer c so that we have a projective unimodular triangulation $B' \to cB$. Now, by Proposition 2.5, there is a positive integer c' and an alteration $B' \to c'cB$ such that this alteration induces with respect to $c'cf$ an alteration $X_1 \to c'cX$ and map $X_1 \to B'$. Hence, we may assume B is a unimodular triangulation. By arbitrarily triangulating X, we may assume X is a triangulation.

2.5.2 Lowering the index

Fix a linear order on the vertices of B. By Lemma 2.9, there exists c such that for each $Q \in B$ we have a projective triangulation

$$\Sigma'(Q, c) \to cQ.$$

Since these triangulations are canonical, this gives a triangulation B_1 of cB. By Proposition 2.5, for some c', the alteration $B_1 \to c'B_1 \to c'cB$ induces an alteration $X_1 \to c'cX$ and a map $f_1 : X_1 \to B_1$. We may assume $c' > \dim X$.

Suppose $Q \in B$ and P is a full-dimensional simplex of the complex $f^{-1}(Q)$. Let v_1, $\ldots$, v_n be the vertices of Q, and let

$$P_i := (f^{-1}(v_i) \cap P, N_P)$$

for all i. Note that the P_i are affinely independent simplices and $P = \mathcal{C}(P_1, \ldots, P_n)$.

Let v be a vertex of B_1 contained in cQ. Define

$$P(v) := f_1^{-1}(v) \cap c'cP.$$

Let $(a_1, \ldots, a_n)$ be the barycentric coordinates of v with respect to the vertices cv_1, $\ldots$, cv_n of cQ. Since Q is unimodular, ca_1, $\ldots$, ca_n are nonnegative integers. From the definition of X_1, we have

$$\begin{aligned} P(v) &= a_1P(cv_1) + a_2P(cv_2) + \cdots + a_kP(cv_n) \\ &= c'c(a_1P_1 + a_2P_2 + \cdots + a_nP_n). \end{aligned}$$

Thus, if $Q_1 \in B_1$ has vertices u_1, $\ldots$, u_m, we have

$$\begin{aligned} f_1^{-1}(Q_1) \cap c'cP &= \mathcal{C}(P(u_1), \ldots, P(u_m)) \\ &= \mathcal{C}(P_1, \ldots, P_n, A) \end{aligned}$$

where A is an $m \times n$ matrix of nonnegative integers divisible by c'. By Lemma 2.9, A is full. Hence $(P_1, \ldots, P_n, A) \in \mathrm{Ob}(\mathcal{F}^*)$.

Let $\mathbf{m}$ be a box point of some $P^0 \in X$. By Lemma 2.11, we have a projective triangulation $Y \to X_1$ where each $P' \in X_1$ is triangulated into $\Sigma^{\mathbf{m}}(P')$. Hence, every simplex Q of Y has either $L_Q = L_P$ for some simplex $P \neq P^0$ of X or $\mathrm{index}(Q) < \mathrm{index}(P)$ where P is a simplex of X and $\mathbf{m}$ is a box point of P.

Now repeat the process of Section 2.5.2 with Y instead of X. Each time we do this procedure, we lower the indices of some of the lattices spanned by elements of X while keeping the other lattices the same. Eventually all lattices will be unimodular, completing the proof.

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