Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #12, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Non-kissing versus non-crossing

Yann Palu*[1], Vincent Pilaud†[2], and Pierre-Guy Plamondon‡[3]

[1] *LAMFA, Université Picardie Jules Verne, Amiens*
[2] *CNRS & LIX, École Polytechnique, Palaiseau, France*
[3] *Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay*

Abstract. Starting from a locally gentle bound quiver, we define on the one hand a simplicial complex, called the non-kissing complex. On the other hand, we construct a punctured, marked, oriented surface with boundary, endowed with a pair of dual dissections. From those geometric data, we define two simplicial complexes: the accordion complex, and the slalom complex, generalizing work of A. Garver and T. McConville in the case of a disk. We show that all three complexes are isomorphic.

Résumé. Étant donné un carquois localement aimable, nous définissons d'une part un complexe simplicial appelé complexe platonique. D'autre part, nous construisons une surface orientable à bord pointée et marquée, munie d'une paire de dissections duales. À partir de cette donnée géométrique, nous définissons deux autres complexes simpliciaux : le complexe d'accordéons et le complexe des slaloms qui généralisent les travaux de A. Garver et T. McConville dans le cas du disque. Nous montrons que ces trois complexes sont isomorphes.

Keywords: Gentle algebras, non-kissing complex, non-crossing complex

1 Introduction

This extended abstract reports on [16], where more details and proofs are available. It shows that two combinatorial objects, the *non-kissing complex* and the *non-crossing complex*, are isomorphic. Both complexes appeared in different works in specific cases: the non-kissing complex of a grid appeared in [13], while the non-crossing complex of a disk appeared in [11, 12]. It was shown in [15] that these complexes are special cases of a more general simplicial complex, defined for any gentle algebra (see also [5]).

In order to unify these two objects, we are led to introduce two generalizations. On the algebraic side, the non-kissing complex is extended from the class of gentle algebras to that of locally gentle algebras (which are an infinite-dimensional version of gentle algebras). On the geometric side, we construct the non-crossing complex of an arbitrary oriented punctured surface endowed with a pair of dual dissections.

*yann.palu@u-picardie.fr. Supp. by ANR SC3A 15 CE40 0004 01.
†vincent.pilaud@lix.polytechnique.fr. Supp. by ANR SC3A 15 CE40 0004 01 and CAPPS 17 CE40 0018.
‡pierre-guy.plamondon@math.u-psud.fr. Supp. by ANR SC3A 15 CE40 0004 01 and a PEPS JCJC.

The two main objects of our study will thus be locally gentle algebras and dissections of surfaces. Our first result is that these two classes of objects are essentially the same.

Theorem (see Theorem 34)**.** *There is an explicit bijection between the set of isomorphism classes of locally gentle bound quivers and the set of homeomorphism classes of oriented punctured marked surfaces with boundary endowed with a pair of dual cellular dissections.*

On the algebraic side, we extend the study of walks from the gentle case [13, 15] to the locally gentle case, and define a notion of compatibility called non-kissing (Section 2). On the geometric side, we extend the study of dissections, accordions and slaloms from the case of the disk [11, 12] to the case of an arbitrary surface, and define a notion of compatibility called non-crossing (Section 3). The combinatorial information contained in these notions is encoded in simplicial complexes: the non-kissing complex $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ and the non-crossing complex $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}, \mathrm{D}^*)$. Our second result is the following.

Theorem (see Theorem 44)**.** *The complexes $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ and $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}, \mathrm{D}^*)$ are isomorphic.*

Let us briefly review the algebraic and geometric objects which appear in this paper. Gentle algebras are a class of finite-dimensional associative algebras over a field, defined by generators and relations. Their representation theory was first systematically investigated in [6], and they have been thoroughly studied since. Locally gentle algebras are obtained by dropping the requirement that gentle algebras be finite-dimensional. It turns out that their representation theory is also well-behaved [9] and that these algebras are Koszul [4]. The τ-tilting theory [1] of gentle algebras was recently studied in [15, 5].

Dissections of surfaces, on the other hand, are certain collections of pairwise non-intersecting curves on an orientable surface. They have been defined and studied in [8, 11, 12] in the case where the surface is an unpunctured disk.

The idea of associating a finite-dimensional algebra to a dissection (or a triangulation) of a surface seems to take its roots in the theory of cluster algebras and cluster categories. This was first done for triangulations of polygons in [7], and then for any orientable surface with boundary in [2]. The algebra of a dissection as we shall use it in this paper has appeared in [10]. In most of the above cases, the algebras obtained are gentle algebras. It has also been shown in [3] that any gentle algebra is obtained from a dissection of a surface, and that the module category of the algebra can be interpreted by using curves on the surface. In the case where the surface is a polygon, the τ-tilting theory of the algebra of a dissection has been studied in [15].

Conversely, the construction of a surface associated to a gentle algebra has appeared in [14]. We give a different construction of the same surface in this paper, which is obtained by "gluing" quadrilaterals to the arrows of the blossoming quiver (as defined in [15], and called "fringed quiver" in [5]). Our construction has the advantage that it easily yields the two dual dissections of the surface at the same time (the dissection and dual lamination of [14]). Note that our dissections are always cellular, while those in [3] can be arbitrary.

2 Non-kissing complex

2.1 Locally gentle bound quivers and their blossoming quivers

We consider a *bound quiver* $\bar{Q} = (Q, I)$, formed by a finite quiver $Q = (Q_0, Q_1, s, t)$ and an ideal I of the path algebra kQ (the k-vector space generated by all paths in Q, including vertices as paths of length zero, with multiplication induced by concatenation of paths) such that I is generated by linear combinations of paths of length at least two. Note that we **do not require** that the quotient algebra kQ/I be finite dimensional. The following definition is adapted from [6]. We note that it is customary to consider non-finite dimensional gentle algebras in the literature, see e.g. [17].

Definition 1. *A locally gentle bound quiver $\bar{Q} := (Q, I)$ is a (finite) bound quiver where*

(i) each vertex $a \in Q_0$ has at most two incoming and two outgoing arrows,

(ii) the ideal I is generated by paths of length exactly two,

(iii) for any arrow $\beta \in Q_1$, there is at most one arrow $\alpha \in Q_1$ with $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$ (resp. $\alpha\beta \in I$) and at most one arrow $\gamma \in Q_1$ with $t(\beta) = s(\gamma)$ and $\beta\gamma \notin I$ (resp. $\beta\gamma \in I$).

The algebra kQ/I is called a locally gentle algebra. A gentle bound quiver is a locally gentle bound quiver $\bar{Q}$ such that the algebra kQ/I is finite-dimensional; kQ/I is then a gentle algebra.

Definition 2. *A locally gentle bound quiver $\bar{Q}$ is complete if any vertex $a \in Q_0$ is incident to either one (a is a leaf) or four arrows (a is an internal vertex). The pruned subquiver of a quiver $\bar{Q}$ is the locally gentle quiver obtained by deleting all leaves of $\bar{Q}$ (degree one vertices) and their incident arrows. The blossoming quiver of a locally gentle bound quiver $\bar{Q}$ is the complete locally gentle bound quiver $\bar{Q}^{❀}$ whose pruned subquiver is $\bar{Q}$. The vertices of $Q_0^{❀} \smallsetminus Q_0$ are called blossom vertices, and the arrows in $Q_1^{❀} \smallsetminus Q_1$ are called blossom arrows. See Figure 1.*

Remark 3. *Note that $\bar{Q}^{❀}$ has $2|Q_0| - |Q_1|$ incoming blossom arrows and $2|Q_0| - |Q_1|$ outgoing blossom arrows. Therefore, it has $|Q_0^{❀}| = |Q_0| + 2(2|Q_0| - |Q_1|) = 5|Q_0| - 2|Q_1|$ vertices and $|Q_1^{❀}| = |Q_1| + 2(2|Q_0| - |Q_1|) = 4|Q_0| - |Q_1|$ arrows.*

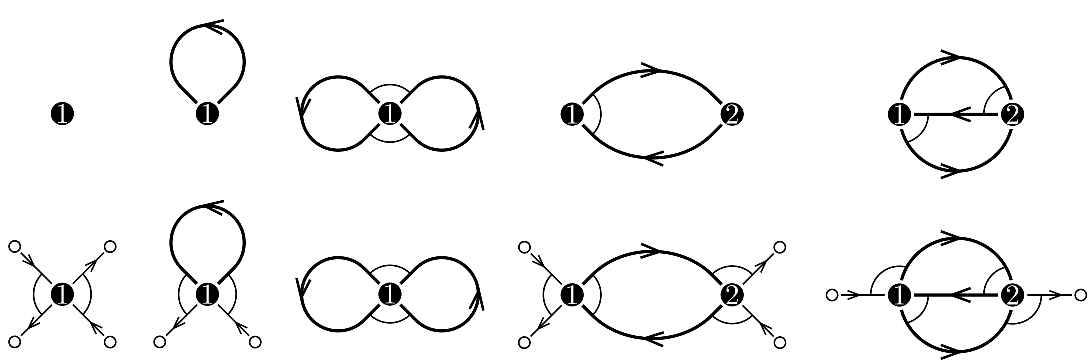


Figure 1: Some locally gentle quivers (top) and their blossoming quivers (bottom).

2.2 Strings and walks

The non-kissing complex is constructed using the combinatorics of strings and walks in the quiver $\bar{Q}$, whose definitions are now briefly recalled. The terminology and notations in the following definitions is borrowed from [6, 9].

For any arrow α of Q, define a formal inverse α^{-1} with the properties that $s(\alpha^{-1}) = t(\alpha)$, $t(\alpha^{-1}) = s(\alpha)$, $\alpha^{-1}\alpha = \varepsilon_{t(\alpha)}$ and $\alpha\alpha^{-1} = \varepsilon_{s(\alpha)}$, where ε_v is the path of length zero starting and ending at the vertex $v \in Q_0$.

Definition 4. *Let $\bar{Q} = (Q, I)$ be a locally gentle bound quiver. A finite string in $\bar{Q}$ is a word of the form $\rho = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_\ell^{\varepsilon_\ell}$, where*

(i) $\alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$ for all $i \in [\ell]$,

(ii) $t(\alpha_i^{\varepsilon_i}) = s(\alpha_{i+1}^{\varepsilon_{i+1}})$ for all $i \in [\ell - 1]$,

(iii) there is no path $\pi \in I$ such that π or π^{-1} appears as a factor of ρ, and

(iv) ρ is reduced, in the sense that no factor $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ appears in ρ, for $\alpha \in Q_1$.

The integer ℓ is called the length of the string ρ. We let $s(\rho) := s(\alpha_1^{\varepsilon_1})$ and $t(\rho) := t(\alpha_\ell^{\varepsilon_\ell})$ denote the source and target of ρ. For each vertex $a \in Q_0$, there is also a string of length zero, denoted by ε_a, that starts and ends at a.

If c is an oriented cycle in Q such that $c, c^2 \notin I$, we write c^∞ for the infinite word $ccc\cdots$, ${}^\infty c$ for the infinite word $\cdots ccc$, and ${}^\infty c^\infty$ for the bi-infinite word ${}^\infty cc^\infty$. We also define ${}^{-\infty}c := {}^\infty(c^{-1}) = (c^\infty)^{-1}$ and similarly $c^{-\infty} := (c^{-1})^\infty = ({}^\infty c)^{-1}$.

Definition 5. *An eventually cyclic string for $\bar{Q}$ is a word ρ of the form ${}^\infty(c_1^{\varepsilon_1})\sigma(c_2^{\varepsilon_2})^\infty$, where c_1, c_2 are oriented cycles in $\bar{Q}$ (possibly of length zero) and $\varepsilon_1, \varepsilon_2$ are signs in $\{\pm 1\}$ such that $c_1^{2\varepsilon_1}\sigma c_2^{2\varepsilon_2}$ is a finite string in $\bar{Q}$.*

Definition 6. *A string for $\bar{Q}$ is a word which is either a finite string or an eventually cyclic infinite string. We often implicitly identify the strings ρ and ρ^{-1}, and call it an undirected string.*

To avoid distinguishing between finite and (bi-)infinite words, we denote strings as products $\rho = \prod_{i<\ell<j} \alpha_\ell^{\varepsilon_\ell}$ where $i < j \in \mathbb{Z} \cup \{\pm\infty\}$, $\varepsilon_\ell \in \{\pm 1\}$ and $\alpha_\ell \in Q_1$ for all $i < \ell < j$.

Definition 7. *A walk of a locally gentle quiver $\bar{Q}$ is a maximal string of its blossoming quiver $\bar{Q}^{❀}$ (meaning that at each end it either reaches a blossom vertex of $\bar{Q}^{❀}$ or enters an infinite oriented cycle). We implicitly identify the walks ω and ω^{-1}, and call it an undirected walk.*

Definition 8. *A substring of a walk $\omega = \prod_{i<\ell<j} \alpha_\ell^{\varepsilon_\ell}$ of $\bar{Q}$ is a string $\sigma = \prod_{i'<\ell<j'} \alpha_\ell^{\varepsilon_\ell}$ of $\bar{Q}$ for some $i \le i' < j' \le j$, where the inequality $i \le i'$ (resp. $j' \le j$) is strict when $i \neq -\infty$ (resp. $j \neq \infty$). In other words, σ is a factor of ω such that*

- *the endpoints of σ are not allowed to be the possible blossom endpoints of ω,*
- *the position of σ as a factor of ω matters.*

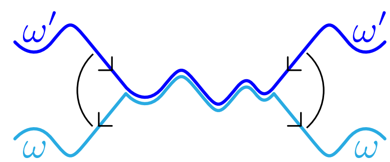
Note that the string ε_a is a substring of ω for each occurrence of a as a vertex of ω. We denote by $\Sigma(\omega)$ the set of substrings of ω. We use the same notation for undirected walks.

Definition 9. *We say that the substring $\sigma = \prod_{i'<\ell<j'} \alpha_\ell^{\varepsilon_\ell}$ is at the bottom (resp. on top) of the walk $\omega = \prod_{i<\ell<j} \alpha_\ell^{\varepsilon_\ell}$ if $i' = -\infty$ or $\varepsilon_{i'} = 1$ and $j' = +\infty$ or $\varepsilon_{j'} = -1$ (resp. if $i' = -\infty$ or $\varepsilon_{i'} = -1$ and $j' = \infty$ or $\varepsilon_{j'} = 1$). In other words the (at most) two arrows of ω incident to the endpoints of σ point towards σ (resp. outwards from σ). We denote by $\Sigma_{\text{bot}}(\omega)$ and $\Sigma_{\text{top}}(\omega)$ the sets of bottom and top substrings of ω. We use the same notations for undirected walks.*

Definition 10. *A walk ω is straight if ω or ω^{-1} is a path in $\bar{Q}^{❀}$, and bending otherwise.*

2.3 Non-kissing complex

We can now define the non-kissing complex of a locally gentle bound quiver following [13, 15, 5]. We start with the kissing relation for walks.



Definition 11. *Let ω and ω' be two undirected walks on $\bar{Q}$. We say that ω kisses ω' if $\Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega')$ contains a* finite *substring. We say that ω and ω' are kissing if ω kisses ω' or ω' kisses ω (or both). We allow the situation where the common finite substring is reduced to a vertex a, meaning that a is a peak of ω and a deep of ω'. Note also that ω can kiss ω' several times, that ω and ω' can mutually kiss, and that ω can kiss itself.*

Definition 12. *The non-kissing complex of $\bar{Q}$ is the simplicial complex $\mathcal{K}_{\text{nk}}(\bar{Q})$ whose faces are the sets of pairwise non-kissing walks of $\bar{Q}$. Note that self-kissing walks are excluded from the vertex set of this complex.*

3 Accordions, slaloms, and non-crossing complex

3.1 Dual dissections of a surface

Before defining accordions and slaloms, we need a strong notion of pairs of dual dissections of an orientable surface. We first review classical definitions of curves, arcs and dissections of a surface adapting it to our setting.

Definition 13. *A marked surface $\bar{\mathcal{S}} := (\mathcal{S}, M)$ is an orientable surface $\mathcal{S}$ with boundaries, together with a set M of marked points which can be on the boundary of $\mathcal{S}$ or not. For $V \subset \mathcal{S}$,*

(i) a V-arc on $\bar{\mathcal{S}}$ is a curve on $\mathcal{S}$ connecting two points of V and whose interior is disjoint from M and the boundary of $\mathcal{S}$.

(ii) a V-curve on $\bar{\mathcal{S}}$ is a curve on $\mathcal{S}$ which at each end either reaches a point of V or infinitely circles around and finally reaches a puncture of M, and whose interior is disjoint from M and the boundary of $\mathcal{S}$.

As usual, curves and arcs are considered up to homotopy relative to their endpoints in $\mathcal{S} \smallsetminus M$, and curves homotopic to a boundary are not allowed.

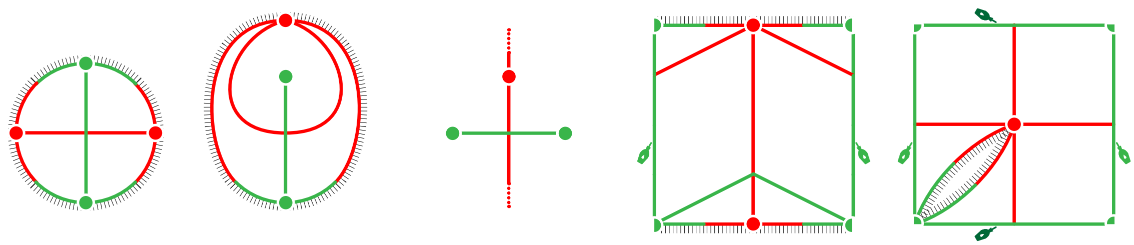


Figure 2: Some pairs $(\mathrm{D}, \mathrm{D}^*)$ of dual cellular dissections on different surfaces. The dissection D is in green while its dual dissection D^* is in red. The boundaries of the surfaces are shaded, and the glue symbols indicate how to identify edges.

Definition 14. *Two curves or arcs cross when they intersect in their interior. We will always assume that collections of arcs on a surface are in minimal position, in the sense that they cross each other transversely, and the number of crossings is minimal.*

Definition 15. *A dissection of $\bar{\mathcal{S}}$ is a collection D of pairwise non-crossing arcs on $\bar{\mathcal{S}}$. The edges of D are its arcs together with the boundary arcs of $\bar{\mathcal{S}}$. The faces of D are the connected components of the complement of the union of the edges of D in the surface $\mathcal{S}$. We denote by $\mathcal{V}(\mathrm{D})$, $\mathcal{E}(\mathrm{D})$ and $\mathcal{F}(\mathrm{D})$ the sets of vertices, edges and faces of D respectively. The dissection D is cellular if all its faces are topological disks. For $V \subseteq M$, a V-dissection is a dissection with only V-arcs.*

Convention 16. ***All throughout the paper, all dissections are considered cellular.***

Definition 17. *Consider a marked surface $\bar{\mathcal{S}} = (\mathcal{S}, V \sqcup V^*)$, where V and V^* are two disjoint sets of marked points so that the points of V and V^* that are on the boundary of $\mathcal{S}$ alternate. A cellular V-dissection D of $\bar{\mathcal{S}}$ and a cellular V^*-dissection D^* of $\bar{\mathcal{S}}$ are dual cellular dissections if there are pairs of mutually inverse bijections $V^* \leftrightarrow \mathcal{F}(\mathrm{D})$ and $V \leftrightarrow \mathcal{F}(\mathrm{D}^*)$, both denoted * in both directions, such that D has an edge joining its vertices $u, v \in V$ and separating its faces $f, g \in \mathcal{F}(\mathrm{D})$ if and only if D^* has an edge joining its vertices $f^*, g^* \in V^*$ and separating its faces $u^*, v^* \in \mathcal{F}(\mathrm{D}^*)$.*

Some examples of dual cellular dissections on different surfaces are represented in Figure 2. Note that contrarily to the usual conventions, the dual vertex f^* of a face f of D is not always in the interior of the face f. More precisely, each face f has either no or exactly one edge on the boundary of $\mathcal{S}$. Its dual vertex f^* then lies in the interior of f and is a puncture of $\bar{\mathcal{S}}$ in the former case, and on the boundary edge of f in the latter case.

Definition 18. *We consider a set B of points on the boundary of the surface $\mathcal{S}$ such that B and $V \cup V^*$ alternate along the boundary of $\mathcal{S}$. The points of B are called the blossom points. See e.g. Figures 3 and 4 where the blossom points appear as white hollow vertices.*

3.2 Accordion complex and slalom complex

Let $\bar{\mathcal{S}} = (\mathcal{S}, V \sqcup V^*)$ be a surface with two disjoint sets V and V^* of marked points, and let B be the corresponding blossom points on the boundary of $\mathcal{S}$. Consider two dual cellular dissections D and D* of $\bar{\mathcal{S}}$. The next definition generalises the one of [12] for the case where $\bar{\mathcal{S}}$ is a disk.

Definition 19. *A D-accordion is a B-curve α of $\bar{\mathcal{S}}$ such that whenever α meets a face f of* D*,*

- *(i) it enters crossing an edge a of f and leaves crossing an edge b of f (in other words, α is not allowed to circle around f^* when f^* is a puncture),*
- *(ii) the two edges a and b of f crossed by α are consecutive along the boundary of f,*
- *(iii) α, a and b bound a disk inside f that does not contain f^*.*

By convention, we also consider that the punctures of V are D*-accordions.*

Figure 3 illustrates some D-accordions for the dissections D of Figure 2.

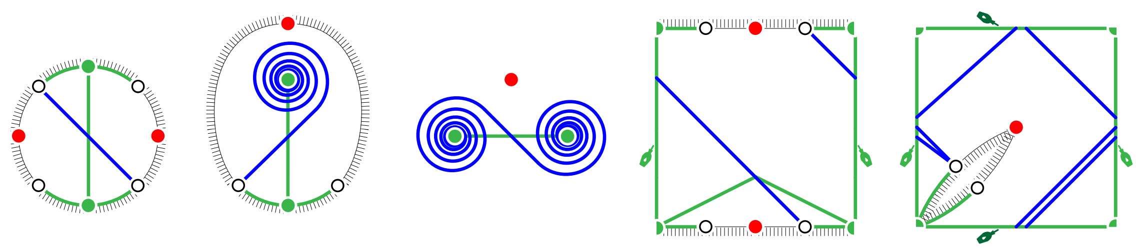


Figure 3: Some D-accordions (in blue) for the dissections D of Figure 2 (in green).

Remark 20. *In Definition 19, observe that*

- *the edges of a and b of f crossed by α might coincide, see the second example in Figure 3.*
- *the first condition is automatically satisfied if f^* is not a puncture.*

Definition 21. *The D-accordion complex $\mathcal{K}_{\text{acc}}(\text{D})$ is the simplicial complex whose faces are the sets of pairwise non-crossing* D*-accordions. Note that self-crossing accordions are excluded from the vertex set of this complex.*

The following definition generalizes the one from [11] for the case where $\bar{\mathcal{S}}$ is a disk.

Definition 22. *A D*-slalom is a B-curve α of $\bar{\mathcal{S}}$ such that, whenever α crosses an edge a^* of* D* *contained in two faces f^*, g^* of* D**, the marked points f and g lie on opposite sides of α in the union of f^* and g^* glued along a^*. Here, we consider that f lies on the right (resp. left) of α when α circles clockwise (resp. counterclockwise) around f. By convention, we also consider that the punctures of V are* D**-slaloms.*

Figure 4 illustrates some D*-slaloms for the dual dissections D* of Figure 2.

Definition 23. *The D*-slalom complex $\mathcal{K}_{\text{sla}}(\text{D}^*)$ is the simplicial complex whose faces are the sets of pairwise non-crossing* D**-slaloms. Note that self-crossing slaloms are forbidden.*

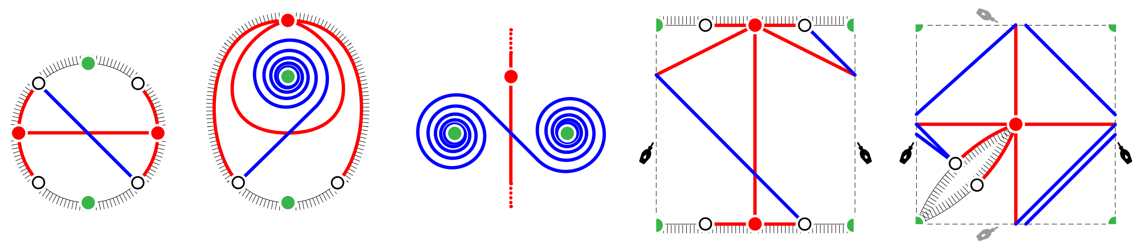


Figure 4: Some D^*-slaloms (in blue) for the dual dissections D^* of Figure 2 (in red).

3.3 Accordions versus slaloms and the non-crossing complex

Proposition 24. *For two dual cellular dissections* D *and* D^**, the* D*-accordions are precisely the* D^**-slaloms and the* D*-accordion complex coincides with the* D^**-slalom complex. See Figures 3 and 4.*

Definition 25. *The* <u>*non-crossing complex*</u> *of the pair of dual cellular dissections* (D, D^*) *is the simplicial complex* $\mathcal{K}_{nc}(D, D^*) := \mathcal{K}_{acc}(D) = \mathcal{K}_{sla}(D^*)$*.*

4 Non-kissing versus non-crossing

4.1 The bound quiver of a dissection

We first associate a quiver to a pair of dual dissections (D, D^*).

Definition 26. *The quiver of the dissection* D *is the quiver* $\bar{Q}_D = (Q_D, I_D)$ *defined by:*

(i) the set of vertices of Q_D *is the set of edges of* D*;*

(ii) there is an arrow from a *to* b *for each common endpoint* v *of* a *and* b *such that* b *comes immediately after* a *in the counterclockwise order around* v*;*

(iii) the ideal I_D *is generated by the paths of length two in* Q_D *obtained by composing arrows which correspond to triples of consecutive edges in a face of* D*.*

The bound quiver of the dissection D^* *is the bound quiver* $\bar{Q}_{D^*} = (Q_{D^*}, I_{D^*})$ *defined by replacing* D *by* D^* *in the above. See Figure 5.*

Remark 27. *The blossoming quiver* $\bar{Q}_D^{\text{❀}}$ *of the quiver* $\bar{Q}_D$ *is obtained with the same procedure by considering additional blossom vertices along the boundary of the surface. See Figure 5.*

Lemma 28. *The bound quiver* $\bar{Q}_D = (Q_D, I_D)$ *is a locally gentle bound quiver.*

Definition 29. *The* <u>*Koszul dual*</u> *of a locally gentle bound quiver* $\bar{Q} = (Q, I)$ *is the bound quiver* $\bar{Q}^! = (Q^!, I^!)$ *defined as follows:*

(i) the quiver $Q^!$ *is equal to the opposite quiver of* Q *(obtained from* Q *by reversing all arrows);*

(ii) the ideal $I^!$ *is generated by the opposites of the paths of length two in* Q *that are not in* I*.*

Proposition 30 ([14, Prop. 1.25])**.** *Let* D *and* D^* *be two dual cellular dissections of a marked surface* $(\mathcal{S}, V \sqcup V^*)$*. The bound quivers* $\bar{Q}_D$ *and* $\bar{Q}_{D^*}$ *are Koszul dual to each other. See Figure 5.*

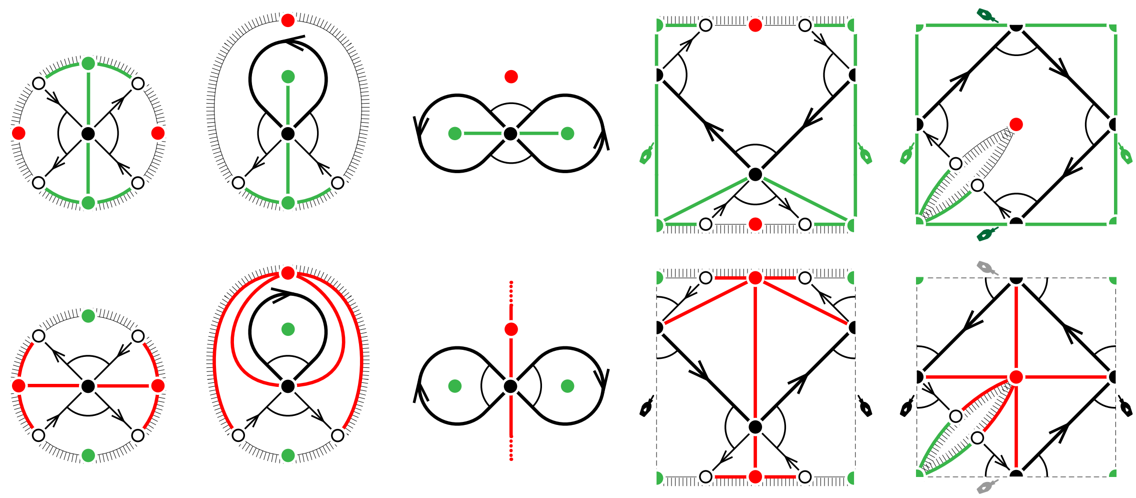
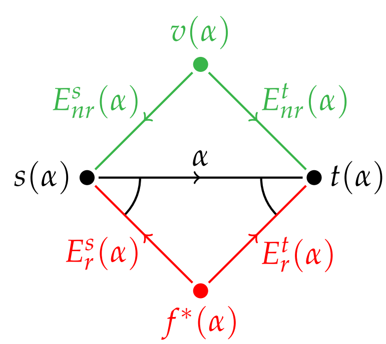


Figure 5: The quivers associated to the dissections of Figure 2.

4.2 The surface of a locally gentle bound quiver

We now associate a surface to a locally gentle bound quiver. This construction yields the same surface as the one constructed in [14].

Definition 31. *The surface $\mathcal{S}_{\bar{Q}}$ of a locally gentle bound quiver $\bar{Q} = (Q, I)$ is the surface obtained from the blossoming quiver $\bar{Q}^{❀}$ as follows:*



(i) for each arrow $\alpha \in Q_1^{❀}$, consider a lozenge $L(\alpha)$ with sides

$E^s_{nr}(\alpha) = [v(\alpha), s(\alpha)]$ $E^t_{nr}(\alpha) = [v(\alpha), t(\alpha)]$ *(green)*
$E^s_r(\alpha) = [f^*(\alpha), s(\alpha)]$ $E^t_r(\alpha) = [f^*(\alpha), t(\alpha)]$ *(red)*

(ii) for any $\alpha, \beta \in Q_1^{❀}$ with $t(\alpha) = s(\beta)$, identify:

- *$E^t_r(\alpha)$ with $E^s_r(\beta)$ if $\alpha\beta \in I$,*
- *$E^t_{nr}(\alpha)$ with $E^s_{nr}(\beta)$ if $\alpha\beta \notin I$.*

The orientations on the edges are only used for identifications and can be immediately forgotten.

Definition 31 constructs an orientable surface $\mathcal{S}_{\bar{Q}}$ with boundaries. Figure 6 illustrates this construction for the quivers of Figure 1.

Definition 32. *The surface $\mathcal{S}_{\bar{Q}}$ is endowed with*

- *the set $V_{\bar{Q}}$ of points $v(\alpha)$ for $\alpha \in Q_1^{❀}$ after the identifications given by (ii),*
- *the $V_{\bar{Q}}$-dissection $\mathrm{D}_{\bar{Q}}$ given by all sides $E^s_{nr}(\alpha)$ and $E^t_{nr}(\alpha)$ for $\alpha \in Q_1^{❀}$ after the identifications given by (ii).*

The set $V^*_{\bar{Q}}$ and the $V^*_{\bar{Q}}$-dissection $\mathrm{D}^*_{\bar{Q}}$ are defined similarly by using $f^*(\alpha)$, $E^s_r(\alpha)$ and $E^t_r(\alpha)$.

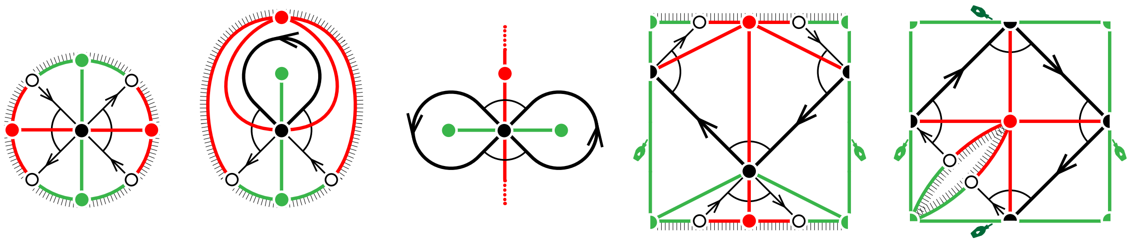


Figure 6: The surface $\mathcal{S}_{\bar{Q}}$ for the quivers $\bar{Q}$ of Figure 1.

Proposition 33. *Let $\bar{Q}$ be a locally gentle bound quiver. Then the dissections $\mathrm{D}_{\bar{Q}}$ and $\mathrm{D}^*_{\bar{Q}}$ are cellular and dual to each other.*

Theorem 34. *Up to isomorphism, the constructions of Definitions 26 and 31 are inverse to each other. They induce a bijection between the set of isomorphism classes of locally gentle bound quivers and the set of homeomorphism classes of marked surfaces with a pair of dual cellular dissections.*

Remark 35. *The following observations are useful for the computation of examples.*

(i) *The set $V_{\bar{Q}}$ has one vertex for each straight walk in $\bar{Q}$ (equivalently, for each maximal path in $\bar{Q}$). Finite straight walks yield vertices on the boundary of $\mathcal{S}_{\bar{Q}}$, while infinite cyclic straight walks in $\bar{Q}$ yield punctures of $\mathcal{S}_{\bar{Q}}$ in $V_{\bar{Q}}$. We denote by p the number of infinite cyclic straight walks in $\bar{Q}$.*

(ii) *The dissection $\mathrm{D}_{\bar{Q}}$ has one edge for each vertex $a \in \bar{Q}_0$, obtained by concatenation of the sides $E^t_{nr}(\alpha) = E^s_{nr}(\beta)$ and $E^t_{nr}(\alpha') = E^s_{nr}(\beta')$ where $a = t(\alpha) = s(\beta) = t(\alpha') = s(\beta')$, $\alpha\beta \notin I$ and $\alpha'\beta' \notin I$. We denote by $\varepsilon(a)$ the edge of $\mathrm{D}_{\bar{Q}}$ corresponding to a.*

(iii) *The dissection $\mathrm{D}_{\bar{Q}}$ has one ℓ-cell for each straight walk of length ℓ in $\bar{Q}^!$.*

(iv) *Similar statements hold dually for $V^*_{\bar{Q}}$ and $\mathrm{D}^*_{\bar{Q}}$, and we define p^* and $\varepsilon^*(a)$ similarly.*

(v) *The number of punctures of $\mathcal{S}_{\bar{Q}}$ is the number $p + p^*$ of infinite straight walks in $\bar{Q}$ and $\bar{Q}^!$.*

(vi) *The number b of boundary components of $\mathcal{S}_{\bar{Q}}$ can be computed as follows. There are two natural perfect matchings whose vertices are the blossom vertices of $\bar{Q}$: one is obtained by joining the endpoints of each finite straight walk of $\bar{Q}$, and the other is obtained similarly from $\bar{Q}^!$. Let G be the superposition of these two perfect matchings. Then the number b of boundary components of $\mathcal{S}_{\bar{Q}}$ is the number of connected components of G.*

(vii) *The genus of the surface $\mathcal{S}_{\bar{Q}}$ is $g = (|Q_1| - |Q_0| - b - p - p^* + 2)/2$, where b is the number of boundary components (see above for a way to compute b) and $p + p^*$ the number of punctures (i.e. infinite straight walks in $\bar{Q}$ and in $\bar{Q}^!$).*

Proposition 36. *For any gentle bound quiver $\bar{Q}$ with Koszul dual $\bar{Q}^!$, the surfaces $\mathcal{S}_{\bar{Q}}$ and $\mathcal{S}_{\bar{Q}^!}$ coincide, but $\mathrm{D}_{\bar{Q}^!} = \mathrm{D}^*_{\bar{Q}}$ and $\mathrm{D}^*_{\bar{Q}^!} = \mathrm{D}_{\bar{Q}}$.*

Example 37. *Consider the family of cycle quivers indicated in Figure 7. We apply Remark 35 to understand the corresponding surfaces. The corresponding perfect matchings (see Remark 35 (vi))*

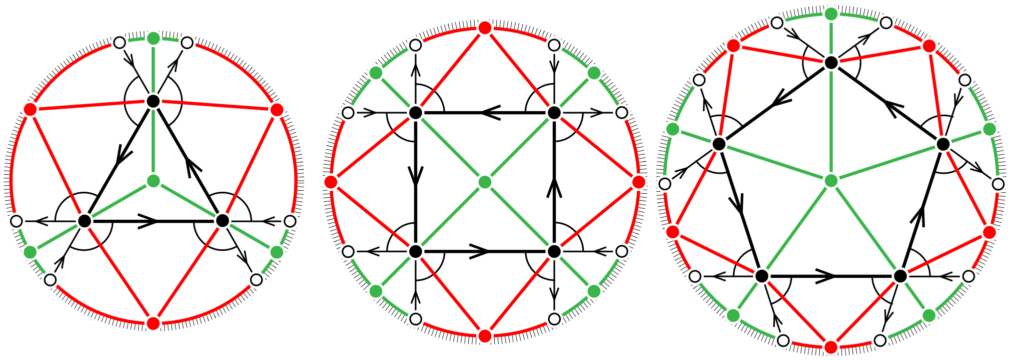


Figure 7: The surface $\mathcal{S}_{\bar{Q}}$ for the cycle quivers with $3, 4, 5$ vertices.

form a cycle with green and red arrows alternating. Moreover, $\bar{Q}$ has 1 infinite straight walk, while $\bar{Q}^!$ has none. Since $|Q_1| = |Q_0|$, the corresponding surfaces are punctured disks (1 boundary component, 1 puncture and genus 0). See Figure 7.

4.3 Non-crossing and non-kissing complexes coincide

Let $\bar{Q}$ be a locally gentle bound quiver. For each edge of the dissection $\mathrm{D}_{\bar{Q}}$ on $\mathcal{S}_{\bar{Q}}$, we fix a point on the interior of this edge, which we call its "middle point" (this is the black vertex on the pictures). To each walk on $\bar{Q}$, we will now associate a $\mathrm{D}_{\bar{Q}}$-accordion.

Definition 38. *For any arrow α on $\bar{Q}^{❀}$, let $\gamma(\alpha)$ be the curve on $\mathcal{S}_{\bar{Q}}$ which goes from the middle point of the edge of D corresponding to $s(\alpha)$ to the middle point of the edge of D corresponding to $t(\alpha)$ by following the angle corresponding to α. Define $\gamma(\alpha^{-1})$ to be $\gamma(\alpha)^{-1}$.*

Definition 39. *Let $\omega = \prod_{i<\ell<j} \alpha_\ell^{\varepsilon_\ell}$ be a (possibly infinite) walk on $\bar{Q}$. Define the curve $\gamma(\omega)$ to be the concatenation of the curves $\gamma(\alpha_\ell^{\varepsilon_\ell})$ of Definition 38.*

In practice, we will represent $\gamma(\omega)$ by a curve which intersects itself only transversely, and such that if it circles infinitely around a puncture, then it spirals towards it.

Lemma 40. *Let ω be a walk on $\bar{Q}$. Then $\gamma(\omega)$ is a $\mathrm{D}_{\bar{Q}}$-accordion.*

Lemma 41. *Let γ be a $\mathrm{D}_{\bar{Q}}$-accordion. There exists a unique undirected walk $\omega(\gamma)$ such that we have $\gamma(\omega(\gamma)) = \gamma$.*

Proposition 42. *The maps $\gamma(-)$ and $\omega(-)$ induce mutually inverse bijections between the set of undirected walks on $\bar{Q}$ and the set of $\mathrm{D}_{\bar{Q}}$-accordions on $\mathcal{S}_{\bar{Q}}$.*

Lemma 43. *Two undirected walks ω_1 and ω_2 on $\bar{Q}$ are non-kissing if and only if the corresponding $\mathrm{D}_{\bar{Q}}$-accordions $\gamma(\omega_1)$ and $\gamma(\omega_2)$ are non-crossing on $\mathcal{S}_{\bar{Q}}$.*

Theorem 44. *The non-kissing complex and the non-crossing complex are isomorphic.*

References

[1] T. Adachi, O. Iyama, and I. Reiten. “τ-tilting theory”. *Compos. Math.* **150**.3 (2014), pp. 415–452. Link.

[2] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, and P.-G. Plamondon. “Gentle algebras arising from surface triangulations”. *Algebra Number Theory* **4**.2 (2010), pp. 201–229. Link.

[3] K. Baur and R. Coelho Simões. “A geometric model for the module category of a gentle algebra”. 2018. arXiv:1803.05802.

[4] C. Bessenrodt and T. Holm. “Weighted locally gentle quivers and Cartan matrices”. *J. Pure and Appl. Algebra* **212**.1 (2008), pp. 204–221. Link.

[5] T. Brüstle, G. Douville, K. Mousavand, H. Thomas, and E. Yıldırım. “On the combinatorics of gentle algebras”. 2017. arXiv:1707.07665.

[6] M. C. R. Butler and C. M. Ringel. “Auslander-Reiten sequences with few middle terms and applications to string algebras”. *Comm. Algebra* **15**.1-2 (1987), pp. 145–179. Link.

[7] P. Caldero, F. Chapoton, and R. Schiffler. “Quivers with relations arising from clusters (A_n case)”. *Trans. Amer. Math. Soc.* **358**.3 (2006), pp. 1347–1364. Link.

[8] F. Chapoton. “Stokes posets and serpent nests”. *Discr. Math. Theor. Comput. Sci.* **18**.3 (2016). Link.

[9] W. Crawley-Boevey. “Classification of modules for infinite-dimensional string algebras”. *Trans. Amer. Math. Soc.* **370**.5 (2018), pp. 3289–3313. Link.

[10] L. David-Roesler and R. Schiffler. “Algebras from surfaces without punctures”. *J. Algebra* **350**.1 (2012), pp. 218–244. Link.

[11] A. Garver and T. McConville. “Oriented flip graphs of polygonal subdivisions and noncrossing tree partitions”. *J. Combin. Theory Ser. A* **158** (2018), pp. 126–175. Link.

[12] T. Manneville and V. Pilaud. “Geometric realizations of the accordion complex of a dissection”. *Discrete Comput. Geom.* **61**.3 (2019), pp. 507–540. Link.

[13] T. McConville. “Lattice structure of Grid-Tamari orders”. *J. Combin. Theory Ser. A* **148** (2017), pp. 27–56. Link.

[14] S. Opper, P.-G. Plamondon, and S. Schroll. “A geometric model for the derived category of gentle algebras”. 2018. arXiv:1801.09659.

[15] Y. Palu, V. Pilaud, and P.-G. Plamondon. “Non-kissing complexes and tau-tilting for gentle algebras”. In press, *Mem. Amer. Math. Soc.* 2017. arXiv:1707.07574.

[16] Y. Palu, V. Pilaud, and P.-G. Plamondon. “Non-kissing and non-crossing complexes for locally gentle algebras”. 2018. arXiv:1807.04730.

[17] J. Schröer. “Modules without self-extensions over gentle algebras”. *J. Algebra* **216**.1 (1999), pp. 178–189. Link.