

The equivariant volumes of the permutahedron

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Abstract. We consider the action of the symmetric group S_n on the permutahedron Π_n . We prove that if σ is a permutation of S_n which has m cycles of lengths l_1, \dots, l_m , then the subset of Π_n fixed by σ is a polytope with normalized volume $n^{m-2} \gcd(l_1, \dots, l_m)$.

Resumen. Consideramos la acción del grupo simétrico S_n sobre el permutaedro Π_n . Demostramos que si σ es una permutación de S_n que tiene m ciclos de longitudes l_1, \dots, l_m , entonces el subconjunto de Π_n que permanece fijo bajo la acción de σ es un politopo cuyo volumen normalizado es igual a $n^{m-2} \text{mcd}(l_1, \dots, l_m)$.

Keywords: permutahedron, volume, symmetric group, tree

1 Introduction

The n -permutahedron is the polytope in \mathbb{R}^n whose vertices are the permutations of $[n]$:

$$\Pi_n := \text{conv} \{(\pi(1), \pi(2), \dots, \pi(n)) : \pi \in S_n\}.$$

The symmetric group S_n acts on $\Pi_n \subset \mathbb{R}^n$ by permuting coordinates; more precisely, a permutation $\sigma \in S_n$ acts on a point $x = (x_1, x_2, \dots, x_n) \in \Pi_n$, by

$$\sigma \cdot x := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}).$$

Definition 1.1. The fixed polytope of the permutahedron Π_n under a permutation σ of $[n]$ is

$$\Pi_n^\sigma = \{x \in \Pi_n : \sigma \cdot x = x\}.$$

Our main result is a generalization of the fact, due to Stanley [4], that $\text{Vol} \Pi_n = n^{n-2}$; see [Theorem 3.1](#).

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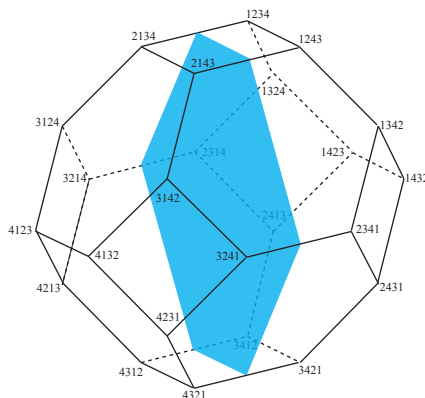


Figure 1: The fixed polytope $\Pi_4^{(12)}$ of the permutahedron Π_4 under $(12) \in S_4$ is a hexagon.

Theorem 1.2. *If σ is a permutation of $[n]$ whose cycles have lengths l_1, \dots, l_m , then the normalized volume of the fixed polytope of Π_n under σ is*

$$\text{Vol } \Pi_n^\sigma = n^{m-2} \gcd(l_1, \dots, l_m).$$

This is the first step towards describing the equivariant Ehrhart theory of the permutahedron, a question posed by Stapledon [6].

1.1 Normalizing the volume

The permutahedron and its fixed polytopes are not full-dimensional. We normalize volumes so that every primitive parallelotope has volume 1. This is the normalization under which the volume of Π_n equals n^{n-2} .

More precisely, let P be a d -dimensional polytope on an affine d -plane $L \subset \mathbb{Z}^n$. Assume L is integral, in the sense that $L \cap \mathbb{Z}^n$ is a lattice translate of a d -dimensional lattice Λ . We call a lattice d -parallelotope in L *primitive* if its edges generate the lattice Λ ; all primitive parallelotopes have the same volume. Then we define the volume of a d -polytope P in L to be $\text{Vol}(P) := \text{EVol}(P) / \text{EVol}(\square)$ for any primitive parallelotope \square in L , where EVol denotes Euclidean volume.

The definition of $\text{Vol}(P)$ makes sense even when P is not an integral polytope. This is important because the fixed polytopes of the permutahedron are not necessarily integral.

1.2 Notation

We identify each permutation $\pi \in S_n$ with the point $(\pi(1), \dots, \pi(n))$ in \mathbb{R}^n . When we write permutations in cycle notation, we do not use commas to separate the entries

of each cycle. For example, we identify the permutation 246513 in S_6 with the point $(2, 4, 6, 5, 1, 3) \in \mathbb{R}^6$, and write it as $(1245)(36)$ in cycle notation.

Our main goal is to find the volume of the fixed polytope Π_n^σ for a permutation $\sigma \in S_n$. We assume that σ has m cycles of lengths $l_1 \geq \dots \geq l_m$. In fact, for the goals of this paper, it suffices to assume

$$\sigma = (1 \ 2 \ \dots \ l_1)(l_1 + 1 \ l_1 + 2 \ \dots \ l_1 + l_2) \cdots (l_1 + \dots + l_{m-1} + 1 \ \dots \ n - 1 \ n).$$

We let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n , and $e_S := e_{s_1} + \dots + e_{s_k}$ for $S = \{s_1, \dots, s_k\} \subseteq [n]$. Recall that the Minkowski sum of polytopes $P, Q \subset \mathbb{R}^n$ is the polytope $P + Q := \{p + q : p \in P, q \in Q\} \subset \mathbb{R}^n$. [3]

1.3 Organization

Section 2 presents **Theorem 2.11**, which describes the fixed polytope Π_n^σ in terms of its vertices, its defining inequalities, and a Minkowski sum decomposition. **Section 3** uses this to prove our main result, **Theorem 1.2**, on the normalized volume of Π_n^σ . This is an extended abstract; for complete statements and proofs, see [1].

2 Describing the fixed polytopes of the permutahedron

Proposition 2.1 ([7]). *The permutahedron Π_n can be described in the following three ways:*

1. (Inequalities) *It is the set of points $x \in \mathbb{R}^n$ satisfying*

(a) $x_1 + x_2 + \dots + x_n = 1 + 2 + \dots + n$, and

(b) for any proper subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$,

$$x_{i_1} + x_{i_2} + \dots + x_{i_k} \geq 1 + 2 + \dots + k.$$

2. (Vertices) *It is the convex hull of the points $(\pi(1), \dots, \pi(n))$ as π ranges over the permutations of $[n]$.*

3. (Minkowski sum) *It is the Minkowski sum: $\sum_{1 \leq j < k \leq n} [e_k, e_j] + \sum_{1 \leq k \leq n} e_k$.*

The n -permutahedron is $(n - 1)$ -dimensional and every permutation of $[n]$ is indeed a vertex.

Our first goal is to prove the analogous result for the fixed polytopes of Π_n ; we do so in **Theorem 2.11**.

2.1 Standardizing the permutation

We define the *cycle type* of a permutation σ to be the partition of n consisting of the lengths $l_1 \geq \dots \geq l_m$ of the cycles of σ .

Lemma 2.2. *The volume of Π_n^σ only depends on the cycle type of σ .*

We wish to measure the various fixed polytopes of Π_n , and by [Lemma 2.2](#) we can focus our attention on the polytopes Π_n^σ fixed by a permutation of the form

$$\sigma = (1 \ 2 \ \dots \ l_1)(l_1 + 1 \ l_1 + 2 \ \dots \ l_1 + l_2) \cdots (l_1 + \dots + l_{m-1} + 1 \ \dots \ n - 1 \ n) \quad (2.1)$$

for a partition $l_1 \geq l_2 \geq \dots \geq l_m$ with $l_1 + \dots + l_m = n$. We do so from now on.

2.2 Towards the inequality description

Lemma 2.3. *For a permutation $\sigma \in S_n$, the fixed polytope Π_n^σ consists of the points $x \in \Pi_n$ satisfying $x_j = x_k$ for any j and k in the same cycle of σ .*

Corollary 2.4. *If a permutation σ of $[n]$ has m cycles then Π_n^σ has dimension $m - 1$.*

2.3 Towards a vertex description

In this section we describe a set $\text{Vert}(\sigma)$ of $m!$ points associated to a permutation σ of S_n . We will show in [Theorem 2.11](#) that this is the set of vertices of the fixed polytope Π_n^σ . For a point $w \in \mathbb{R}^n$, let \bar{w} be the average of the σ -orbit of w , that is,

$$\bar{w} := \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot w, \quad (2.2)$$

where $|\sigma|$ is the order of σ as an element of the symmetric group S_n .

Definition 2.5. *Given $\sigma \in S_n$, we say a permutation $v = (v_1, \dots, v_n)$ of $[n]$ is σ -standard if it satisfies the following property: for each cycle $(j_1 \ j_2 \ \dots \ j_r)$ of σ , $(v_{j_1}, v_{j_2}, \dots, v_{j_r})$ is a sequence of consecutive integers in increasing order. We define the set of σ -vertices to be*

$$\text{Vert}(\sigma) := \{\bar{w} : w \text{ is a } \sigma\text{-standard permutation of } [n]\}.$$

These points should not be confused with the vertices of the ambient permutahedron Π_n . Let us illustrate this definition in an example and prove some preliminary results.

Example 2.6. For $\sigma = (1234)(567)(89)$, the σ -standard permutations in S_9 are

$$\begin{aligned} (1,2,3,4, 5,6,7, 8,9), & \quad (1,2,3,4, 7,8,9, 5,6), \\ (4,5,6,7, 1,2,3, 8,9), & \quad (3,4,5,6, 7,8,9, 1,2), \\ (6,7,8,9, 1,2,3, 4,5), & \quad (6,7,8,9, 3,4,5, 1,2), \end{aligned}$$

and the corresponding σ -vertices are

$$\begin{aligned} \frac{1+2+3+4}{4} e_{1234} + \frac{5+6+7}{3} e_{567} + \frac{8+9}{2} e_{89}, & \quad \frac{1+2+3+4}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{5+6}{2} e_{89}, \\ \frac{4+5+6+7}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{8+9}{2} e_{89}, & \quad \frac{3+4+5+6}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{1+2}{2} e_{89}, \\ \frac{6+7+8+9}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{4+5}{2} e_{89}, & \quad \frac{6+7+8+9}{4} e_{1234} + \frac{3+4+5}{3} e_{567} + \frac{1+2}{2} e_{89}. \end{aligned}$$

Let us give a more explicit description of \bar{w} in general, and of the σ -vertices in particular, which will be important in the proof of [Theorem 2.11](#).

Lemma 2.7. For any $w \in \mathbb{R}^n$, the average of the σ -orbit of w is

$$\bar{w} = \sum_{k=1}^m \frac{\sum_{j \in \sigma_k} w_j}{l_k} e_{\sigma_k}.$$

Notice that the entries of \bar{w} within each cycle σ_k are constant, bearing witness to the fact that \bar{w} , being the average of a σ -orbit, must be in the fixed polytope Π_n^σ .

Corollary 2.8. The set $\text{Vert}(\sigma)$ of σ -vertices consists of the $m!$ points

$$\bar{v}_{\prec} := \sum_{k=1}^m \left(\frac{l_k + 1}{2} + \sum_{j: \sigma_j \prec \sigma_k} l_j \right) e_{\sigma_k}$$

as \prec ranges over the $m!$ possible linear orderings of $\sigma_1, \sigma_2, \dots, \sigma_m$.

2.4 Towards a zonotope description

We will show in [Theorem 2.11](#) that the fixed polytope Π_n^σ is the zonotope given by the following Minkowski sum.

Definition 2.9. Let M_σ denote the Minkowski sum

$$\begin{aligned} M_\sigma &:= \sum_{1 \leq j < k \leq m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^m \frac{l_k + 1}{2} e_{\sigma_k} \\ &= \sum_{1 \leq j < k \leq m} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^m \left(\frac{l_k + 1}{2} + \sum_{j < k} l_j \right) e_{\sigma_k}. \end{aligned} \tag{2.3}$$

Proposition 2.10. The zonotope M_σ is combinatorially equivalent to the standard permutahedron Π_m , where m is the number of cycles of σ .

2.5 Three descriptions of the fixed polytope of the permutahedron

Theorem 2.11. *Let σ be a permutation of $[n]$ whose cycles $\sigma_1, \dots, \sigma_m$ have respective lengths l_1, \dots, l_m . The fixed polytope Π_n^σ can be described in the following ways:*

0. *It is the set of points x in the permutahedron Π_n such that $\sigma \cdot x = x$.*

1. *It is the set of points $x \in \mathbb{R}^n$ satisfying*

(a) $x_1 + x_2 + \dots + x_n = 1 + 2 + \dots + n,$

(b) *for any proper subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\},$*

$$x_{i_1} + x_{i_2} + \dots + x_{i_k} \leq 1 + 2 + \dots + k, \text{ and}$$

(c) *for any i and j which are in the same cycle of σ , $x_i = x_j$.*

2. *It is the convex hull of the set $\text{Vert}(\sigma)$ of σ -vertices, as described in [Corollary 2.8](#).*

3. *It is the Minkowski sum M_σ of [Definition 2.9](#)*

Consequently, the fixed polytope Π_n^σ is a zonotope that is combinatorially isomorphic to the permutahedron Π_m . It is $(m - 1)$ -dimensional and every σ -vertex is indeed a vertex of Π_n^σ .

Proof. Description 0. is the definition of the fixed polytope Π_n^σ , and we already observed in [Lemma 2.3](#) that description 1. is accurate. Recall that we denoted the polytopes described in 2. and 3. by $\text{conv}(\text{Vert}(\sigma))$ and M_σ , respectively. It remains to prove that

$$\Pi_n^\sigma = \text{conv}(\text{Vert}(\sigma)) = M_\sigma.$$

We proceed in three steps as follows:

$$A. \quad \text{conv}(\text{Vert}(\sigma)) \subseteq \Pi_n^\sigma \qquad B. \quad M_\sigma \subseteq \text{conv}(\text{Vert}(\sigma)) \qquad C. \quad \Pi_n^\sigma \subseteq M_\sigma$$

A. $\text{conv}(\text{Vert}(\sigma)) \subseteq \Pi_n^\sigma$: It suffices to show that Π_n^σ contains any point in $\text{Vert}(\sigma)$, say

$$\overline{v_{\prec}} = \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot v_{\prec},$$

where \prec is a total order of $\sigma_1, \dots, \sigma_m$ and v_{\prec} is the associated σ -standard permutation. Since v_{\prec} is a vertex of Π_n , we conclude that $\sigma^i \cdot v_{\prec}$ is a vertex of Π_n for all i , and hence their average $\overline{v_{\prec}}$ is in Π_n . Also, since $\sigma^{|\sigma|} = 1$, we have that $\sigma \cdot \overline{v_{\prec}} = \overline{v_{\prec}}$. Therefore, $\overline{v_{\prec}}$ is in Π_n^σ by 0., as desired.

B. $M_\sigma \subseteq \text{conv}(\text{Vert}(\sigma))$: It suffices to show that any vertex of M_σ is in $\text{Vert}(\sigma)$.

For a polytope $P \subset \mathbb{R}^n$ and a linear functional $c \in (\mathbb{R}^n)^*$, we let P_c denote the face of P where c is maximized. In particular, for any given vertex v of M_σ , consider a linear functional $c = (c_1, c_2, \dots, c_n) \in (\mathbb{R}^n)^*$ such that $v = (M_\sigma)_c$ is the unique point in M_σ maximizing c . For $k = 1, \dots, m$, let $c_{\sigma_k} := \frac{1}{l_k} \sum_{i \in \sigma_k} c_i$. One can verify that

(a) $c_{\sigma_j} \neq c_{\sigma_k}$ for $j \neq k$, and

(b) $v = \overline{v}_{\prec}$ for the linear order \prec on $\sigma_1, \sigma_2, \dots, \sigma_m$ where $\sigma_j \prec \sigma_k$ if and only if $c_{\sigma_j} < c_{\sigma_k}$.

This shows that every vertex of M_σ is a σ -vertex, as desired.

C. $\Pi_n^\sigma \subseteq M_\sigma$: Any point $p \in \Pi_n^\sigma$ can be written as a convex combination $p = \sum_{\tau \in S_n} \lambda_\tau \tau$ of the $n!$ permutations of $[n]$, where $\lambda_\tau \geq 0$ for all τ and $\sum_{\tau \in S_n} \lambda_\tau = 1$. Recall from (2.2) that \overline{w} represents the average of the σ -orbit of $w \in \mathbb{R}^n$. Since p is fixed by σ we have

$$p = \overline{p} = \sum_{\tau \in S_n} \lambda_\tau \overline{\tau}.$$

It follows that $\Pi_n^\sigma \subseteq \text{conv}\{\overline{\tau} : \tau \in S_n\}$. Therefore, to show that $\Pi_n^\sigma \subseteq M_\sigma$, it suffices to show that $\overline{\tau} \in M_\sigma$ for all permutations τ . To do so, let us first derive an alternative expression for $\overline{\tau}$.

Let us begin with the vertex $\text{id} = (1, 2, \dots, n)$ of Π_n corresponding to the identity permutation. As described in Corollary 2.8, this is the σ -standard permutation corresponding to the order $\sigma_1 \prec \sigma_2 \prec \dots \prec \sigma_m$, so

$$\overline{\text{id}} = \sum_{k=1}^m \left(\frac{l_k + 1}{2} + \sum_{j < k} l_j \right) e_{\sigma_k}. \quad (2.4)$$

Notice that this is the translation vector for the Minkowski sum of (2.3).

Now, let us compute $\overline{\tau}$ for any permutation τ . Let

$$l = \text{inv}(\tau) = |\{(a, b) : 1 \leq a < b \leq n, \tau(a) > \tau(b)\}|$$

be the number of inversions of τ . Consider a minimal sequence $\text{id} = \tau_0, \tau_1, \dots, \tau_l = \tau$ of permutations such that τ_{i+1} is obtained from τ_i by exchanging the positions of numbers p and $p + 1$, thus introducing a single new inversion without affecting any existing inversions. Such a sequence corresponds to a minimal factorization of τ as a product of simple transpositions $(p \ p + 1)$ for $1 \leq p \leq n - 1$. We have $\text{inv}(\tau_i) = i$ for $1 \leq i \leq l$.

Now we compute $\overline{\tau}$ by analyzing how $\overline{\tau_i}$ changes as we introduce new inversions, using that

$$\overline{\tau} - \overline{\text{id}} = (\overline{\tau_l} - \overline{\tau_{l-1}}) + \dots + (\overline{\tau_1} - \overline{\tau_0}). \quad (2.5)$$

If $a < b$ are the positions of the numbers p and $p + 1$ that we switch as we go from τ_i to τ_{i+1} , then regarding τ_i and τ_{i+1} as vectors in \mathbb{R}^n we have

$$\tau_{i+1} - \tau_i = e_a - e_b.$$

If σ_j and σ_k are the cycles of σ containing a and b , respectively, we have

$$\overline{\tau_{i+1}} - \overline{\tau_i} = \overline{e_a} - \overline{e_b} = \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} = \frac{1}{l_j l_k} (l_k e_{\sigma_j} - l_j e_{\sigma_k}) \quad (2.6)$$

in light of [Lemma 2.7](#). This is the local contribution to (2.5) that we obtain when we introduce a new inversion between a position a in cycle σ_j and a position b in cycle σ_k in our permutation. Notice that this contribution is 0 when $j = k$. Also notice that we will still have an inversion between positions a and b in all subsequent permutations, due to the minimality of the sequence. We conclude that

$$\overline{\tau} - \overline{\text{id}} = \sum_{j < k} \frac{\text{inv}_{j,k}(\tau)}{l_j l_k} (l_k e_{\sigma_j} - l_j e_{\sigma_k}) \quad (2.7)$$

where

$$\text{inv}_{j,k}(\tau) = |\{(a, b) : 1 \leq a < b \leq n, a \in \sigma_j, b \in \sigma_k \text{ and } \tau(a) > \tau(b)\}|$$

is the number of inversions in τ between a position in σ_j and a position in σ_k for $j < k$.

Equations (2.4) and (2.7) give us an alternative description for $\overline{\tau}$. This description makes it apparent that $\overline{\tau} \in M_\sigma$: Notice that $|\sigma_j| = l_j$ and $|\sigma_k| = l_k$ imply that $0 \leq \text{inv}_{j,k}(\tau) \leq l_j l_k$, so

$$\overline{\tau} - \overline{\text{id}} \in \sum_{1 \leq j < k \leq n} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}];$$

combining this with (2.3) and (2.4) gives the desired result. \square

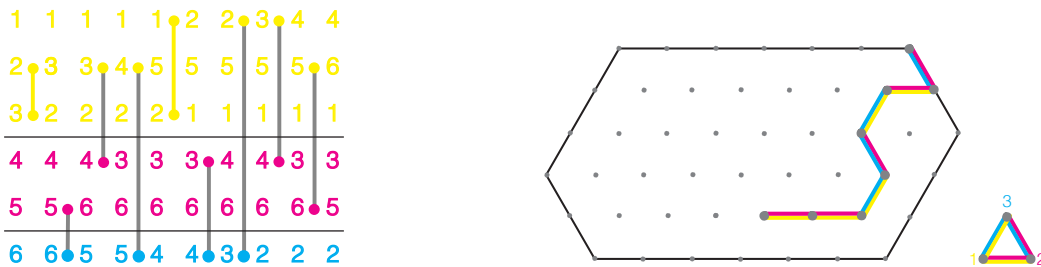


Figure 2: (a) A minimal sequence of permutations $\text{id} = \tau_0, \tau_1, \dots, \tau_9 = 461352$ adding one inversion at a time and (b) the corresponding path from $\overline{\text{id}}$ to $\overline{\tau}$ in the zonotope M_σ .

Example 2.12. [Figure 2](#) illustrates part C of the proof above for $n = 6$, $\sigma = (123)(45)(6)$, and the permutation $\tau = 461352$. This permutation has $\text{inv}(\tau) = 9$ inversions, and the columns of the left panel show a minimal sequence of permutations $\text{id} = \tau_0, \tau_1, \dots, \tau_9 =$

τ where each τ_{i+1} is obtained from τ_i by swapping two consecutive numbers, thus introducing a single new inversion.

The rows of the diagram are split into three groups 1, 2, and 3, corresponding to the support of the cycles of σ . Out of the $\text{inv}(\tau) = 9$ inversions of τ , there are $\text{inv}_{1,2}(\tau) = 3$ involving groups 1 and 2, $\text{inv}_{1,3}(\tau) = 2$ involve groups 1 and 3, and $\text{inv}_{2,3}(\tau) = 2$ involving groups 2 and 3.

This sequence of permutations gives rise to a walk from $\overline{\text{id}}$, which is the top right vertex of the zonotope M_σ , to $\bar{\tau}$. In the rightmost triangle, which is not drawn to scale, vertex i represents the point e_{σ_i}/l_i for $1 \leq i \leq 3$. Whenever two numbers in groups $j < k$ are swapped in the left panel, to get from permutation τ_i to τ_{i+1} , we take a step in direction $e_{\sigma_j}/l_j - e_{\sigma_k}/l_k$ in the right panel, to get from point $\bar{\tau}_i$ to $\bar{\tau}_{i+1}$. This is the direction of edge jk in the triangle, and its length is $1/l_j l_k$ of the length of the generator $l_k e_{\sigma_j} - l_j e_{\sigma_k}$ of the zonotope. Then

$$\bar{\tau} - \overline{\text{id}} = \frac{3}{l_1 l_2} (l_2 e_{\sigma_1} - l_1 e_{\sigma_2}) + \frac{2}{l_1 l_3} (l_3 e_{\sigma_1} - l_1 e_{\sigma_3}) + \frac{2}{l_2 l_3} (l_3 e_{\sigma_2} - l_2 e_{\sigma_3}).$$

Since $3 = \text{inv}_{1,2}(\tau) \leq l_1 l_2 = 6$, $2 = \text{inv}_{1,3}(\tau) \leq l_1 l_3 = 3$ and $2 = \text{inv}_{2,3}(\tau) \leq l_2 l_3 = 2$, the resulting point $\bar{\tau}$ is in the zonotope M_σ .

3 The volumes of the fixed polytopes of the permutahedron

To compute the volume of Π_n^σ we use its description as a zonotope, recalling that a zonotope can be tiled by parallelotopes as follows. If A is a set of vectors, then $B \subseteq A$ is called a *basis* for A if B is linearly independent and $\text{rank}(B) = \text{rank}(A)$. We define the parallelotope $\square B$ to be the Minkowski sum of the segments in B , that is,

$$\square B := \left\{ \sum_{b \in B} \lambda_b b : 0 \leq \lambda_b \leq 1 \text{ for each } b \in B \right\}.$$

Theorem 3.1 ([2, 4, 7]). *Let $A \subset \mathbb{Z}^n$ be a set of lattice vectors of rank d .*

1. *The zonotope $Z(A)$ can be tiled using one translate of the parallelotope $\square B$ for each basis B of A . Therefore, the volume of the d -dimensional zonotope $Z(A)$ is*

$$\text{Vol}(Z(A)) = \sum_{\substack{B \subseteq A \\ B \text{ basis}}} \text{Vol}(\square B).$$

2. *For each $B \subset \mathbb{Z}^n$ of rank d , $\text{Vol}(\square B)$ equals the index of $\mathbb{Z}B$ as a sublattice of $(\text{span } B) \cap \mathbb{Z}^n$. Using the vectors in B as the columns of an $n \times d$ matrix, $\text{Vol}(B)$ is the greatest common divisor of the minors of rank d .*

By [Theorem 2.11](#), the fixed polytope Π_n^σ is a translate of the zonotope generated by the set

$$F_\sigma = \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} ; 1 \leq j < k \leq m \right\}.$$

This set of vectors has a nice combinatorial structure, allowing us to describe the bases B and the volumes $\text{Vol}(\square B)$ combinatorially. We do this in the next two lemmas. For a tree T whose vertex set is $[m]$, let

$$\begin{aligned} F_T &= \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : j < k \text{ and } jk \text{ is an edge of } T \right\}, \\ E_T &= \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} : j < k \text{ and } jk \text{ is an edge of } T \right\}. \end{aligned}$$

Lemma 3.2 ([4]). *The vector configuration*

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : 1 \leq j < k \leq m \right\}$$

has exactly m^{m-2} bases: they are the sets F_T as T ranges over the spanning trees on $[m]$.

Lemma 3.3. *For any tree T on $[m]$ we have*

$$\begin{aligned} 1. \quad \text{Vol}(\square F_T) &= \prod_{i=1}^m l_i^{\deg_T(i)} \text{Vol}(E_T), \\ 2. \quad \text{Vol}(\square E_T) &= \frac{\gcd(l_1, \dots, l_m)}{l_1 \cdots l_m}, \end{aligned}$$

where $\deg_T(i)$ is the number of edges containing vertex i in T .

Proof. 1. Since $l_k e_{\sigma_j} - l_j e_{\sigma_k} = l_j l_k \left(\frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \right)$ for each edge jk of T , and volumes scale linearly with respect to each edge length of a parallelotope, we have

$$\text{Vol}(\square F_T) = \left(\prod_{jk \text{ edge of } T} l_j l_k \right) \text{Vol}(\square E_T) = \prod_{i=1}^m l_i^{\deg_T(i)} \text{Vol}(\square E_T).$$

2. The parallelotopes $\square E_T$ are the images of the parallelotopes $\square A_T$ under the linear bijective map

$$\begin{aligned} \phi : \mathbb{R}^m &\rightarrow (\mathbb{R}^n)^\sigma \\ f_i &\mapsto \frac{e_{\sigma_i}}{l_i}, \end{aligned}$$

where

$$A_T := \{ f_j - f_k : j < k, jk \text{ is an edge of } T \}.$$

Since the vector configuration $\{f_j - f_k : 1 \leq j < k \leq m\}$ is unimodular, all parallelotopes $\square A_T$ have unit volume. Therefore, the parallelotopes $\square E_T = \phi(\square A_T)$ have the same normalized volume, so $\text{Vol}(E_T)$ is independent of T .

It follows that we can use any tree T to compute $\text{Vol}(E_T)$ or, equivalently, $\text{Vol}(F_T)$. We choose the tree $T = \text{Claw}_m$ with edges $1m, 2m, \dots, (m-1)m$. Writing the $m-1$ vectors of

$$F_{\text{Claw}_m} = \{l_m e_{\sigma_i} - l_i e_{\sigma_m} : 1 \leq i \leq m-1\}$$

as the columns of an $n \times (m-1)$ matrix, then $\text{Vol}(F_{\text{Claw}_m})$ is the greatest common divisor of the non-zero maximal minors of this matrix. This quantity does not change when we remove duplicate rows; the result is the $m \times (m-1)$ matrix

$$\begin{bmatrix} l_m & 0 & 0 & \cdots & 0 \\ 0 & l_m & 0 & \cdots & 0 \\ 0 & 0 & l_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & l_m \\ -l_1 & -l_2 & -l_3 & \cdots & -l_{m-1} \end{bmatrix}.$$

This matrix has m maximal minors, whose absolute values equal $l_m^{m-2} l_1, \dots, l_m^{m-2} l_{m-1}, l_m^{m-1}$. Therefore,

$$\text{Vol}(\square F_{\text{Claw}_m}) = l_m^{m-2} \text{gcd}(l_1, \dots, l_{m-1}, l_m)$$

and part 1 then implies that

$$\text{Vol}(\square E_{\text{Claw}_m}) = \frac{\text{Vol}(\square F_{\text{Claw}_m})}{l_1 \cdots l_{m-1} l_m^{m-1}} = \frac{\text{gcd}(l_1, \dots, l_m)}{l_1 \cdots l_m}$$

as desired. □

Lemma 3.4. For any positive integer $m \geq 2$ and unknowns x_1, \dots, x_m , we have

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^m x_i^{\deg_T(i)-1} = (x_1 + \cdots + x_m)^{m-2}.$$

Sketch of proof. This is a variant of the analogous result for rooted trees [5, Theorem 5.3.4], which states that

$$\sum_{\substack{(T,r) \text{ rooted} \\ \text{tree on } [m]}} \prod_{i=1}^m x_i^{\text{children}_{(T,r)}(i)} = (x_1 + \cdots + x_m)^{m-1}$$

where $\text{children}_{(T,r)}(v)$ counts the children of v . It can be proved similarly, or derived directly from it. □

Theorem 1.2. *If σ is a permutation of $[n]$ whose cycles have lengths l_1, \dots, l_m , then the normalized volume of the fixed polytope of Π_n under σ is*

$$\text{Vol } \Pi_n^\sigma = n^{m-2} \gcd(l_1, \dots, l_m).$$

Proof. Since Π_n^σ is a translate of the zonotope for $F_\sigma := \{l_k e_{\sigma_j} - l_j e_{\sigma_k} : 1 \leq j < k \leq m\}$, we invoke [Theorem 3.1](#). Using [Lemmas 3.2](#) to [3.4](#), it follows that

$$\begin{aligned} \text{Vol } \Pi_n^\sigma &= \sum_{T \text{ tree on } [m]} \text{Vol}(\square F_T) \\ &= \sum_{T \text{ tree on } [m]} \prod_{i=1}^m l_i^{\deg_T(i)-1} \gcd(l_1, \dots, l_m) \\ &= (l_1 + \dots + l_m)^{m-2} \gcd(l_1, \dots, l_m) = n^{m-2} \gcd(l_1, \dots, l_m). \quad \square \end{aligned}$$

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