Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #15, 12 pp.

Proceedings of the 31^{st} Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Spanning line configurations

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Abstract. We define and study a variety $X_{n,k}$ which depends on two positive integers $k \leq n$. When $k = n$, the variety $X_{n,k}$ is homotopy equivalent to the *flag variety* $\mathcal{F}\ell(n)$ of complete flags in $\mathbb{C}^n$. We describe an affine paving of $X_{n,k}$, present its cohomology, and describe the cellular cohomology classes in terms of Schubert polynomials. Just as the geometry of $\mathcal{F}\ell(n)$ is governed by the combinatorics of permutations in S_n, the geometry of $X_{n,k}$ is governed by length n words on the alphabet $\{1, 2, \ldots, k\}$ in which each letter appears at least once. The space $X_{n,k}$ carries a natural action of S_n, and we relate the induced cohomology representation to Macdonald theory via the Delta Conjecture of Haglund, Remmel, and Wilson.

Keywords: Fubini word, flag variety, symmetric function, coinvariant ring

1 Introduction

In this extended abstract we introduce and study a variety $X_{n,k}$ depending on two positive integers $k \leq n$. Our goal is to provide a geometric context to study the *Delta Conjecture* of Haglund, Remmel, and Wilson [8] which extends the role played by the classical flag variety $\mathcal{F}\ell(n)$ in the study of diagonal coinvariants and the *Shuffle Theorem* [3]. We introduce our variety $X_{n,k}$ in Section 2 below; the remainder of the introduction is devoted to connections with the Delta Conjecture and related work of Haglund, Rhoades, and Shimozono [9] on *generalized coinvariant rings*. We solve the problem [9, Prob. 7.2] of finding a flag variety for the Delta Conjecture.

Consider the action of the symmetric group S_n on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by subscript permutation. The invariant subring $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ is the ring of *symmetric polynomials*. Let $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}_+$ be the family of symmetric polynomials with vanishing constant term. The *invariant ideal* $I_n \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ is the ideal generated by $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}_+$. If $e_d = \sum_{1 \leq i_i < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}$ is the degree d elementary symmetric polynomial, we have $I_n = \langle e_1, e_2, \ldots, e_n \rangle$. The *coinvariant ring* is

$$R_n := \mathbb{Z}[x_1, \ldots, x_n]/I_n = \mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle. \tag{1.1}$$

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The ring R_n is a graded $\mathbb{Z}$-algebra with a graded action of S_n.

Let $\mathbb{C}^n$ be the standard n-dimensional complex vector space. A *(complete) flag* in $\mathbb{C}^n$ is a maximal sequence $V_\bullet = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ of nested subspaces of $\mathbb{C}^n$ such that $\dim(V_i) = i$ for $1 \leq i \leq n$. The *flag variety* $\mathcal{F}\ell(n)$ is the family of complete flags in $\mathbb{C}^n$. The identification $\mathcal{F}\ell(n) = \mathrm{GL}_n(\mathbb{C})/B$, where $B \subseteq \mathrm{GL}_n(\mathbb{C})$ is the upper triangular subgroup, endows $\mathcal{F}\ell(n)$ with the structure of a complex algebraic variety. Borel proved [2] that the (singular, integral) cohomology of $\mathcal{F}\ell(n)$ is presented by the coinvariant ring:

$$H^\bullet(\mathcal{F}\ell(n)) = R_n. \tag{1.2}$$

Algebraic properties of R_n and geometric properties of $\mathcal{F}\ell(n)$ are governed by combinatorial properties of permutations in S_n. In no small part for this reason, R_n is one of the most well-studied rings and $\mathcal{F}\ell(n)$ is one of the most well-studied varieties in algebraic combinatorics.

Consider a polynomial ring in two sets of n variables $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over the rational field $\mathbb{Q}$. This ring carries a *diagonal* action of S_n, viz. $w.x_i = x_{w(i)}, w.y_i := y_{w(i)}$ for $w \in S_n$ and $1 \leq i \leq n$. The *diagonal coinvariant ring* [7] is the bigraded S_n-module

$$DR_n^{\mathbb{Q}} := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] / \langle \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{S_n} \rangle \tag{1.3}$$

obtained by modding out by invariants with vanishing constant term. Setting the y-variables equal to zero recovers (up to ground ring) the classical coinvariant ring R_n which presents the cohomology of $\mathcal{F}\ell(n)$.

It is natural to ask for the bigraded S_n-isomorphism type of DR_n. We recall the basics of the Frobenius map connecting S_n-modules and symmetric functions.

The irreducible representations of S_n are in bijective correspondence with partitions of n. Given a partition $\lambda \vdash n$, let S^λ be the corresponding irreducible $\mathfrak{S}_n$-module. If V is any finite-dimensional S_n-module, there exist unique multiplicities $c_\lambda \geq 0$ so that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius image* of V is the symmetric function $\mathrm{Frob}(V) = \sum_{\lambda \vdash n} c_\lambda s_\lambda$, where s_λ is the Schur function.

Going further, if $V = \bigoplus_{i \geq 0} V_i$ is a *graded* S_n-module with each graded piece V_i finite-dimensional, the *graded Frobenius image* is $\mathrm{grFrob}(V; q) := \sum_{i \geq 0} \mathrm{Frob}(V_i) \cdot q^i$. Finally, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ is a *bigraded* S_n-module with each $V_{i,j}$ finite-dimensional, the *bigraded Frobenius image* is $\mathrm{grFrob}(V; q, t) = \sum_{i,j \geq 0} \mathrm{Frob}(V_{i,j}) \cdot q^i t^j$.

Haiman [10] proved that the bigraded Frobenius image of DR_n is given by

$$\mathrm{grFrob}(DR_n; q, t) = \nabla e_n, \tag{1.4}$$

where ∇ is the Bergeron-Garsia nabla operator on symmetric functions and e_n is the elementary symmetric function. Finding the bigraded isomorphism type of DR_n therefore reduces to finding a positive formula for the Schur expansion of ∇e_n. While there is not even a conjecture in this direction, Carlsson and Mellit [3] proved the *Shuffle Theorem* which gives a monomial expansion of ∇e_n.

The *Delta Conjecture* of Haglund, Remmel, and Wilson [8] predicts a generalization of the Shuffle Theorem which depends on two positive integers $k \leq n$. It reads

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t). \tag{1.5}$$

Here $\Delta'_{e_{k-1}}$ is the primed *delta operator* labeled by e_{k-1} and Rise and Val are two formal power series arising from lattice path combinatorics depending on an infinite set of variables $\mathbf{x} = (x_1, x_2, \dots)$ and two additional parameters q, t. The Delta Conjecture reduces to the Shuffle Theorem when $k = n$.

Although the Delta Conjecture is open in general, it is proven when one of the parameters q, t is set to zero. Combining results of [6, 9, 14, 16], we have

$$\Delta'_{e_{k-1}} e_n \mid_{t=0} = \mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q). \tag{1.6}$$

Let $C_{n,k}(\mathbf{x}; q)$ be the common symmetric function of Equation (1.6).

Haglund, Rhoades, and Shimozono [9] defined an extension of the coinvariant ring which applies to the Delta Conjecture. If $k \leq n$ are positive integers, let $I_{n,k} \subseteq \mathbb{Z}[x_1, \dots, x_n]$ be the ideal

$$I_{n,k} := \langle x_1^k, x_2^k, \dots, x_n^k, e_n, e_{n-1}, \dots, e_{n-k+1} \rangle \tag{1.7}$$

and let $R_{n,k} := \mathbb{Z}[x_1, \dots, x_n] / I_{n,k}$ be the corresponding quotient. The ring $R_{n,k}$ is a graded S_n-module. If we let $R_{n,k}^{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} R_{n,k}$, we have the graded Frobenius image [9]

$$\mathrm{grFrob}(R_{n,k}^{\mathbb{Q}}; q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q), \tag{1.8}$$

where rev_q reverses the coefficient sequences of polynomials in q and ω is the symmetric function involution trading e_n and h_n.

Equation (1.8) says that the generalized coinvariant ring $R_{n,k}$ plays the same role for the Delta Conjecture as the classical coinvariant ring R_n for the Shuffle Theorem on the level of graded S_n-modules. Haglund, Rhoades, and Shimozono left open the problem [9, Prob. 7.2] of finding a corresponding generalization of the flag variety: a variety $X_{n,k}$ whose cohomology is presented by $R_{n,k}$. We solve this problem here.

A word $w_1 \dots w_n$ over the positive integers is *Fubini* (or *packed*) if for any $i > 1$ such that i appears as a letter in $w_1 \dots w_n$, so does $i - 1$. Let $\mathcal{W}_{n,k}$ be the family of length n Fubini words with maximum letter k; when $k = n$ we have $\mathcal{W}_{n,k} = S_n$. **The geometry of our variety $X_{n,k}$, like the algebra of the ring $R_{n,k}$, is governed by the combinatorics of Fubini words in $\mathcal{W}_{n,k}$.** In addition to presenting the cohomology of $X_{n,k}$, we generalize classical Schubert calculus theorems of Ehresmann [4] and Lascoux-Schützenberger [11] from the flag variety $\mathcal{F}\ell(n)$ to the more general spaces $X_{n,k}$. It is the hope of the authors that this will inspire a generalization of Schubert calculus with Fubini words as its basis.

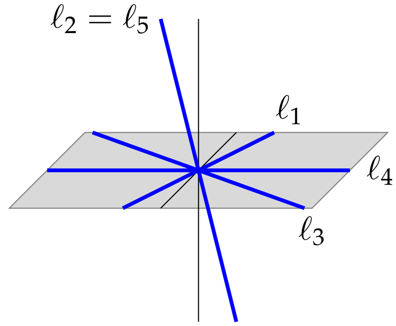

Figure 1: A point in $X_{5,3}$.

2 The spanning moduli space

Our object of study is the following moduli space of line configurations [1] which depends on two positive integers $k \leq n$ and a field $\mathbb{F}$.

Definition 1. *Let $k \leq n$ be positive integers and let $\mathbb{F}$ be a field. We define*

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \subseteq \mathbb{F}^k \text{ a 1-dimensional subspace and } \ell_1 + \cdots + \ell_n = \mathbb{F}^k\} \quad (2.1)$$

to be the set of all n-tuples of lines through the origin in $\mathbb{F}^k$ whose span equals $\mathbb{F}^k$.

Warning. *Do not confuse $X_{n,k}$ with the Grassmannian of k-dimensional subspaces of $\mathbb{F}^n$. These objects have very different combinatorial and geometric properties.*

A point in the space $X_{5,3}$ is shown in Figure 1. We have an ordered quintuple of lines through the origin which together have full span $\mathbb{F}^3$. We leave $\mathbb{F}$ general for now, but we specialize to the finite field $\mathbb{F}_q$ at the end of Section 3 and the complex field $\mathbb{C}$ in Sections 4 to 6.

Let $\mathbb{P}^{k-1}$ stand for the projective space of lines through the origin in $\mathbb{F}^k$ and let $(\mathbb{P}^{k-1})^n$ be its n-fold Cartesian product. The natural inclusion $X_{n,k} \subset (\mathbb{P}^{k-1})^n$ realizes $X_{n,k}$ as a Zariski open subset of $(\mathbb{P}^{k-1})^n$, and therefore a smooth complex manifold when $\mathbb{F} = \mathbb{C}$.

The set $X_{n,k}$ carries an action of the symmetric group S_n by the rule

$$w.(\ell_1, \ldots, \ell_n) := (\ell_{w(1)}, \ldots, \ell_{w(n)}) \quad (2.2)$$

for all $w \in S_n$ and $(\ell_1, \ldots, \ell_n) \in X_{n,k}$. When $\mathbb{F} = \mathbb{C}$, this action is continuous and so endows the (singular, integral) cohomology ring $H^\bullet(X_{n,k})$ with the structure of a graded S_n-module.

[1] We use 'configuration' rather than 'arrangement' because we are considering *ordered* tuples of lines.

We view our moduli space $X_{n,k}$ as a generalization of the flag variety. To justify this, observe that when $k = n$, we have a natural surjection

$$X_{n,n} = GL_n/T \twoheadrightarrow GL_n/B = \mathcal{F}\ell(n), \tag{2.3}$$

where $T \subseteq GL_n$ is the diagonal torus. When $\mathbb{F} = \mathbb{C}$, this is a homotopy equivalence [2], so that $X_{n,n}$ agrees with $\mathcal{F}\ell(n)$ up to homotopy and $H^\bullet(X_{n,n}) = H^\bullet(\mathcal{F}\ell(n))$. At the other extreme, the space $X_{n,1} = \{*\}$ is a single point.

3 The orbit set $X_{n,k}$

In order to understand the combinatorics of $X_{n,k}$ and the geometry of its embedding inside $(\mathbb{P}^{k-1})^n$, we will need matrices. If $\mathrm{Mat}_{k\times n}$ is the affine space of $k \times n$ matrices over $\mathbb{F}$, we introduce the Zariski open subsets $\mathcal{U}_{n,k} \subseteq \mathcal{V}_{n,k}$ by

$$\mathcal{U}_{n,k} := \{A \in \mathrm{Mat}_{k\times n} \,:\, A \text{ has no zero columns and has full rank}\}, \tag{3.1}$$
$$\mathcal{V}_{n,k} := \{A \in \mathrm{Mat}_{k\times n} \,:\, A \text{ has no zero columns}\}. \tag{3.2}$$

Let $T \subseteq GL_n$ be the diagonal subgroup and let $U \subseteq GL_k$ be the group of *lower* triangular $k \times k$ matrices with 1's on the diagonal. The product group $U \times T$ acts on both $\mathcal{U}_{n,k}$ and $\mathcal{V}_{n,k}$ by the rule $(u,t).A := uAt$ for all $(u,t) \in U \times T$. We have the orbit set identifications $X_{n,k} = \mathcal{U}_{n,k}/T$ and $(\mathbb{P}^{k-1})^n = \mathcal{V}_{n,k}/T$.

Proposition 1. *The action of $U \times T$ on the set $\mathcal{U}_{n,k}$ is free.*

What do the $U \times T$-orbits in $\mathcal{U}_{n,k}$ look like? Given any length n word $w = w_1 \dots w_n$, a position $1 \leq j \leq n$ is *initial* if w_j is the first occurrence of its letter. Let $\mathrm{in}(w)$ be the set of initial positions, so that $\mathrm{in}(2331231) = \{1,2,4\}$. If $w \in \mathcal{W}_{n,k}$ is Fubini, the *pattern matrix* $\mathrm{PM}(w)$ is the $k \times n$ matrix with entries in $\{0,1,\star\}$ whose entries $\mathrm{PM}(w)_{i,j}$ (for $1 \leq i \leq k$ and $1 \leq j \leq n$) are as follows.

- We have $\mathrm{PM}(w)_{i,j} = 1$ if and only if $w_j = i$.
- Suppose $j \in \mathrm{in}(w)$ is an initial position of w and $w_j \neq i$. If $w_j > i$ and there exists $j' < j$ with $w_{j'} = i$ then $\mathrm{PM}(w)_{i,j} = \star$. Otherwise $\mathrm{PM}(w)_{i,j} = 0$.
- Suppose $j \notin \mathrm{in}(w)$ is not an initial position of w and $w_j \neq i$. If the first occurrence of i in $w = w_1 \dots w_n$ is before the first occurrence of w_j in $w = w_1 \dots w_n$ then $\mathrm{PM}(w)_{i,j} = \star$. Otherwise $\mathrm{PM}(w)_{i,j} = 0$.

[2] It is a fiber bundle over a Hausdorff base space whose fiber – homeomorphic to the group of upper triangular matrices with 1's on the diagonal – is contractible.

In our example $w = 2331231 \in \mathcal{W}_{7,3}$ the pattern matrix is

$$\mathrm{PM}(w) = \mathrm{PM}(2331231) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & \star & \star & 0 & 1 & \star & \star \\ 0 & 1 & 1 & 0 & 0 & 1 & \star \end{pmatrix}.$$

The *dimension* $\dim(w)$ of a Fubini word $w \in \mathcal{W}_{n,k}$ is the number of $\star$'s in its pattern matrix, so that $\dim(2331231) = 5$.

A matrix $A \in \mathcal{U}_{n,k}$ *fits the pattern* of a Fubini word $w \in \mathcal{W}_{n,k}$ if A can be obtained by replacing the $\star$'s in $\mathrm{PM}(w)$ with field elements. The following is another application of linear algebra.

Proposition 2. *For any $U \times T$-orbit $\mathcal{O}$ in $\mathcal{U}_{n,k}$, there exists a unique Fubini word $w \in \mathcal{W}_{n,k}$ and a unique matrix A which fits the pattern of w such that $A \in \mathcal{O}$.*

Propositions 1 and 2 yield a disjoint union decomposition of $X_{n,k}$. Let $\widehat{C}_w \subseteq \mathcal{U}_{n,k}$ be the set of matrices which fit the pattern of a Fubini word $w \in \mathcal{W}_{n,k}$; this is an affine space of dimension $\dim(w)$. Define $C_w \subseteq X_{n,k}$ by

$$C_w := \text{image of } U\widehat{C}_w \text{ in } X_{n,k}. \tag{3.3}$$

We have

$$X_{n,k} = \bigsqcup_{w \in \mathcal{W}_{n,k}} C_w. \tag{3.4}$$

There is an enumerative result over the finite field $\mathbb{F}_q$. Recall the q-analogs

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]!_q := [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}. \tag{3.5}$$

The *q-Stirling number* $\mathrm{Stir}_q(n,k)$ is defined recursively by $\mathrm{Stir}_q(0,k) = \delta_{0,k}$ and

$$\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k). \tag{3.6}$$

The polynomial $[k]!_q \cdot \mathrm{Stir}_q(n,k)$ is called the *Mahonian distribution* on $\mathcal{W}_{n,k}$. Any statistic $\mathrm{stat} : \mathcal{W}_{n,k} \to \mathbb{Z}_{\geq 0}$ which satisfies $\sum_{w \in \mathcal{W}_{n,k}} q^{\mathrm{stat}(w)} = [k]!_q \cdot \mathrm{Stir}_q(n,k)$ is called a *Mahonian statistic*; see [1, 13, 14] for examples.

Proposition 3. *The dimension statistic* dim *is Mahonian.*

Propositions 1 to 3 combine to yield the following interpretation of the Mahonian distribution on $\mathcal{W}_{n,k}$ in terms of finite fields. It is our analog of the result that the number of flags in $\mathbb{F}_q^n$ is $[n]!_q$.

Corollary 1. *Let q be a prime power. Over the field $\mathbb{F}_q$ with q elements, there are $[k]!_q \cdot \mathrm{Stir}_q(n,k)$ orbits in the $U \times T$-set $\mathcal{U}_{n,k}$.*

Billey and Coskun [1] relate the Mahonian distribution on $\mathcal{W}_{n,k}$ to *rank varieties*. The authors do not know a geometric connection between rank varieties and $X_{n,k}$.

4 A cellular decomposition and the Poincaré series of $X_{n,k}$

For the rest of the extended abstract, we work over the complex field $\mathbb{C}$. We exploit the decomposition (3.4) to understand the geometry of $X_{n,k}$.

Let X be a complex algebraic variety. A *cellular decomposition* (a.k.a. *affine paving*) of X is a filtration $X_\bullet = (X = X_0 \supset X_1 \supset \cdots \supset X_m = \varnothing)$ of X, where each X_i is a closed subvariety and each difference $X_i - X_{i+1}$ is nonempty and isomorphic (as a variety) to a disjoint union of affine spaces. If we express $X_i - X_{i+1} = \bigsqcup_j A_{ij}$ as such a disjoint union, the A_{ij} are called the *cells* of the decomposition. We say that the partition of X formed by the collection of all cells $\{A_{ij}\}$ *induces* the decomposition $X_\bullet$. The following generalizes Ehresmann's CW decomposition of $\mathcal{F}\ell(n)$.

Theorem 1. *The set of cells $\{C_w : w \in \mathcal{W}_{n,k}\}$ induces a cellular decomposition of $X_{n,k}$.*

Theorem 1 determines the structure of $H^\bullet(X_{n,k})$ as a graded abelian group. Let $X^+_{n,k} = X_{n,k} \cup \{\infty\}$ be the one-point compactification of $X_{n,k}$. The *Borel-Moore homology* $\bar{H}_\bullet(X_{n,k})$ is the homology of the pair $(X^+_{n,k}, \{\infty\})$. By Theorem 1, $\bar{H}_d(X_{n,k})$ vanishes when d is odd and is free abelian with basis $\{[\overline{C_w}] : w \in \mathcal{W}_{n,k}, 2 \cdot \dim(w) = d\}$ when d is even.

The reader might ask whether the cellular decomposition of Theorem 1 can be replaced by the less technical notion of a CW decomposition. This is impossible because the space $X_{n,k}$ is not compact. Indeed, we will show that the Hilbert series of the cohomology ring $H^\bullet(X_{n,k})$ is not always palindromic (it equals $2q^4 + 3q^2 + 1$ when $n = 3$ and $k = 2$). Since $X_{n,k}$ is smooth, this means that $X_{n,k}$ must be noncompact. [3]

The variety $X_{n,k}$ is irreducible. To see this, observe that the (affine) cell C_w for $w = 123\ldots kk\ldots k \in \mathcal{W}_{n,k}$ is (Zariski) dense in $X_{n,k}$. Poincaré duality asserts the isomorphism of abelian groups $\bar{H}_d(X_{n,k}) \cong H^{\dim(X_{n,k})-d}(X_{n,k})$.

Theorem 2. *Let $k \leq n$ be positive integers. The cohomology ring $H^\bullet(X_{n,k})$ is free abelian as a graded group, with $\mathbb{Z}$-basis given by the classes $\{[\overline{C_w}] : w \in \mathcal{W}_{n,k}\}$. Furthermore, the Poincaré polynomial of $X_{n,k}$ is given by*

$$\sum_{d \geq 0} \mathrm{rank}(H^d(X_{n,k})) \cdot q^d = \mathrm{rev}_q([k]!_{q^2} \cdot \mathrm{Stir}_{q^2}(n,k)). \tag{4.1}$$

In particular, $\mathrm{rank}(H^\bullet(X_{n,k})) = |\mathcal{W}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$ *where* $\mathrm{Stir}(n,k)$ *is the Stirling number of the second kind.*

[3] The authors do not know whether the one-point compactification $X^+_{n,k}$ of $X_{n,k}$ admits a finite CW structure given by the cells $\{C_w : w \in \mathcal{W}_{n,k}\}$ together with an additional 0-cell for the added point ∞.

5 The cohomology of $X_{n,k}$

Theorem 2 describes the structure of $H^\bullet(X_{n,k})$ as a graded group. We go further and present $H^\bullet(X_{n,k})$ as a graded ring. The first step is an extension of the cellular decomposition of Theorem 1 from $X_{n,k}$ to the larger space $(\mathbb{P}^{k-1})^n$.

Recall that the space $\mathcal{V}_{n,k}$ of $k \times n$ matrices with no zero columns carries an action of the product group $U \times T$. Let $w = w_1 \dots w_n$ be an arbitrary word in $[k]^n$ (which may not be Fubini). The notion of 'pattern matrix' may be extended to define $\mathrm{PM}(w)$ as the $k \times n$ matrix over $\{0, 1, \star\}$ whose entries are the same as in the Fubini case, except that any index $1 \leq i \leq n$ which does not appear indexes a row of zeros. We refer the reader to [12, Sec. 5] for a more precise definition. As an example, if $k = 4$ we have

$$\mathrm{PM}(441121) = \begin{pmatrix} 0 & 0 & 1 & 1 & \star & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \star & 0 & \star \end{pmatrix}.$$

As before, we say that a $k \times n$ matrix *fits the pattern* of a word $w \in [k]^n$ if it can be obtained by replacing the $\star$'s in $\mathrm{PM}(w)$ with complex numbers. If $\widehat{C}_w$ is the set of matrices which fit the pattern of w, we define $C_w \subseteq (\mathbb{P}^{k-1})^n$ by the rule

$$C_w := \text{image of } U\widehat{C}_w \text{ in } (\mathbb{P}^{k-1})^n. \tag{5.1}$$

Proposition 2 extends to the $U \times T$-set $\mathcal{V}_{n,k}$ to give the disjoint union decomposition

$$(\mathbb{P}^{k-1})^n = \bigsqcup_{w \in [k]^n} C_w. \tag{5.2}$$

The decomposition (5.2) of $(\mathbb{P}^{k-1})^n$ extends the decomposition (3.4) of $X_{n,k}$ set theoretically. This statement can be strengthened to cellular decompositions as follows.

Lemma 1. *There is a cellular decomposition $X_\bullet = (X_0 \supset X_1 \supset \cdots \supset X_m)$ of $(\mathbb{P}^{k-1})^n$ with cells $\{C_w : w \in [k]^n\}$ such that $X_i = \bigsqcup_{w \in [k]^n - \mathcal{W}_{n,k}} C_w = (\mathbb{P}^{k-1})^n - X_{n,k}$ for some $0 \leq i \leq m$.*

Let $\iota : X_{n,k} \hookrightarrow (\mathbb{P}^{k-1})^n$ be the inclusion map. Lemma 1 and the general theory of cellular decompositions imply that the induced map $\iota^* : H^\bullet((\mathbb{P}^{k-1})^n) \twoheadrightarrow H^\bullet(X_{n,k})$ is surjective. In fact, if $J_{n,k} \subseteq H^\bullet((\mathbb{P}^{k-1})^n)$ is the ideal generated by the classes of cell closures $\{[\overline{C_w}] : w \in [k]^n - \mathcal{W}_{n,k}\}$ corresponding to non-Fubini words, then ι^* induces an isomorphism of graded rings

$$H^\bullet(X_{n,k}) \cong H^\bullet((\mathbb{P}^{k-1})^n)/J_{n,k}. \tag{5.3}$$

To exploit the isomorphism (5.3) and present the cohomology of $X_{n,k}$, we need a better understanding of the classes $[\overline{C_w}]$ inside $H^\bullet((\mathbb{P}^{k-1})^n)$.

For $1 \leq i \leq n-1$, the *divided difference operator* ∂_i on $\mathbb{Z}[x_1, \dots, x_n]$ is given by

$$\partial_i : f(x_1, \dots, x_n) \mapsto \frac{f(x_1, \dots, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}}. \tag{5.4}$$

Schubert polynomials $\{\mathfrak{S}_w : w \in S_n\}$ are defined recursively by $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0$ when $w_0 = n(n-1)\dots 1$ and

$$\mathfrak{S}_{w_1 \dots w_{i+1} w_i \dots w_n} = \partial_i(\mathfrak{S}_{w_1 \dots w_i w_{i+1} \dots w_n}) \quad \text{when } w_i > w_{i+1}. \tag{5.5}$$

A word is *convex* if it does not have a subword of the form $\dots i \dots j \dots i \dots$ for $i \neq j$. If $w = w_1 \dots w_n \in [k]^n$, the *convexification* $\text{conv}(w)$ is the unique convex word with the same letter multiplicities as w in which the initial letters appear in the same order. We let $\sigma(w) \in S_n$ be the unique Bruhat-minimal permutation such that $\sigma(w).\text{conv}(w) = w$. For example, if $w = 215235 \in [5]^6$ then $\text{conv}(w) = 221553$ so that $\sigma(w) = 142365 \in S_6$.

Let $w = w_1 \dots w_n \in [k]^n$ be a word with m distinct letters. The *standardization* $\text{st}(w) \in S_{n+k-m}$ is given by replacing the letters in noninitial positions of w from left to right with $k+1, k+2, \dots, k+n-m$, and then appending the letters in $[k]$ which do not appear in w to the end in increasing order. For example, if $w = 215235 \in [5]^6$ (so that $m = 4$) then $\text{st}(w) = 2156374 \in S_7$. We extend the Schubert polynomials to words as follows.

Definition 2. *Let $k \leq n$ be positive integers and let $w \in [k]^n$ be a word. Define a polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, \dots, x_n]$ by*

$$\mathfrak{S}_w := \sigma(w)^{-1}.\mathfrak{S}_{\text{st}(\text{conv}(w))}. \tag{5.6}$$

For $1 \leq i \leq n$, let ℓ_i be the i^{th} tautological line bundle over the projective space product $(\mathbb{P}^{k-1})^n$. By the Künneth Theorem, we have the presentation

$$H^\bullet((\mathbb{P}^{k-1})^n) = \mathbb{Z}[x_1, \dots, x_n] / \langle x_1^k, \dots, x_n^k \rangle, \tag{5.7}$$

where x_i represents the Chern class $c_1(\ell_i^*) \in H^2((\mathbb{P}^{k-1})^n)$ (and so $\deg(x_i) = 2$). This presentation interacts with the cellular decomposition of Lemma 1 as follows; the proof uses Fulton's theory of *degeneracy loci* [5].

Lemma 2. *Let $k \leq n$ and let $w \in [k]^n$ be a word. The class $[\overline{C_w}] \in H^\bullet((\mathbb{P}^{k-1})^n)$ is represented by the polynomial $\mathfrak{S}_w$ under the presentation (5.7).*

The connection between $X_{n,k}$, the ring $R_{n,k}$ of Haglund, Rhoades, and Shimozono, and the Delta Conjecture is as follows.

Theorem 3. *Let $k \leq n$ be positive integers. The cohomology of $X_{n,k}$ may be presented as*

$$H^\bullet(X_{n,k}) = R_{n,k}. \tag{5.8}$$

Under this presentation, the variable x_i represents the Chern class $c_1(\ell_i^*) \in H^2(X_{n,k})$, where $\ell_i \twoheadrightarrow X_{n,k}$ is the i^{th} tautological line bundle. If $w \in \mathcal{W}_{n,k}$, the class $[\overline{C_w}] \in H^\bullet(X_{n,k})$ is represented by the polynomial $\mathfrak{S}_w$ of Definition 2.

Proof. (Sketch.) Applying Lemmas 1 and 2, we have the presentation

$$H^\bullet(X_{n,k}) = \mathbb{Z}[x_1, \ldots, x_n]/K_{n,k}, \tag{5.9}$$

where $K_{n,k} := \langle \mathfrak{S}_w : w \in [k]^n - \mathcal{W}_{n,k} \rangle + \langle x_1^k, \ldots, x_n^k \rangle$. For $1 \leq i \leq k$, let $w^i \in [k]^n$ be the unique weakly increasing word with letters $[k] - \{i\}$ whose first $k-1$ letters are distinct. For example, the word $w^3 \in [6]^7$ is $w^3 = 1245666$. Then w^i is not Fubini, so that $\mathfrak{S}_{w^i}$ is a generator of $K_{n,k}$. One shows that $\mathfrak{S}_{w^i} = e_{n-i+1}$, so that we have $I_{n,k} \subseteq K_{n,k}$.

The containment $I_{n,k} \subseteq K_{n,k}$ of ideals means that we have a canonical surjection of rings

$$\pi : R_{n,k} = \mathbb{Z}[x_1, \ldots, x_n]/I_{n,k} \twoheadrightarrow \mathbb{Z}[x_1, \ldots, x_n]/K_{n,k} = H^\bullet(X_{n,k}). \tag{5.10}$$

By Theorem 2, the target of π is a free $\mathbb{Z}$-module of rank $k! \cdot \mathrm{Stir}(n,k)$. One shows that the domain $R_{n,k}$ is also a free $\mathbb{Z}$-module of rank $k! \cdot \mathrm{Stir}(n,k)$; *Demazure characters* play a key role in this argument.

Since any surjection between $\mathbb{Z}$-modules of the same finite rank is an isomorphism, the map π is an isomorphism of rings and (5.8) is proven. The remainder of the theorem comes from the corresponding statements about $(\mathbb{P}^{k-1})^n$. □

Line permutation endows the rational cohomology ring $H^\bullet(X_{n,k}; \mathbb{Q})$ with the structure of a graded S_n-module which is concentrated in even degree. Theorem 3 implies that

$$\mathrm{grFrob}(H^\bullet(X_{n,k}; \mathbb{Q}); \sqrt{q}) = \mathrm{grFrob}(R_{n,k}^{\mathbb{Q}}; q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q), \tag{5.11}$$

justifying our assertion that $X_{n,k}$ is the flag variety for the Delta Conjecture.

Haglund, Rhoades, and Shimozono discovered extensions of various monomial bases of $\mathbb{Q} \otimes_{\mathbb{Z}} R_n$ to $\mathbb{Q} \otimes_{\mathbb{Z}} R_{n,k}$. They asked [9, Prob. 7.2] for an extension of the Schubert basis; such an extension (valid over the integers) is given as follows.

Corollary 2. *The set $\{\mathfrak{S}_w : w \in \mathcal{W}_{n,k}\}$ descends to a $\mathbb{Z}$-basis of $R_{n,k}$.*

The structure constants involved in the basis of Corollary 2 can in general be negative.

6 Stability for $X_{n,k}$

There are two ways to grow a pair of integers (n,k) subject to the condition $k \leq n$:

$$(n,k) \rightsquigarrow (n+1,k) \text{ and } (n,k) \rightsquigarrow (n+1,k+1). \tag{6.1}$$

In this section we describe stability results for these two growth rules.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition and $n > 0$. If $n \geq |\lambda| + \lambda_1$, the *padded partition* is $\lambda[n] := (n - |\lambda|, \lambda_1, \lambda_2, \ldots) \vdash n$. Any partition of n has the form $\lambda[n]$ for a unique partition λ, so that any finite-dimensional S_n-module V has the form $V \cong \bigoplus_\lambda c_\lambda S^{\lambda[n]}$, where the direct sum is over *all* partitions λ.

Let $(V_n)_{n>0}$ be a sequence of finite-dimensional S_n-modules. For each $n > 0$ we can write $V_n \cong \bigoplus_\lambda c_{\lambda,n} S^{\lambda[n]}$ for some unique integers $c_{\lambda,n}$. We call the sequence V_n *multiplicity stable* [15] if for any partition λ, the sequence $c_{\lambda,n}$ is eventually constant.

Theorem 4. *Fix a cohomological degree d. Either of the module sequences*

$$\ldots,\ H^d(X_{n-1,k};\mathbb{Q}),\ H^d(X_{n,k};\mathbb{Q}),\ H^d(X_{n+1,k};\mathbb{Q}),\ \ldots \quad \textit{or} \tag{6.2}$$

$$\ldots,\ H^d(X_{n-1,k-1};\mathbb{Q}),\ H^d(X_{n,k};\mathbb{Q}),\ H^d(X_{n+1,k+1};\mathbb{Q}),\ \ldots \tag{6.3}$$

is multiplicity stable.

Proof. (Sketch.) Both of these module sequences are identically zero when d is odd, so assume $d = 2m$ is even.

Let $\mathrm{SYT}(n)$ be the family of standard Young tableaux with n boxes. Given a tableau $T \in \mathrm{SYT}(n)$, let $\mathrm{des}(T)$ be the number of descents in T and let $\mathrm{maj}(T)$ be the major index of T. Work of Haglund, Rhoades, and Shimozono [9] yields the tableau formula

$$\mathrm{grFrob}(H^\bullet(X_{n,k};\mathbb{Q});\sqrt{q}) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n-k \end{bmatrix}_q s_{\mathrm{shape}(T)} \tag{6.4}$$

for fixed $k \leq n$. A tableau T only contributes to this sum when $\mathrm{des}(T) < k$. Since $m = d/2$ is fixed, the representation stability asserted in the theorem follows from the standard combinatorial interpretation of the q-binomial $\left[{n-\mathrm{des}(T)-1 \atop n-k}\right]_q$ in terms of partitions inside a box of size $(n-k) \times (k - \mathrm{des}(T) - 1)$. □

We also mention that there exist growth rules for Fubini words $\mathcal{W}_{n,k} \to \mathcal{W}_{n+1,k}$ and $\mathcal{W}_{n,k} \to \mathcal{W}_{n+1,k+1}$ which give rise to stability results for the word Schubert polynomials $\mathfrak{S}_w$. The space constraints of this extended abstract preclude us from expanding on this, but see [12] for more information.

Acknowledgements

The authors are grateful to Sara Billey, Thomas Lam, and Vic Reiner for helpful conversations. B. Rhoades was partially supported by NSF Grant DMS-1500838. The authors thank two anonymous referees for their careful reading of the manuscript and valuable suggestions which improved our exposition.

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