Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #9, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Skew hook formula for d-complete posets via equivariant K-theory

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Abstract. Peterson and Proctor obtained a product formula for the multivariate generating function of P-partitions on a d-complete poset P in terms of hooks in P. In this article, we give a skew generalization of Peterson–Proctor's hook formula, i.e., a subtraction-free formula for the generating function of $(P \setminus F)$-partitions for a d-complete poset P and its order filter F. Our proof uses the equivariant K-theory of Kac–Moody partial flag varieties, and this generalization provides an alternate proof of Peterson–Proctor's hook formula.

Keywords: d-complete posets, hook formulas, P-partitions, equivariant K-theory

1 Introduction

The origin of hook formulas is the Frame–Robinson–Thrall hook formula [1], which asserts that, for a partition λ, the number f^λ of standard tableaux of shape λ is given by

$$f^\lambda = \frac{|\lambda|!}{\prod_{v \in D(\lambda)} h_{D(\lambda)}(v)}, \tag{1.1}$$

where $h_{D(\lambda)}(v)$ denotes the hook length of the cell v in the Young diagram $D(\lambda)$. Later Stanley [13] obtained a hook formula for the univariate generating function of reverse plane partitions of shape λ with respect to $|\sigma| = \sum_{v \in D(\lambda)} \sigma(v)$:

$$\sum_{\sigma \in \mathcal{A}(D(\lambda))} q^{|\sigma|} = \frac{1}{\prod_{v \in D(\lambda)} (1 - q^{h_{D(\lambda)}(v)})}, \tag{1.2}$$

and Gansner [2] gave a multivariate generalization of (1.2). Similar formulas hold for shifted Young diagrams and rooted trees.

Standard tableaux and reverse plane partitions of shape λ can be regarded as linear extensions and P-partitions of the poset $P = D(\lambda)$ respectively. Given an n-element poset

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P, a linear extension of P is an order-preserving bijection from P to $\{1, 2, \dots, n\}$, and a P-partition is an order-reversing map σ from P to $\mathbb{N}$, the set of nonnegative integers. We denote by $\mathcal{A}(P)$ the set of all P-partitions. Using Stanley's theory of P-partitions, we can derive (1.1) from (1.2).

Proctor [10, 11] introduced a wide class of posets, called d-complete posets, enjoying "hook-length property", as a generalization of Young diagrams, shifted Young diagrams and rooted trees. Peterson and Proctor obtained the following theorem, which is a far-reaching generalization of the hook formulas (1.1) and (1.2).

Theorem 1.1. (Peterson–Proctor, see [12]) Let P be a d-complete poset. The multivariate generating function of P-partitions is given by

$$\sum_{\sigma \in \mathcal{A}(P)} z^{\sigma} = \frac{1}{\prod_{v \in P} (1 - z[H_P(v)])}. \tag{1.3}$$

(Refer to Section 2 for undefined notations.)

However the original proof, based on representation theory, of this theorem is not yet published, though an outline of their proof is given in [12]. Different proofs are sketched by Ishikawa–Tagawa [3] (using Schur function identities) and Nakada [7] (using combinatorics of root systems). Our skew generalization (Theorem 1.2 below) provides an alternate proof of Theorem 1.1, which is based on equivariant Schubert calculus.

Another direction of generalizing the Frame–Robinson–Thrall hook formula (1.1) is to consider skew shapes. However one cannot expect a nice product formula for the number $f^{\lambda/\mu}$ of standard tableaux of skew shape λ/μ in general. Naruse [8] presented and sketched a proof of a subtraction-free formula for $f^{\lambda/\mu}$:

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}_{D(\lambda)}(D(\mu))} \frac{1}{\prod_{v \in D(\lambda) \setminus D} h_\lambda(v)}, \tag{1.4}$$

where and D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$. Morales–Pak–Panova [6] gave a q-analogue of Naruse's skew hook formula for the univariate generating functions for P-partitions on $P = D(\lambda) \setminus D(\mu)$.

The main result of this article is the following skew generalization of Peterson–Proctor's hook formula (Theorem 1.1). A subset F of a poset P is called an *order filter* of P if $x < y$ in P and $x \in F$ imply $y \in F$.

Theorem 1.2. Let P be a connected d-complete poset and F an order filter of P. Then the multivariate generating function of $(P \setminus F)$-partitions, where $P \setminus F$ is viewed as an induced subposet of P, is given by

$$\sum_{\sigma \in \mathcal{A}(P \setminus F)} z^{\sigma} = \sum_{D \in \mathcal{E}_P(F)} \frac{\prod_{v \in B(D)} z[H_P(v)]}{\prod_{v \in P \setminus D} (1 - z[H_P(v)])}, \tag{1.5}$$

where D runs over all excited diagrams of F in P. (See Section 2 for undefined notations.)

If $F = \emptyset$, then our main theorem (Theorem 1.2) gives Theorem 1.1. If $P = D(\lambda)$ and $F = D(\mu)$ are the Young diagrams of partitions $\lambda \supset \mu$, then (1.5) reduces to Morales–Pak–Panova's q-hook formula [6, Corollary 6.17] after specializing all variables z_i to q, and then to Naruse's hook formula (1.4) by the theory of P-partitions.

This article is an extended abstract of [9] and is organized as follows. In Section 2, we give basic definitions and notations for d-complete posets and excited diagrams/peaks. In Section 3, we provide Lie theoretical interpretations of the combinatorial notions for d-complete posets. In Section 4, we give a sketch of the proof of Theorem 1.2 by using the Billey-type formula and the Chevalley-type formula for the equivariant K-theory of the Kac–Moody partial flag variety.

2 Basic definitions and notations

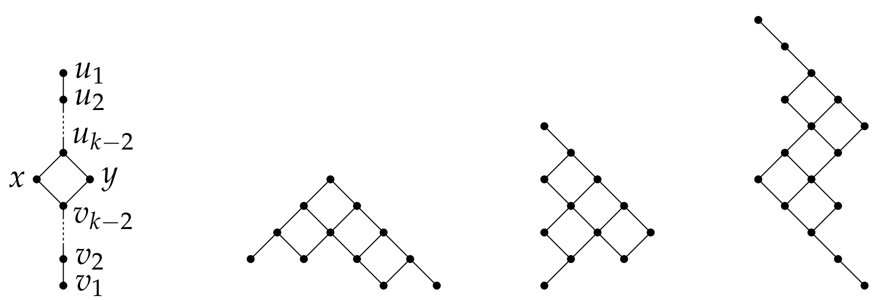
We give several definitions concerning d-complete posets and introduce the notion of excited diagrams and excited peaks.

2.1 Definition of d-complete posets

For an integer $k \geq 3$, we denote by $d_k(1)$ the poset consisting of $2k-2$ elements $u_1, \cdots, u_{k-2}, x, y, v_{k-2}, \cdots, v_1$ with covering relations

$$u_1 \gtrdot u_2 \gtrdot \cdots \gtrdot u_{k-2}, \quad u_{k-2} \gtrdot x \gtrdot v_{k-2}, \quad u_{k-2} \gtrdot y \gtrdot v_{k-2}, \quad v_{k-2} \gtrdot \cdots \gtrdot v_2 \gtrdot v_1.$$

Note that x and y are incomparable. The poset $d_k(1)$ is called the *double-tailed diamond*. The Hasse diagram of $d_k(1)$ is shown in Figure 1 (a).



(a) $d_k(1)$ (b) $D(5,4,2,1)$ (c) $S(5,4,2,1)$ (d) $e_6(1)$

Figure 1: Double-tailed diamond, shape, shifted shape and swivel

Let P be a poset. An interval $[v, u] = \{x \in P : v \leq x \leq u\}$ is called a d_k-*interval* if it is isomorphic to $d_k(1)$. Then v and u are called the *bottom* and *top* of $[v, u]$ respectively, and the two incomparable elements of $[v, u]$ are called the *sides*. A subset I of P is called *convex* if $x < y < z$ in P and $x, z \in I$ imply $y \in I$. A convex subset I is called a *d_k^--convex set* if it is isomorphic to the poset obtained by removing the top element from $d_k(1)$.

Definition 2.1. A poset P is *d-complete* if it satisfies the following three conditions for every $k \geq 3$:

(D1) If I is a d_k^--convex set, then there exists an element u such that u covers the maximal elements of I and $I \cup \{u\}$ is a d_k-interval.

(D2) If $I = [v, u]$ is a d_k-interval and the top u covers u' in P, then $u' \in I$.

(D3) There are no d_k^--convex sets which differ only in the minimal elements.

It is clear that rooted trees, viewed as posets with their roots being the maximum elements, are d-complete posets.

Example 2.2. We regard $\mathbb{Z}^2$ as a poset by defining $(i, j) \leq (i', j')$ if and only if $i \geq i'$ and $j \geq j'$. The following induced subposets of $\mathbb{Z}^2$ are d-complete:

$$\begin{aligned} D(\lambda) &= \{(i,j) \in \mathbb{Z}^2 : i \geq 1,\ 1 \leq j \leq \lambda_i\}, \\ S(\mu) &= \{(i,j) \in \mathbb{Z}^2 : i \geq 1,\ i \leq j \leq \mu_i + i - 1\}, \\ e_6(1) &= \left\{ \begin{array}{c} (1,1),(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5), \\ (3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(4,7),(4,8) \end{array} \right\}, \end{aligned}$$

where λ is a partition and μ is a strict partition. These posets are called a *shape*, a *shifted shape* and a *swivel* respectively. Figure 1 (b), (c), (d) illustrate the Hasse diagrams of $D(5,4,2,1)$, $S(5,4,2,1)$ and $e_6(1)$. Sometimes we represent subposets of $\mathbb{Z}^2$ as collections of unit cells like Young diagrams.

A poset P is called *connected* if the Hasse diagram of P is a connected graph. It is easy to see that, if P is a d-complete poset, then each connected component of P is d-complete. Hence, when considering the generating functions of P-partitions, we may assume that a d-complete poset is connected.

Proposition 2.3. ([10, §3]) If a d-complete poset P is connected, then P has a unique maximal element.

2.2 Top tree and d-complete coloring

Let P be a poset with a unique maximal element. The *top tree* Γ of P is an induced subgraph of the Hasse diagram of P, whose vertex set consists of all elements $x \in P$ such that the order filter generated by x is a chain. We will regard the top tree as a simply-laced Dynkin diagram. For example, the top trees of Figure 1 (a), (b), (c) and (d) are of type D_k, A_8, D_6 and E_6 respectively.

Proposition 2.4. ([11, Proposition 8.6]) Let P be a connected d-complete poset and Γ its top tree. Let I be a set of colors whose cardinality is the same as Γ. Then a bijective labeling $c : \Gamma \to I$ can be uniquely extended to a map, called a *d-complete coloring*, $c : P \to I$ satisfying the following three conditions:

(C1) If x and y are incomparable, then $c(x) \neq c(y)$.

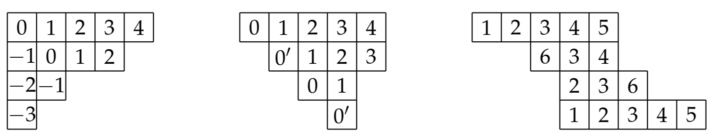
(C2) If an interval $[v, u]$ is a chain, then the colors $c(x)$ ($x \in [v, u]$) are distinct.

(C3) If $[v, u]$ is a d_k-interval then $c(v) = c(u)$.

Example 2.5. The assignments given in Figure 2 are d-complete colorings. In general, for a shape $D(\lambda)$ and a shifted shape $S(\mu)$ with $l(\mu) \geq 2$, the maps $c_{D(\lambda)} : D(\lambda) \to \{-(\lambda'_1 - 1), \dots, -1, 0, 1, \dots, \lambda_1 - 1\}$ and $c_{S(\mu)} : S(\mu) \to \{0, 0', 1, 2, \dots, \mu_1 - 1\}$ given by

$$c_{D(\lambda)}(i,j) = j - i, \quad c_{S(\mu)}(i,j) = \begin{cases} j - i & \text{if } i < j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd}, \\ 0' & \text{if } i = j \text{ and } i \text{ is even}. \end{cases}$$

are d-complete colorings.



(a) $D(5,4,2,1)$ (b) $S(5,4,2,1)$ (c) $e_6(1)$

Figure 2: d-complete colorings

2.3 Hook monomials

In the rest of this paper, we assume that P is a connected d-complete poset with top tree Γ, and fix a d-complete coloring $c : P \to I$, where I is identified with vertex set of Γ. Take a set of indeterminates $\boldsymbol{z} = (z_i)_{i \in I}$ indexed by I.

Given an order filter F of P, we regard $P \setminus F$ as the induced subposet. For a $(P \setminus F)$-partition $\sigma \in \mathcal{A}(P \setminus F)$, we put

$$\boldsymbol{z}^\sigma = \prod_{v \in P \setminus F} z_{c(v)}^{\sigma(v)},$$

and we are interested in the multivariate generating function $\sum_{\sigma \in \mathcal{A}(P \setminus F)} \boldsymbol{z}^\sigma$ of $(P \setminus F)$-partitions.

Definition 2.6. For each element $u \in P$, we define the monomial $\boldsymbol{z}[H_P(u)]$, called the *hook monomial* of u, inductively as follows:

(i) If u is not the top of any d_k-interval, then we define $z[H_P(u)] = \prod_{w \le u} z_{c(w)}$.

(ii) If u is the top of a d_k-interval $[v,u]$, then we define

$$z[H_P(u)] = \frac{z[H_P(x)] \cdot z[H_P(y)]}{z[H_P(v)]},$$

where x and y are the sides of $[v,u]$.

Example 2.7. If P is a shape $D(\lambda)$ or a shifted shape $S(\mu)$, then there is the classical notion of hooks $H_{D(\lambda)}(u) \subset D(\lambda)$ and $H_{S(\mu)}(u) \subset S(\mu)$ given by

$$H_{D(\lambda)}(i,j) = \{(i,l) \in D(\lambda) : l \ge j\} \cup \{(k,j) \in D(\lambda) : k > i\},$$
$$H_{S(\mu)}(i,j) = \{(i,l) \in S(\mu) : l \ge j\} \cup \{(k,j) \in S(\mu) : k > j\} \cup \{(j+1,l) \in S(\mu) : l > j\}.$$

Then the hook monomial $z[H_P(u)]$ in Definition 2.6 coincides with the product $\prod_{v \in H_P(u)} z_{c(v)}$.

2.4 Excited diagrams and excited peaks

In order to formulate a skew hook formula for d-complete posets, we need to generalize the notion of excited diagrams and excited peaks used in [8] and [6] to general d-complete posets.

For $i \in I$, let N_i be the subset of P consisting of elements $x \in P$ whose color $c(x)$ is adjacent to i in the Dynkin diagram Γ. Note that, if $[v,u]$ is a d_k-interval, then $[v,u] \cap N_{c(u)}$ consists of elements $x \in [v,u]$ such that x is covered by u or covers v.

Definition 2.8. Let P be a connected d-complete poset and let F be an order filter of P.

(a) Let D be a subset of P and $u \in D$. We say that u is *D-active* if there exists an element $v \in P \setminus D$ such that $v < u$, $[v,u]$ is a d_k-interval and $[v,u] \cap D \cap N_{c(u)} = \emptyset$.

(b) Let D be a subset of P and $u \in D$. If u is D-active, then we define $\alpha_u(D)$ to be the subset of P obtained from D by replacing $u \in D$ by the bottom element v of the d_k-interval $[v,u]$. We call this replacement an *elementary excitation*.

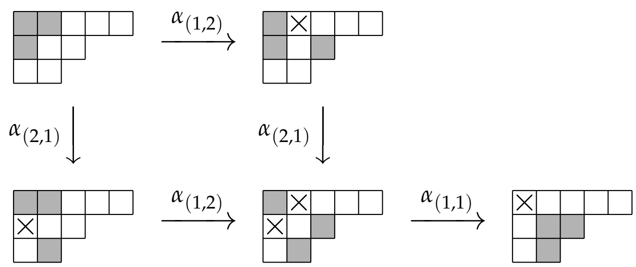
(c) An *excited diagram* of F in P is a subset of P obtained from F after a sequence of elementary excitations on active elements. Let $\mathcal{E}_P(F)$ be the set of all excited diagrams of F in P.

(d) To an excited diagram $D \in \mathcal{E}_P(F)$ we associate a subset $B(D) \subset P$ as follows: If $D = F$, then $B(F) = \emptyset$. If D is an excited diagram with an active element u, then we define

$$B(\alpha_u(D)) = \Big(B(D) \setminus ([v,u] \cap N_{c(u)})\Big) \cup \{u\},$$

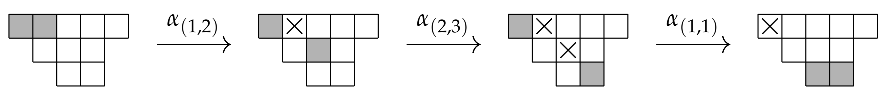
where $[v, u]$ is the d_k-interval with top element u. We call $B(D)$ the set of *excited peaks* of D. (It can be shown that $B(D)$ is a well-defined subset of $P \setminus D$.)

Example 2.9. (1) If $P = D(5, 3, 2)$ and $F = D(2, 1)$, then there are five excited diagrams of F in P.

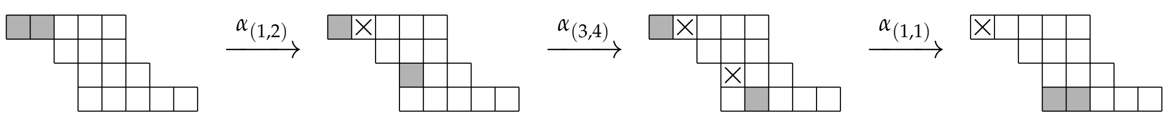


Here the shaded cells form an exited diagram and a cell with $\times$ is an excited peak.

(2) If $P = S(5, 3, 2)$ and $F = S(2)$, then there are four excited diagrams of F in P.



(3) If $P = e_6(1)$ and F is the order filter consisting of two elements, then there are four excited diagrams of F in P.



3 d-Complete posets, Weyl groups and root systems

In this section, we provide connections of combinatorics of d-complete posets with Lie theory involving Weyl groups and root systems.

3.1 Lie theoretical interpretations

Let P be a connected d-complete poset with top tree Γ, and regard Γ as a (simply-laced) Dynkin diagram with node set I. We fix an associated root datum $(\Lambda, \Lambda^*, \Pi, \Pi^\vee)$ consisting of a free $\mathbb{Z}$-module Λ (the weight lattice), its dual lattice Λ^* (the coweight lattice), a subset $\Pi = \{\alpha_i : i \in I\} \subset \Lambda$ (the set of simple roots), and a subset $\Pi^\vee = \{\alpha_i^\vee : i \in I\} \subset \Lambda^*$ (the set of simple coroots) subject to certain conditions. Let W be the corresponding Weyl group generated by the simple reflections $\{s_i : i \in I\}$, $l : W \to \mathbb{N}$ the length function and $<$ the Bruhat order on W. Let $\Phi = W\Pi$ and $\Phi^\vee = W\Pi^\vee$ be the sets of real roots and real coroots respectively. The simple system Π (resp. $\Pi^\vee$) determines the decomposition of Φ (resp. $\Phi^\vee$) into the positive system Φ_+ (resp. $\Phi_+^\vee$) and the negative system Φ_- (resp. $\Phi_-^\vee$). Then the standard partial ordering $>$ on Φ_+

(resp. $\Phi_+^\vee$) is defined by setting $\alpha > \beta$ if $\alpha - \beta$ is a sum of simple roots (resp. simple coroots).

By using the d-complete coloring $c : P \to I$, we write $\alpha(p) = \alpha_{c(p)}$, $\alpha^\vee(p) = \alpha^\vee_{c(p)}$ and $s(p) = s_{c(p)}$ for each $p \in P$. Take a linear extension and label the elements of P with $p_1, \cdots, p_N$ ($N = \#P$) so that $p_i < p_j$ in P implies $i < j$. Then, for a subset $D = \{p_{i_1}, \dots, p_{i_r}\}$ ($i_1 < \cdots < i_r$) of P, we define

$$w_D = s(p_{i_1}) \dots s(p_{i_r}) \in W.$$

If $p = p_k \in P$, then we define

$$\beta(p_k) = s(p_1) \cdots s(p_{k-1}) \alpha(p_k) \in \Phi, \quad \gamma^\vee(p_k) = s(p_N) \cdots s(p_{k+1}) \alpha^\vee(p_k) \in \Phi^\vee.$$

It turns out that these elements w_D, $\beta(p)$ and $\gamma^\vee(p)$ are independent of the choices of linear extensions of P.

Let i_P be the color of the unique maximal element of P, and $\lambda_P \in \Lambda$ the corresponding fundamental weight. Let W_{λ_P} be the stabilizer of λ_P in W, which is the maximal parabolic subgroup corresponding to $I \setminus \{i_P\}$. Let W^{λ_P} be the set of minimum length coset representatives of W/W_{λ_P}, viewed as an induced subposet of W with respect to the Bruhat order.

Then we summarize connections of combinatorics of d-complete posets with Weyl groups and root systems. Proofs can be found in [10], [12] and [14].

Proposition 3.1. (a) The element $w_P \in W$ is λ_P-minuscule, i.e., $\langle \gamma^\vee(p), \lambda_P \rangle = 1$ for all $p \in P$, where $\langle \ , \ \rangle : \Lambda^* \times \Lambda \to \mathbb{Z}$ is the canonical pairing.

(b) w_P is fully commutative, i.e., any reduced expression of w can be obtained from any other by using only the Coxeter relations of the form $st = ts$.

(c) The poset P is isomorphic to the order dual of $\Phi_+^\vee \cap w_P^{-1} \Phi_-^\vee$ via the correspondence $p \mapsto \gamma^\vee(p)$.

(d) Under the identification $z_i = e^{\alpha_i}$ ($i \in I$), we have $\boldsymbol{z}[H_P(p)] = e^{\beta(p)}$ for any $p \in P$.

(e) The map $F \mapsto w_F$ gives a poset isomorphism from the set of all order filters of P ordered by inclusion to the Bruhat interval $[e, w_P]$ in W^{λ_P}.

(f) If F is an order filter, then w_F is λ_P-minuscule and $w_F \lambda_P = \lambda_P - \sum_{p \in F} \alpha(p)$.

3.2 Excited diagrams and Weyl groups

Let $* : W \times W \to W$ be the associative product, called the Demazure product, defined by

$$s_i * w = \begin{cases} s_i w & \text{if } l(s_i w) = l(w) + 1, \\ w & \text{if } l(s_i w) = l(w) - 1. \end{cases}$$

For a fixed linear extension of P and a subset $D = \{p_{i_1}, \cdots, p_{i_r}\}$ ($i_1 < \cdots < i_r$) of P, we define an element $w^*_D \in W$ by putting

$$w^*_D = s(p_{i_1}) * s(p_{i_2}) * \cdots * s(p_{i_r}).$$

It follows from Proposition 3.1(b) that the element w^*_D is independent of the choices of linear extensions of P The following proposition is one of the key ingredients of the proof of Theorem 1.2. See [9, Section 3] for the proof.

Proposition 3.2. Let F be an order filter of a connected d-complete poset P and $E \subset P$.

(a) E is an excited diagram of F in P, i.e., $E \in \mathcal{E}_P(F)$, if and only if $\#E = \#F$ and $w_E = w_F$.

(b) E is of the form $E = D \sqcup S$ for some $D \in \mathcal{E}_P(F)$ and $S \subset B(D)$ if and only if $w^*_E = w_F$.

4 Equivariant K-theory and proof of Theorem 1.2

In this section, we use the equivariant K-theory of a thick partial flag variety (see [4] for example) to give a sketch of the proof of our main theorem (Theorem 1.2).

4.1 Equivariant K-theory of Kac–Moody partial flag varieties

Let P be a d-complete poset with top tree Γ. From a fixed root datum $(\Lambda, \Lambda^*, \Pi, \Pi^\vee)$ associated to the Dynkin diagram Γ, we can construct the Kac–Moody group $\mathcal{G}$ over $\mathbb{C}$, its Borel subgroup $\mathcal{B}_-$ (corresponding to Φ_-) and maximal torus $\mathcal{T} \subset \mathcal{B}_-$. Let $i_P \in I$ be the color of the maximum element of P and λ_P the corresponding fundamental weight. Let $\mathcal{P}_- \supset \mathcal{B}_-$ be the parabolic subgroup of $\mathcal{G}$ corresponding to $J = I \setminus \{i_P\}$. Then we consider the Kashiwara thick partial flag variety $\mathcal{X} = \mathcal{G}/\mathcal{P}_-$.

Let $K_{\mathcal{T}}(\mathcal{X})$ be the $\mathcal{T}$-equivariant K-theory of $\mathcal{X}$. Then $K_{\mathcal{T}}(\mathcal{X})$ has a commutative associative $K_{\mathcal{T}}(\mathrm{pt})$-algebra structure. Here the $\mathcal{T}$-equivalent K-theory $K_{\mathcal{T}}(\mathrm{pt})$ of a point is isomorphic to the group algebra $\mathbb{Z}[\Lambda]$ with basis $\{e^\lambda : \lambda \in \Lambda\}$. In the following, we identify e^{α_i} with the indeterminates z_i corresponding to the color $i \in I$. Any elements of $K_{\mathcal{T}}(\mathcal{X})$ is a (possibly infinite) $K_{\mathcal{T}}(\mathrm{pt})$-linear combination of the equivariant Schubert classes $\{[\mathcal{O}_v] : v \in W^{\lambda_P}\}$, where $[\mathcal{O}_v]$ is the class of the structure sheaf $\mathcal{O}_v$ of the Schubert subvariety $\mathcal{X}_v$ and W^{λ_P} is the set of minimum length coset representatives in W/W_{λ_P}.

Each $w \in W^{\lambda_P}$ gives a $\mathcal{T}$-fixed point $e_w = w\mathcal{P}_-/\mathcal{P}_- \in \mathcal{X}$, and the inclusion map $\iota_w : \{e_w\} \to \mathcal{X}$ induces the pull-back ring homomorphism $\iota^*_w : K_{\mathcal{T}}(\mathcal{X}) \to K_{\mathcal{T}}(e_w) \cong \mathbb{Z}[\Lambda]$, called the localization map at w. For two elements $v, w \in W^{\lambda_P}$, we define

$$\xi^v|_w = \iota^*_w([\mathcal{O}_v]) \in \mathbb{Z}[\Lambda].$$

Then, by using Proposition 3.1 (d), we can derive the following explicit expression from the Billey-type formula [4, Proposition 2.10].

Proposition 4.1. For a connected d-complete poset P and its order filter F, we have

$$\xi^{w_F}|_{w_P} = \sum_{E : w_E^* = w_F} (-1)^{\#E - \#F} \prod_{p \in E} (1 - \boldsymbol{z}[H_P(p)]), \tag{4.1}$$

where the summation is taken over all subsets $E \subset P$ satisfying $w_E^* = w_F$.

4.2 Equivariant K-theoretical Littlewood–Richardson coefficients

We consider the structure constants for the multiplication in $K_{\mathcal{T}}(\mathcal{X})$ with respect to the equivariant Schubert classes. For $u, v, w \in W^J$, we denote by $c_{u,v}^w \in K_{\mathcal{T}}(\mathrm{pt})$ the structure constant determined by

$$[\mathcal{O}_u][\mathcal{O}_v] = \sum_{w \in W^{\lambda_P}} c_{u,v}^w [\mathcal{O}_w]. \tag{4.2}$$

Then we have $c_{u,v}^w = 0$ unless $u \le w$ and $v \le w$. It is not difficult to prove the following proposition.

Proposition 4.2. (a) For $v, w \in W^{\lambda_P}$, we have $c_{v,w}^w = \xi^v|_w$.

(b) Let $u, v, w \in W^{\lambda_P}$ and put $s = s_{i_P} \in W^{\lambda_P}$. If $c_{s,w}^w \neq c_{s,u}^u$, then we have

$$c_{u,w}^w = \frac{1}{c_{s,w}^w - c_{s,u}^u} \sum_{u < x \le w} c_{s,u}^x c_{x,w}^w. \tag{4.3}$$

By using Proposition 3.1 (c), (e) and (f), we can prove the following explicit formula from the Chevalley-type formula [5, Theorem 4.8]. (See [9, Section 4] for the proof.)

Proposition 4.3. Let P be a connected d-complete poset and put $s = s_{i_P}$. For two order filters F and F' of P, we have

$$c_{s,w_F}^{w_{F'}} = \begin{cases} 1 - \boldsymbol{z}[F] & \text{if } F' = F, \\ (-1)^{\#(F' \setminus F) - 1} \boldsymbol{z}[F] & \text{if } F' \supsetneq F \text{ and } F' \setminus F \text{ is an antichain}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

where $\boldsymbol{z}[F] = \prod_{p \in F} z_{c(p)}$.

4.3 Outline of the Proof of the Main Theorem

Now we are ready to prove our Main Theorem 1.2. Theorem 1.2 follows from the following two identities:

$$\sum_{\sigma \in \mathcal{A}(P \setminus F)} z^{\sigma} = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}, \quad \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}} = \sum_{D \in \mathcal{E}_P(F)} \frac{\prod_{q \in B(D)} z[H_P(q)]}{\prod_{p \in P \setminus D} (1 - z[H_P(p)])}. \tag{4.5}$$

We use Proposition 4.3 to prove the first identity of (4.5). We proceed by induction on $\#(P \setminus F)$. For an order filter F of P, we put

$$G_{P/F}(z) = \sum_{\sigma \in \mathcal{A}(P \setminus F)} z^{\sigma}, \quad Z_{P/F}(z) = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}.$$

Since $Z_{P/P}(z) = G_{P/P}(z) = 1$, it is enough to show that $Z_{P/F}(z)$ and $G_{P/F}(z)$ satisfy the same recurrence of the form

$$X_{P/F}(z) = \frac{1}{1 - z[P \setminus F]} \sum_{F'} (-1)^{\#(F' \setminus F) - 1} X_{P/F'}(z), \tag{4.6}$$

where F' runs over all order filters such that $F \subsetneq F' \subset P$ and $F' \setminus F$ is an antichain.

First we show that $G_{P/F}$ satisfies (4.6). Let M be the set of all maximal elements of $P \setminus F$. For a subset $I \subset M$, let

$$\mathcal{A}(P \setminus F)_I = \{\sigma \in \mathcal{A}(P \setminus F) : \sigma(x) = 0 \text{ for all } x \in I\},$$
$$\mathcal{A}'(P \setminus F) = \{\sigma \in \mathcal{A}(P \setminus F) : \sigma(x) = 0 \text{ for some } x \in M\}.$$

Then $G_{P/(F \sqcup I)}(z) = \sum_{\sigma \in \mathcal{A}(P \setminus F)_I} z^{\sigma}$ and by using the Inclusion-Exclusion Principle we have

$$\sum_{F'} (-1)^{\#(F' \setminus F) - 1} G_{P/F'}(z) = \sum_{I \subset M,\ I \neq \emptyset} (-1)^{\# I - 1} \sum_{\sigma \in \mathcal{A}(P \setminus F)_I} z^{\sigma} = \sum_{\sigma \in \mathcal{A}'(P \setminus F)} z^{\sigma}.$$

Also it is not difficult to show

$$\sum_{\sigma \in \mathcal{A}(P \setminus F)} z^{\sigma} = \frac{1}{1 - z[P \setminus F]} \sum_{\sigma \in \mathcal{A}'(P \setminus F)} z^{\sigma}.$$

Hence $G_{P/F}(z)$ satisfies (4.6). On the other hand, by using Proposition 3.1 (e), Proposition 4.2 (a) and Proposition 4.3, we can rewrite (4.3) to show that $Z_{P/F}(z)$ satisfies the recurrence (4.6). Hence we obtain the first identity of (4.5).

The second identity of (4.5) is derived by using Proposition 4.1 and Proposition 3.2 as follows:

$$\begin{aligned} \xi^{w_F}|_{w_P} &= \sum_{D \in \mathcal{E}_P(F)} \prod_{p \in D} (1 - z[H_P(p)]) \sum_{S \subset B(D)} (-1)^{\# S} \prod_{p \in S} (1 - z[H_P(p)]) \\ &= \sum_{D \in \mathcal{E}_P(F)} \prod_{p \in D} (1 - z[H_P(p)]) \prod_{p \in B(D)} z[H_P(p)]. \end{aligned}$$

By dividing the both sides by $\xi^{w_P}|_{w_P} = \prod_{p \in P}(1 - z[H_P(p)])$, we obtain the second identity in (4.5). This completes the proof of Theorem 1.2.

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