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Triangulations of the product of spheres with few vertices

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Abstract. A small triangulation of sphere products can be found in lower dimensional cases by computer search and is known in few other cases: Klee and Novik constructed a balanced triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $3d$ vertices and a centrally symmetric triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ with $2d+2$ vertices for all $d \geq 3$ and $1 \leq i \leq d-2$. In this paper, we provide an alternative centrally symmetric $(2d+2)$-vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. We also construct the first balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices, using a sphere decomposition inspired by handle theory.

Keywords: sphere products, minimal triangulations of manifolds, balanced complexes, centrally symmetric complexes

1 Introduction

Minimal triangulations of manifolds are an important research object in combinatorial and computational topology. What is the minimal number of vertices required to triangulate a given manifold? How do we construct a vertex-minimal triangulation and is this triangulation unique?

In this paper, we focus on the triangulation of sphere products. From a result of [2], it is known that a combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ has at least $2d-i+2$ vertices. In 1986, [7] constructed a triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $2d+1$ vertices for odd d. Later, two groups of researchers, [1] as well as [3], found in 2008 that Kühnel's construction is indeed the unique minimal triangulation for odd d. For even d, they showed that the minimal triangulation requires $2d+2$ vertices and is not unique.

The minimal triangulations of other sphere products are less well-understood. The best general result is from [4], where a centrally symmetric triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ with $2d+2$ vertices is constructed as a subcomplex of the boundary of the $(d+1)$-cross-polytope. In general, a result of [2] states that a triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ requires at least $i+2j+4$ vertices for $i \geq j$. In addition, the minimal triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \leq 6$ as well as the minimal triangulation of $\mathbb{S}^3 \times \mathbb{S}^3$ are found by the computer

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program BISTELLAR [8], which has shown that this lower bound is not always tight, as a triangulation of $\mathbb{S}^2 \times \mathbb{S}^2$ requires at least 11 vertices. In this paper, we give an alternative centrally symmetric $(2d+2)$-vertex triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for all $d \geq 5$. The construction is based on finding two shellable balls in the d-sphere whose intersection triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$, where $\mathbb{D}^{d-2}$ is the $(d-2)$-dimensional disk. By an inductive argument, we also obtain the triangulation of other sphere products in higher dimensions, see Section 3.2.

In recent years, balanced triangulated manifolds have caught much attention. A $(d-1)$-dimensional simplicial complex is *balanced* provided that its graph is d-colorable. Many important classes of complexes arise as balanced complexes, such as barycentric subdivisions of regular CW complexes and Coxeter complexes. As taking barycentric subdivisions of a complex would generate a lot of new vertices, one would ask if there is a more efficient way to construct the balanced triangulated manifold from a non-balanced one.

In much of the same spirit as Kühnel's construction, [5] provided a balanced triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $3d$ vertices for odd d and with $3d+2$ vertices otherwise. Furthermore, [10] showed that the number of vertices for the minimal triangulation is indeed $3d$ for odd d and $3d+2$ otherwise. However, as of yet, no small balanced triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ for $2 \leq i \leq d-3$ exist in literature. In this paper, we construct the first balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices. The construction uses a sphere decomposition inspired by handle theory.

The extended abstract is structured as follows. In Section 2, we review the basics of simplicial complexes, balanced triangulations, and other relevant definitions. In Section 3, we present our centrally symmetric $(2d+2)$-vertex triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ and construct other sphere products inductively. In Section 4, the balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices is constructed, followed by a discussion of its properties.

2 Preliminaries

A *simplicial complex* Δ with vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, such that for every $v \in V$, $\{v\} \in \Delta$. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximum dimension of the faces of Δ. A face $\sigma \in \Delta$ is said to be a *facet* provided that it is a face which is maximal with respect to inclusion. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension. If Δ is $(d-1)$-dimensional and $-1 \leq i \leq d-1$, then the *f-number* $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. The *star* and *link* of a face σ in Δ is defined as follows:

$$\mathrm{st}(\sigma, \Delta) := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}, \quad \mathrm{lk}(\sigma, \Delta) := \{\tau \in \mathrm{st}_\Delta \sigma : \tau \cap \sigma = \emptyset\}.$$

When the context is clear, we may simply denote the star and link of σ as $\mathrm{st}(\sigma)$ and $\mathrm{lk}(\sigma)$ respectively. We also define the *restriction* of Δ to a vertex set W as $\Delta[W] := \{\sigma \in \Delta : \sigma \subseteq W\}$. A subcomplex $\Omega \subset \Delta$ is said to be *induced* provided that for all faces $F \in \Delta$, if every vertex $v \in F$ is a vertex of Ω, then F is a face in Ω. The *i-skeleton* of a simplicial complex Δ is the subcomplex containing all faces of Δ which have dimension at most i. In particular, the 1-skeleton of Δ is the graph of Δ.

Denote by σ^d the d-simplex. A *combinatorial* $(d-1)$*-sphere* (respectively, a *combinatorial* $(d-1)$*-ball*) is a simplicial complex PL homeomorphic to $\partial\sigma^d$ (respectively, σ^{d-1}). A closed *combinatorial* $(d-1)$*-manifold* is a connected simplicial complex with the property that the link of each vertex is a combinatorial $(d-2)$-sphere. A simplicial complex Δ is a *simplicial manifold*, if the geometric realization of Δ is homeomorphic to a manifold. The boundary complex of a simplicial d-ball is a simplicial $(d-1)$-sphere. In general, a simplicial manifold need not be combinatorial.

A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if the graph of Δ is d-colorable; that is, there exists a coloring map $\kappa : V \to \{1, 2, \cdots, d\}$ such that $\kappa(x) \neq \kappa(y)$ for all edges $\{x, y\} \in \Delta$. A simplicial complex is *centrally symmetric* or *cs* if it is endowed with a free involution $\alpha : V(\Delta) \to V(\Delta)$ that induces a free involution on the set of all non-empty faces.

Let ∂C_d^* be the boundary complex of the d-cross-polytope. It is cs, and furthermore, it is a balanced vertex-minimal triangulation of the $(d-1)$-sphere. Label the vertex set of ∂C_d^* as $\{x_1, \dots, x_d, y_1, \dots, y_d\}$ such that x_i, y_i form a pair of antipodal vertices for all i. Every facet of ∂C_d^* can be written in the form $u_1 u_2 \dots u_d$, where each $u_i \in \{x_i, y_i\}$. We say a facet has a switch at position i if u_i and u_{i+1} have different labels. Let $B(i, d)$ be the pure subcomplex of ∂C_d^* that contains all facets with at most i switches. For example, $B(0, d)$ consists of the two disjoint facets $\{x_1, \dots, x_d\}$ and $\{y_1, \dots, y_d\}$. If Γ is a subcomplex of ∂C_d^*, we let the *complement* of Γ in ∂C_d^* be the complex generated by those facets that are not in Γ. Denote by $\mathcal{D}_d$ the dihedral group of order $2d$.

The following lemma is essentially Theorem 1.2 in [4].

Lemma 2.1. *For $0 \leq i < d-1$, the complex $B(i, d)$ satisfies the following properties:*

1. *$B(i, d)$ contains the entire i-skeleton of ∂C_d^* as a subcomplex.*
2. *The boundary of $B(i, d)$ is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.*
3. *$B(i, d)$ is a balanced centrally symmetric combinatorial manifold whose integral (co)homology groups coincide with those of $\mathbb{S}^i$. Also, $B(0, d) \cong \mathbb{D}^{d-1} \times \mathbb{S}^0$ and $B(1, d) \cong \mathbb{D}^{d-2} \times \mathbb{S}^1$.*
4. *The complement of $B(i, d)$ in ∂C_d^* is simplicially isomorphic to $B(d-i-2, d)$.*
5. *B(i,d) admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_d$ if i is even and of $\mathcal{D}_{2d}$ if i is odd.*

Finally, we define shellability.

Definition 2.2. Let Δ be a pure d-dimensional simplicial complex. A *shelling* of Δ is a linear ordering of the facets $F_1, F_2, \dots, F_s$ such that $F_k \cap (\cup_{i=1}^{k-1} F_i)$ is a pure $(d-1)$-dimensional complex for all $2 \le k \le s$, and Δ is called *shellable* if it has a shelling.

3 The cs triangulations of the sphere products

It is known that for $i \le j$, the minimal triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ requires at least $i + 2j + 4$ vertices, see [2]. Such triangulations are constructed by [8] in lower dimensional cases but not known in general. We aim at finding an alternative triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $2d+2$ vertices for $d \ge 5$. The following theorem is Theorem 7 in [6].

Theorem 3.1. *Let M be a simply connected codimension-1 submanifold of $\mathbb{S}^d$, where $d \ge 5$. If M has the homology of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ and $1 < i \le \frac{d-1}{2}$, then M is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.*

Proposition 3.2. *Fix d and $i \le \frac{d-1}{2}$. Let D_1 and D_2 be two combinatorial d-balls such that*

1. *$\partial(D_1 \cup D_2)$ is $(d-1)$-dimensional submanifold of a combinatorial d-sphere.*
2. *$D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ is a path-connected combinatorial $(d-1)$-manifold (with boundary) that has the same homology as $\mathbb{S}^{i-1}$.*
3. *$\partial(D_1 \cap D_2)$ has the same homology as $\mathbb{S}^{i-1} \times \mathbb{S}^{d-i-1}$.*

Then $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ for $d \ge 5$.

Proof: First note that $D_1 \cup D_2$ is the union of two combinatorial d-balls that intersect along the combinatorial $(d-1)$-manifold $D_1 \cap D_2$. Hence $D_1 \cup D_2$ is a combinatorial d-manifold, and $\partial(D_1 \cup D_2)$ is a combinatorial $(d-1)$-manifold.

Since $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$, we have that the intersection of $\overline{\partial D_1 \backslash \partial D_2}$ and $D_1 \cap D_2$ is exactly $\partial(D_1 \cap D_2)$. We apply the Mayer-Vietoris sequence on $(\overline{\partial D_1 \backslash \partial D_2}, D_1 \cap D_2, \partial D_1)$, and by condition (2) we obtain that $\overline{\partial D_1 \backslash \partial D_2}$ has the same homology as $\mathbb{S}^{d-i-1}$.

Note that the intersection of

$$(\overline{\partial D_1 \backslash \partial D_2}) \cap (\overline{\partial D_2 \backslash \partial D_1}) = \partial(\partial D_1 \cap \partial D_2) = \partial(D_1 \cap D_2).$$

We then apply the Mayer-Vietoris sequence to $(\overline{\partial D_1 \backslash \partial D_2}, \overline{\partial D_2 \backslash \partial D_1}, \partial(D_1 \cup D_2))$. By condition (3), we find that $\partial(D_1 \cup D_2)$ has the same homology as $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.

Finally, the complex $D_1 \cup D_2$ is simply connected, since the union of two simply connected open subsets $\mathrm{int} D_1, \mathrm{int} D_2$ with path-connected intersection $D_1 \cap D_2$ is simply connected. We conclude from condition (1) and Theorem 3.1 that $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. □

The above proposition provides us with a general method of constructing a triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.

3.1 A triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$

Let τ be a face of ∂C_d^* and let $\kappa(\tau)$ count the number of y labels in τ. Define Γ_j to be the union of facets τ in ∂C_d^* that have at most 2 switches and with $\kappa(\tau) = j$. Hence for $1 \leq j \leq d-1$, the complex Γ_j consists of d facets $\tau_j^k = \{x_1, \dots, x_d\} \backslash \{x_k, \dots x_{k+j-1}\} \cup \{y_k, \dots, y_{k+j-1}\}$ for $1 \leq k \leq d$.

Lemma 3.3. *The complex $\cup_{k=0}^{i} \Gamma_k$ is a shellable $(d-1)$-ball for all $0 \leq i \leq \left\lceil \frac{d+1}{2} \right\rceil$.*

Proof: The $(d-1)$-ball $\cup_{k=0}^{i} \Gamma_k$ has a shelling order $\{x_1 \dots . x_d\}, \tau_1^1, \dots, \tau_1^d, \dots \tau_i^1, \dots, \tau_i^d$. □

We propose the candidates $D_1, D_2 \subseteq \partial C_{d+1}^*$ that satisfy the conditions in Proposition 3.2.

Construction 3.4. For $d \geq 3$, define two simplicial d-balls D_1, D_2 as a subcomplex of the octahedral d-sphere on vertex set $\{x_1, y_1, \dots, x_{d+1}, y_{d+1}\}$ as follows:

1. For d is odd, let $m = \frac{d-1}{2}$. Define $D_1 = (\cup_{k=0}^{m+1} \Gamma_k) * \{x_{d+1}\}$ and $D_2 = (\cup_{k=m}^{d} \Gamma_k) * \{y_{d+1}\}$. In particular, $D_1 \cap D_2 = \Gamma_m \cup \Gamma_{m+1}$ is cs.
2. For d is even, let $m = \frac{d}{2}$ and $\gamma := \cup_{i=1}^{m} \tau_{m-1}^i$ be a subcomplex of Γ_{m-1}. By the definition, τ_j^k and τ_{d-j}^{k+j} are antipodal facets for any k, j. So $-\gamma = \cup_{i=m}^{d-1} \tau_{m+1}^i \subseteq \Gamma_{m+1}$. In this case we let

$$D_1 = \left((\cup_{k=0}^{m} \Gamma_k) \cup (-\gamma)\right) * \{x_{d+1}\}, \quad D_2 = \left((\cup_{k=m}^{d} \Gamma_k) \cup \gamma\right) * \{y_{d+1}\}.$$

   In particular, $D_1 \cap D_2 = \Gamma_m \cup \gamma \cup (-\gamma)$ is centrally symmetric.

Next we show that $\partial(D_1 \cap D_2) \cong \mathbb{S}^1 \times \mathbb{S}^{d-3}$. Given two facets $F_1, F_2 \in \partial C_d^*$, let $d(F_1, F_2)$ be the distance from F_1 to F_2 in the facet-ridge graph of ∂C_d^*.

Lemma 3.5. *Let Δ be a combinatorial $(d-1)$-manifold in ∂C_d^* whose facet-ridge graph is a $2d$-cycle. Enumerate its facets as $\sigma_1, \sigma_2, \dots \sigma_{2d}$ such that σ_i, σ_{i+1} are adjacent for $1 \leq i \leq 2d$. If $\sigma_i = -\sigma_{d+i}$ for all i, then Δ triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$.*

Proof: Let $\sigma_1 = \{u_1, \dots, u_d\}$. By the assumption, $\sigma_{d+1} = \{-u_1, \dots, -u_d\}$. Since $d(\sigma_1, \sigma_{d+1}) = d$ in ∂C_d^*, the sequence $\sigma_1, \sigma_2, \dots, \sigma_{d+1}$ gives the shortest path from σ_1 to σ_{d+1}. So it follows that there is an ordering of the vertices, say $(u_1, \dots, u_d)$, such that $\sigma_{i+1} = \sigma_i \backslash \{u_i\} \cup \{-u_i\}$. Together with $\sigma_i = -\sigma_{i+d}$ for all i, we see that $\Delta \cong B(1, d)$ as defined in [4]. Hence as $B(1, d)$, Δ also triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$. □

Lemma 3.6. *The complex $D_1 \cap D_2$ constructed above triangulates $\mathbb{S}^1 \times \mathbb{D}^{d-2}$.*

Proof: For odd d and $m = \frac{d-1}{2}$, we enumerate the facets of $D_1 \cap D_2 = \Gamma_m \cup \Gamma_{m+1}$ as $(\sigma_1 \dots, \sigma_{2d}) :=$

$$(\tau_m^1, \tau_{m+1}^1, \tau_m^2, \tau_{m+1}^2, \dots, \tau_m^d, \tau_{m+1}^d).$$

Each σ_i has exactly two adjacent facets $\sigma_{i-1}, \sigma_{i+1}$, and so the facet-ridge graph of $D_1 \cup D_2$ is a $2d$-cycle. Furthermore, $\tau_m^j = -\tau_{m+1}^{j+m}$ by the definition. So the claim follows from Lemma 3.5. The proof is similar for d is even. □

Theorem 3.7. *The complex $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \geq 5$.*

Proof: This follows from Lemmas 3.3 and 3.6 and Proposition 3.2. □

Property 3.8. *For the complex $\partial(D_1 \cup D_2)$ in Construction 3.4:*

1. *It has $2d + 2$ vertices.*
2. *It contains the 2-skeleton of ∂C_{d+1}^*.*
3. *It admits vertex-transitive actions by the group $\mathbb{Z}_2 \times \mathcal{D}_d$ if d is odd, and by $\mathbb{Z}_2$ if d is even.*

Remark 3.9. For d is odd, three types of vertex-transitive action on $\partial(D_1 \cup D_2)$ are given by

- D maps x_j to y_j, and y_j to x_j, for $1 \leq j \leq d+1$.
- R fixes x_{d+1}, y_{d+1}, and maps x_j, y_j to x_{d-j+1}, y_{d-j+1} respectively, for $1 \leq j \leq d$.
- S fixes x_{d+1}, y_{d+1}, and maps x_j, y_j to x_{j+1}, y_{j+1} (modulo d) respectively.

Comparing with the group actions on $B(2, d+1)$ in [4], we see that $\partial(D_1 \cup D_2)$ and $\partial B(2, d+1)$ are combinatorially distinct.

Remark 3.10. There are many other ways to construct D_1', D_2' as the subcomplex of ∂C_{d+1}^* that satisfies the conditions in Proposition 3.2. For example, when $d = 2m+1$, let $\tau = \{x_1, \dots x_d\}$, $-\tau = \{y_1, \dots, y_d\}$. It is possible to construct a simplicial $(d-1)$-ball B in ∂C_d^* and simplicial d-balls

$$D_1' = (B \cup \Gamma_m \cup \Gamma_{m+1}) * \{x_{d+1}\},\ D_2' = \Big((-B) \cup \Gamma_m \cup \Gamma_{m+1}\Big) * \{y_{d+1}\}$$

such that

$$B \cup (-B) = \partial C_d^*,\ \ B \supseteq \{\sigma : |\sigma \cap \tau| \geq m\},\ \ -B = \{\sigma : |\sigma \cap (-\tau)| \geq m\}.$$

Furthermore, $D_1' \cap D_2' = \Gamma_m \cup \Gamma_{m+1}$. When $d = 2m$, D_1', D_2' can also be defined in the same spirit as Construction 3.4.

3.2 The triangulation of other sphere products

The goal of this section is to construct a triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ as a subcomplex of ∂C^*_{d+2} from a given triangulation of $\mathbb{S}^{i-1} \times \mathbb{S}^{d-i-1}$ in ∂C^*_{d+1}, for $i \leq \frac{d-1}{2}$.

Proposition 3.11. *Let D_1 and D_2 be cones over two combinatorial $(d-1)$-balls in ∂C^*_d whose coning points are x_{d+1}, y_{d+1} respectively. Furthermore,*

1. *The union of $D_1 \backslash \{x_{d+1}\}$ and $D_2 \backslash \{y_{d+1}\}$ covers ∂C^*_d.*
2. *$D_1 \cap D_2$ is a path-connected combinatorial $(d-1)$-manifold that has the same homology as $\mathbb{S}^{i-1}$ for some $2 \leq i \leq d-2$.*
3. *$\partial(D_1 \cap D_2)$ has the same homology as $\mathbb{S}^{i-1} \times \mathbb{S}^{d-i-1}$.*

Let

$$E_1 = (\mathrm{st}(y_{d+1}, \partial C^*_{d+1}) \cup D_1) * \{x_{d+2}\}, \quad E_2 = (\mathrm{st}(x_{d+1}, \partial C^*_{d+1}) \cup D_2) * \{y_{d+2}\}.$$

Then the union of $E_1 \backslash \{x_{d+2}\}$ and $E_2 \backslash \{y_{d+2}\}$ covers ∂C^*_{d+1}, $E_1 \cap E_2$ is a combinatorial d-manifold that has the same homology as $\mathbb{S}^i$ and $\partial(E_1 \cup E_2)$ triangulates $\mathbb{S}^{i+1} \times \mathbb{S}^{d-i-1}$.

Proof: By condition (1), $E_1 \cap E_2 = D_1 \cup D_2$. Then we use the Mayer-Vietoris sequence on the triple $(D_1, D_2, D_1 \cup D_2)$ and conclude from Theorem 3.1. The proof is similar to that of Proposition 3.2. □

Construction 3.12. We take our base construction $\partial(D'_1 \cup D'_2)$ as given in Remark 3.10 (Note that the complexes D_1, D_2 in Construction 3.4 does not satisfy the condition that the union of $D_1 \backslash \{x_{d+1}\}$ and $D_2 \backslash \{y_{d+1}\}$ covers ∂C^*_d.) and apply the above proposition inductively. This gives us a family of cs triangulations of sphere products $\mathbb{S}^2 \times \mathbb{S}^{d-3}$, $\mathbb{S}^3 \times \mathbb{S}^{d-3}, \ldots, \mathbb{S}^{d-3} \times \mathbb{S}^{d-3}$. Each triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-3}$ has $2d + 2i - 2$ vertices.

4 A balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$

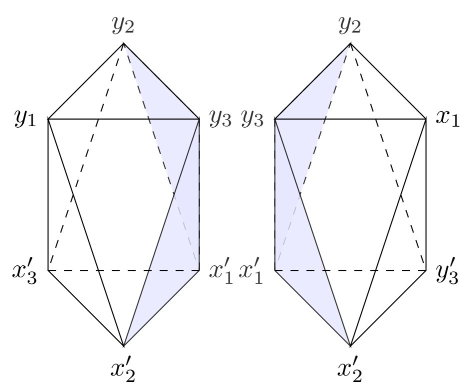
In this section, we present our main construction for a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. The geometric intuition of our construction comes from handle theory. The sphere $\mathbb{S}^d$ admits the following decomposition, see [9]:

$$\mathbb{S}^{d-1} = (\mathbb{S}^1 \times \mathbb{D}^{d-2}) \cup (\mathbb{D}^2 \times \mathbb{S}^{d-3}).$$

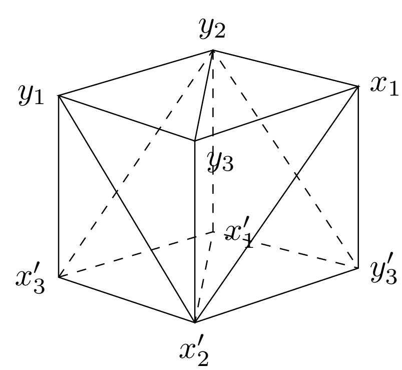
Let S be a triangulated $(d-1)$-sphere that has the decomposition $S = B_1 \cup_{\partial B_1 = \partial B_2} B_2$, where $B_1 \cong \mathbb{S}^1 \times \mathbb{D}^{d-2}$, $B_2 \cong \mathbb{D}^2 \times \mathbb{S}^{d-3}$, and $\partial B_1 \cong \partial B_2 \cong \mathbb{S}^1 \times \mathbb{S}^{d-3}$. Note that $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ admits the decomposition into $(\mathbb{D}^2 \times \mathbb{S}^{d-3}) \cup (\mathbb{D}^2 \times \mathbb{S}^{d-3}) \cong B_2 \cup B_2$. Then, from S we can form a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-2}$ in the following way: take two copies of B_2 and denote them as B_2 and B_2'. If ∂B_2 is an induced subcomplex in B_2, then we glue B_2 and B_2' along their boundaries. The resulting complex is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. However, if ∂B_2 is not an induced subcomplex of B_2, then usually we cannot glue B_2 and B_2' by identifying their boundaries directly and still obtain a triangulated manifold. An alternative method is to find a complex $N \cong \partial B_2 \times \mathbb{D}^1$ with $\partial N = \partial B_2 \cup \partial B_2'$ so that N serves as a tubular neighborhood of both ∂B_2 and $\partial B_2'$. Finally the complex $B_2 \cup N \cup B_2'$ is a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$.

Our approach of constructing a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ is by finding suitable balanced candidates of B_2 and N as described above.

Definition 4.1. Consider (Γ_1, σ_1) and (Γ_2, σ_2), where Γ_i is the boundary complex of the d-cross-polytope, and σ_i is a fixed facet of Γ_i. Let κ be the coloring map on $\Gamma_1 \cup \Gamma_2$. If e_i is an edge in Γ_i but not in $\pm\sigma_i$ and $\kappa(e_1) = \kappa(e_2)$, then the $\diamond$-connected sum $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ is obtained by deleting e_i from Γ_i, and gluing $\Gamma_1 - e_1$ with $\Gamma_2 - e_2$ by identifying $\mathrm{st}(e_1)[V(\sigma_1)]$ with $\mathrm{st}(e_2)[V(\sigma_2)]$, and $\mathrm{st}(e_1)[V(-\sigma_1)]$ with $\mathrm{st}(e_2)[V(-\sigma_2)]$.



(a) Γ_1 and Γ_2; here $\sigma_1 = \{y_1, y_2, y_3\}$ and $\sigma_2 = \{x_1, y_2, y_3\}$



(b) $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$

Figure 1: The $\diamond$-connected sum $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$: delete the edge $\{y_3 x_1'\}$ in both Γ_1 and Γ_2, then glue Γ_1 and Γ_2 along the 4-cycle (y_3, x_2', x_1', y_2).

The following properties of the $\diamond$-connected sum justify the notation $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ in the definition.

Property 4.2. *Let Γ_1 and Γ_2 be two d-crosspolytopes. Furthermore, Γ_1 has antipodal facets $\sigma_1 = \{x_1, \ldots, x_d\}$, $-\sigma_1 = \{y_1, \ldots, y_d\}$, and Γ_2 has antipodal facets $\sigma_2 = \{x_{d+1}, \ldots, x_{2d}\}$, $-\sigma_2 = \{y_{d+1}, \ldots, y_{2d}\}$. Then $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ satisfies the following properties:*

1. *The complex is a balanced triangulation of $\mathbb{S}^{d-1}$.*

2. *The restriction of* $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ *to* $V(\sigma_1) \cup V(\sigma_2)$ *is the usual connected sum of simplices* $\sigma_1 \# \sigma_2$.

3. *The link of every edge* $e = \{x_i, y_j\}$ *in* $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ *is the boundary complex of a* $(d-2)$*-crosspolytope.*

The above properties ensure that it is possible to take the $\diamond$-connected sum inductively. Also recall that if Γ is a pure simplicial complex, then as long as there exist two facets F and F' on Γ and a map $\phi : F \to F'$ so that v and $\phi(v)$ do not have a common neighbor for every $v \in F$, then we can remove F, F' and identify ∂F with $\partial F'$ to get Γ^{ϕ}. This is called a *handle addition*. Similarly, assume that there are two edges e_1 and e_2 of the same color in $(\Gamma_1 \# \ldots \# \Gamma_k, \sigma_1 \# \ldots \# \sigma_k)$ but not in $A := \sigma_1 \# \sigma_2 \ldots \# \sigma_k$ or $-A$. Note that $\mathrm{st}(e_i)$ is a cross-polytope with antipodal facets $\mathrm{st}(e_i)[V(A)]$ and $\mathrm{st}(e_i)[V(-A)]$ for $i = 1, 2$. If the identification maps

$$\phi : \mathrm{st}(e_1)[V(A)] \to \mathrm{st}(e_2)[V(A)] \quad \text{and} \quad \phi' : \mathrm{st}(e_1)[V(-A)] \to \mathrm{st}(e_2)[V(-A)]$$

are well-defined, then the maps ϕ and ϕ' naturally extend to a map

$$\bar{\phi} : \mathrm{st}(e_1) \to \mathrm{st}(e_2)$$

if for every $v \in \mathrm{st}(e_1)$, v and $\phi(v)$(or $\phi'(v)$) do not have a common neighbor. In this way we obtain a balanced simplicial complex $((\Gamma_1 \# \Gamma_2 \ldots \# \Gamma_k)^{\bar{\phi}}, (\sigma_1 \# \sigma_2 \ldots \# \sigma_k)^{\phi})$ by removing e_1, e_2 and identifying $\mathrm{lk}(e_1)$ with $\bar{\phi}(\mathrm{lk}(e_1)) = \mathrm{lk}(e_2)$. We call this the $\diamond$-handle addition.

We are now ready to construct a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices. We will write $\Gamma_1 \# \Gamma_2$ to denote the $\diamond$-connected sum if σ_1 and σ_2 are clear from the context. Also, to simplify notation, we will sometimes write $x_1 \ldots x_m$ to denote the face $\{x_1, \ldots, x_m\}$.

Construction 4.3. Let $d \geq 3$. Take two d-crosspolytopes P and P'. The vertex sets of P and P' are $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ and $\{x'_1, \ldots, x'_d, y'_1, \ldots, y'_d\}$ respectively. We let $\sigma_i = x_1 \ldots x_i y_{i+1} \ldots y_d$ for $1 \leq i \leq d$ and let $\sigma_i = y_1 \ldots y_i x_{i+1} \ldots x_d$ for $d+1 \leq i \leq 2d$. Then the complex $\Delta_1 := \cup_{i=1}^{2d} \sigma_i$ is exactly $B(1, d)$. We further partition the boundary of P as $\partial P = \Delta_1 \cup_{\partial \Delta_1} \Delta_2$. By Lemma 1.1, $\Delta_2 \cong B(d-3, d)$ and $\Delta_1 \cap \Delta_2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{d-3}$.

Next, define a simplicial map $f : \partial P \to \partial P'$ induced by the following bijection on the vertex sets:

$$x_i \mapsto x'_{i+1},\ y_i \mapsto y'_{i+1} \ \text{ for } 1 \leq i \leq d-1;\ x_d \mapsto y'_1,\ y_d \mapsto x'_1.$$

By Lemma 2.1, the complex Δ_1 admits a vertex-transitive action by the dihedral group $\mathcal{D}_{2d}$ of order $4d$, where a generator is given by the map we have chosen (Theorem 1.2 of [4]). Hence f is a simplicial isomorphism and $f(\Delta_1) \cong B(1, d)$. For each i, there is a unique d-cross-polytope Γ_i containing σ_i and $f(\sigma_i)$ as antipodal facets. Next, we check that we can take the $\diamond$-connected sum of Γ_i and Γ_{i+1} inductively. Without loss of generality, assume that $1 \leq i \leq d$; otherwise, we can relabel by switching x and y. Note that for $i \leq d-2$,

$$\sigma_i \cap \sigma_{i+1} = x_1 x_2 \dots x_i y_{i+2} \dots y_d, \text{ and } f(\sigma_i) \cap f(\sigma_{i+1}) = x'_2 x'_3 \dots x'_{i+1} y'_{i+3} \dots y'_d y'_1.$$

The missing indices are $i+1$ and $i+2$ respectively, so we let $e_i = x'_{i+1} y_{i+2}$. It follows that $\Gamma_i \cap \Gamma_{i+1} = \mathrm{st}(e_i, \Gamma_i) = \mathrm{st}(e_i, \Gamma_{i+1})$ and hence the $\diamond$-connected sum is well defined. Similarly, $\Gamma_{d-1} \cap \Gamma_d = \mathrm{st}(\{x'_d, x_1\}, \Gamma_d)$ and $\Gamma_d \cap \Gamma_{d+1} = \mathrm{st}(\{y'_1, x_2\}, \Gamma_d)$. Inductively, we form a complex $\Gamma = ((\Gamma_1 \# \Gamma_2 \dots \# \Gamma_{2d})^{\bar{\phi}}, \Delta_1)$ which contains Δ_1 and $f(\Delta_1)$ as subcomplexes.

We partition Γ as $\Gamma = \Delta_1 \cup f(\Delta_1) \cup N$, so that $N \cap \Delta_1 = \partial \Delta_1$ and $N \cap f(\Delta_1) = \partial f(\Delta_1)$. N is then the tubular neighborhood that we would like to construct. Finally, let $\Sigma = \Delta_2 \cup_{\partial \Delta_1} N \cup_{\partial f(\Delta_1)} f(\Delta_2)$. (This is well defined as by Lemma 2.1, $\partial \Delta_1 \cong \partial \Delta_2$.) As shown in Figure 2, when $d = 3$, σ gives the minimal balanced triangulation of $\mathbb{S}^0 \times \mathbb{S}^2$.

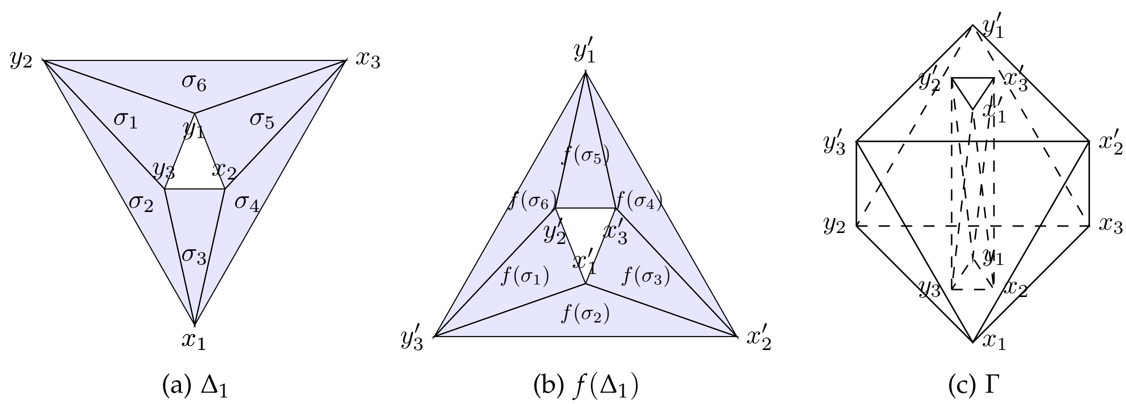


Figure 2: The complexes Δ_1 and $f(\Delta_1)$ when $d = 3$, and the resulting Γ constructed using the previously described sequence of connected sums.

Next to prove Σ indeed triangulates $\mathbb{S}^2 \times \mathbb{S}^{d-3}$, we check that Σ satisfies all the conditions as described in Theorem 3.1.

Lemma 4.4. *The complex Σ in Construction 4.3 is simply connected codimension-1 submanifold of $\mathbb{S}^d$ for $d \geq 5$.*

Proposition 4.5. *The complex Σ in Construction 4.3 is a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \geq 3$.*

Proof: Applying the Mayer-Vietoris sequence to the triple $(\Delta_2 \cup f(\Delta_2), N, \Sigma)$, we find that Σ has the same homology as $\mathbb{S}^2 \times \mathbb{S}^{d-3}$, and so the result follows by Lemma 4.4 and Theorem 3.1. $\square$

Property 4.6. *For $d \geq 5$, the complex Σ in Construction 4.3 satisfies:*

1. $f_0(\Sigma) = 4d$;
2. $f_1(\Sigma) = 4d(2d-3)$;
3. $f_{d-1}(\Sigma) = (d+2)2^d - 8d$;
4. $\mathrm{Aut}(\Sigma)$ *admits a vertex-transitive action of* $\mathbb{Z}_2 \times \mathcal{D}_{2d}$.

Remark 4.7. Working with Lorenzo Venturello, we created a CROSSFLIP program for balanced complexes to attempt to reduce the number of vertices of a given triangulation. However, the complexity of finding shellable subcomplexes in the d-cross-polytope grows exponentially with d, and so the program is highly inefficient for $d > 4$. [4] showed that a balanced triangulation of a non-sphere $(d-1)$-manifold requires at least $3d$ vertices. It is not known that apart from the sphere bundle over the circle, if there are other manifolds that admit balanced triangulations with $3d$ vertices.

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