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Combinatorics of cluster structures in Schubert varieties

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Abstract. We give an explicit combinatorial description of cluster structures in Schubert varieties of the Grassmannian in terms of (target labelings of) Postnikov's *plabic graphs*. This description is a natural generalization of the description given by (Scott 2006) for the Grassmannian and has been believed by experts essentially since (Scott 2006), though the statement was not formally written down until (Muller–Speyer 2016). To prove this conjecture we use a result of (Leclerc 2016), who used the module category of the preprojective algebra to prove that coordinate rings of many Richardson varieties in the complete flag variety admit cluster structures. We also adapt a construction of (Karpman 2016) to build cluster seeds associated to reduced expressions. Further, we explicitly describe cluster structures in skew Schubert varieties using plabic graphs whose boundary vertices need not be labeled in cyclic order.

Résumé. Nous donnons une description combinatoire explicite des structures en amas des variétés de Schubert des grassmanniennes en termes de graphes plabiques. Cette description est une généralisation naturelle de la description donnée par (Scott 2006) pour les grassmanniennes, et correspond à ce qui était attendu par les experts depuis (Scott 2006), bien que l'énoncé n'ait pas été formellement écrit avant (Muller–Speyer 2016). Nous décrivons aussi explicitement les structures en amas des variétés de Schubert généralisées en utilisant des graphes plabiques dont les sommets ne sont pas numérotés dans l'ordre cyclique.

Keywords: cluster algebra, Schubert variety, positroid, plabic graph

1 Introduction

The main result of this extended abstract is that the coordinate ring of (the affine cone over) any (open) Schubert variety of the Grassmannian (embedded into projective space

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via the Plücker embedding) admits a cluster algebra structure, which is described explicitly in terms of plabic graphs. In the first section, we give some background on Schubert varieties, state our main result, and discuss some applications. Background material on cluster algebras and plabic graphs is delayed to Section 2.

Cluster algebras are a class of commutative rings which were introduced by Fomin and Zelevinsky [1]; they are connected to many fields of mathematics including Teichmüller theory and quiver representations, and their generators satisfy many nice properties, including the *Laurent phenomenon* [1] and *positivity theorem* [8, 3]. *Plabic graphs* are certain planar bicolored graphs which were introduced by Postnikov [12]; plabic graphs (or rather an equivalent object, namely alternating strand diagrams) were subsequently used by Scott [15] to show that the coordinate ring of the affine cone over the Grassmannian in its Plücker embedding admits a cluster algebra structure.

There is a natural plabic graph generalization of Scott's construction which experts have believed for some time should give a cluster structure for Schubert varieties (and more generally, positroid varieties). This construction was stated explicitly as a conjecture in a recent paper of Muller–Speyer [10], who additionally provided some evidence in [11]. The conjecture can be stated roughly as follows.

Conjecture 1.1. *Let G be a reduced plabic graph corresponding to an (open) Schubert (or more generally an open positroid) variety. Then the target labeling of the faces of G (which we identify with a collection of Plücker coordinates) together with the dual quiver of G gives rise to a seed for a cluster structure in the coordinate ring of the open Schubert (or positroid) variety.*

Meanwhile, Leclerc [7] constructed a subcategory $\mathcal{C}_{v,w}$ of the module category of the preprojective algebra that has a cluster structure, to show that the coordinate ring of each Richardson variety $\mathcal{R}_{v,w}$ of the complete flag variety contains a subalgebra which is a cluster algebra; when w has a factorization of the form $w = xv$ with $\ell(w) = \ell(x) + \ell(v)$, he showed that this subalgebra coincides with the coordinate ring. Because Schubert varieties are isomorphic to Richardson varieties with the above property, Leclerc's result implies that their coordinate rings admit a cluster structure. However, Leclerc's description of the cluster structure is very different from the plabic graph description and is far from explicit: e.g. his cluster quiver is defined in terms of morphisms between modules of the preprojective algebra.

In this paper we prove Conjecture 1.1 for Schubert varieties by relating Leclerc's cluster structure to the conjectural one coming from plabic graphs. We also generalize our result to the setting of *skew Schubert varieties*; interestingly, these cluster structures for skew Schubert varieties depart from the one in Conjecture 1.1, since they use *generalized* plabic graphs (with boundary vertices which are no longer cyclically labeled).

Once we have proved that the coordinate rings of Schubert and skew Schubert varieties have cluster structures, we obtain a number of results "for free" from the cluster theory, including the Laurent phenomenon and the positivity theorem for cluster variables. We also obtain many combinatorially explicit cluster seeds for each Schubert and skew Schubert variety. Other applications of our results, including a characterization of which Schubert varieties exhibit finite type cluster structures, are described in Section 1.5.

1.1 Notation for the flag variety

Let GL_n denote the general linear group, B the Borel subgroup of lower triangular matrices, B^+ the opposite Borel subgroup of upper triangular matrices, and $W = S_n$ the Weyl group (which in this case is the symmetric group on n letters). W is generated by the simple reflections s_i for $1 \le i \le n-1$, where s_i is the transposition exchanging i and $i+1$, and it contains a longest element, which we denote by w_0, with $\ell(w_0) = \binom{n}{2}$. The *complete flag variety* Fl_n is the homogeneous space $B \setminus \mathrm{GL}_n$. Concretely, each element g of GL_n gives rise to a flag of subspaces $\{V_1 \subset V_2 \subset \cdots \subset V_n\}$, where V_i denotes the span of the top i rows of g. The action of B on the left preserves the flag, so we can identify Fl_n with the set of *flags* $\{V_1 \subset V_2 \subset \cdots \subset V_n\}$ where $\dim V_i = i$.

Let $\pi : \mathrm{GL}_n \to \mathrm{Fl}_n$ denote the natural projection $\pi(g) := Bg$. The Bruhat decomposition

$$\mathrm{GL}_n = \bigsqcup_{w \in W} BwB$$

projects to the Schubert decomposition

$$\mathrm{Fl}_n = \bigsqcup_{w \in W} C_w$$

where $C_w = \pi(BwB)$ is the *Schubert cell* associated to w, isomorphic to $\mathbb{C}^{\ell(w)}$. We also have the Birkhoff decomposition

$$\mathrm{GL}_n = \bigsqcup_{w \in W} BwB^+,$$

which projects to the opposite Schubert decomposition

$$\mathrm{Fl}_n = \bigsqcup_{w \in W} C^w$$

where $C^w = \pi(BwB^+)$ is the *opposite Schubert cell* associated to w, isomorphic to $\mathbb{C}^{\ell(w_0)-\ell(w)}$.

The intersection

$$\mathcal{R}_{v,w} := C^v \cap C_w$$

has been considered by Kazhdan and Lusztig [5] in relation to Kazhdan-Lusztig polynomials. $\mathcal{R}_{v,w}$ is nonempty only if $v \le w$ in the Bruhat order of W, and it is then a smooth irreducible locally closed subset of C_w of dimension $\ell(w) - \ell(v)$. Sometimes $\mathcal{R}_{v,w}$ is called an *open Richardson variety* [6] because its closure is a *Richardson variety* [13]. We have a stratification of the complete flag variety

$$\mathrm{Fl}_n = \bigsqcup_{v \le w} \mathcal{R}_{v,w}.$$

1.2 Notation for the Grassmannian

Fix $1 < k < n$. The parabolic subgroup $W_K = \langle s_1, \dots, s_{k-1}\rangle \times \langle s_{k+1}, s_{k+2}, \dots, s_{n-1}\rangle < W$ gives rise to a parabolic subgroup $P_K = \bigsqcup_{w \in W_K} B\dot{w}B$ in GL_n, where $\dot{w}$ is a matrix representative for w in GL_n. The longest element w_K of W_K has length $\ell(w_K) = \binom{k}{2} + \binom{n-k}{2}$.

The *Grassmannian* $Gr_{k,n}$ is the homogeneous space $P_K \setminus \mathrm{GL}_n$. We can think of the Grassmannian $Gr_{k,n} = P_K \setminus \mathrm{GL}_n$ more concretely as the set of all k-planes in an n-dimensional vector space $\mathbb{C}^n$. An element of $Gr_{k,n}$ can be viewed as a $k \times n$ matrix of rank k, modulo left multiplication by invertible $k \times k$ matrices. That is, two $k \times n$ matrices of rank k represent the same point in $Gr_{k,n}$ if and only if they can be obtained from each other by invertible row operations.

For a positive integer a and $w \in W$, let $[a] := [1, a]$ and $w[a] := \{w(1), \dots, w(a)\}$. Let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n]$.

Given $V \in Gr_{k,n}$ represented by a $k \times n$ matrix A, for $I \in \binom{[n]}{k}$ we let $\Delta_I(V)$ be the maximal minor of A located in the column set I. The $\Delta_I(V)$ do not depend on our choice of matrix A (up to simultaneous rescaling by a nonzero constant), and are called the *Plücker coordinates* of V. The Plücker coordinates give an embedding of $Gr_{k,n}$ into projective space of dimension $\binom{n}{k} - 1$.

We have the usual projection π_k from the complete flag variety Fl_n to the Grassmannian $Gr_{k,n}$. Let $W^K_{\max}$ denote the set of maximal-length coset representatives for $W_K \setminus W$.

Rietsch studied the projections of the open Richardson varieties in the complete flag variety to partial flag varieties [14]. In particular, when $v \in W^K_{\max}$, the projection π_k is an isomorphism from $\mathcal{R}_{v,w}$ to $\pi_k(\mathcal{R}_{v,w})$. We obtain a stratification

$$Gr_{k,n} = \bigsqcup_{v \le w} \pi_k(\mathcal{R}_{v,w})$$

where (v, w) range over all $v \in W^K_{\max}$, $w \in W$, such that $v \le w$. Following work of Postnikov [12, 6], the strata $\pi_k(\mathcal{R}_{v,w})$ are sometimes called *open positroid varieties*, while their closures are called *positroid varieties*. It follows from the definitions (see e.g. [6, Section 6]) that positroid varieties include Schubert varieties in the Grassmannians, which we now define.

Definition 1.2. Let $I \in \binom{[n]}{k}$. The *Schubert cell* Ω_I is defined to be the set

$$\{A \in Gr_{k,n} \mid \text{the lexicographically minimal nonvanishing Plücker coordinate of } A \text{ is } \Delta_I(A)\}.$$

The *Schubert variety* X_I is the closure $\overline{\Omega_I}$ of Ω_I.

When $v \in W^K_{\max}$, $\overline{\pi_k(\mathcal{R}_{v,w_0})}$ is isomorphic to the Schubert variety $X_{v^{-1}[k]}$ in the Grassmannian, which has dimension $\ell(w_0) - \ell(v)$. We therefore refer to $\pi_k(\mathcal{R}_{v,w_0})$ as an *open*

Schubert variety. More generally, if $v \in W^K_{max}$ and $w \in W$ has a factorization of the form $w = xv$ which is *length-additive*, i.e. where $\ell(w) = \ell(x) + \ell(v)$, then we refer to $\overline{\pi_k(\mathcal{R}_{v,w})}$ (respectively, $\pi_k(\mathcal{R}_{v,w})$) as a *skew Schubert variety* (respectively, open skew Schubert variety).

Let λ denote a Young diagram contained in a $k \times (n-k)$ rectangle. We can identify λ with its southeast boundary; we think of the boundary as a lattice path $L^{\swarrow}_{\lambda}$ in the rectangle from the northeast corner to the southwest corner taking steps west and south. Labeling the steps of $L^{\swarrow}_{\lambda}$ from 1 to n, the labels of the south steps give a k-element subset of $[n]$. Conversely, each $I \in \binom{[n]}{k}$ can be identified with a Young diagram, denoted $\lambda^{\swarrow}(I)$. (Later we will also need an analogous partition $\lambda^{\nearrow}(I)$ obtained by reading the boundary of λ from southwest to northeast.) The map $\lambda^{\swarrow}()$ allows us to index open and closed Schubert varieties by Young diagrams, denoting them X_λ and X°_λ respectively. The dimension of X_λ and X°_λ is $|\lambda|$, the number of boxes of λ.

1.3 The main result

We now state the main result. Note that the definitions of plabic graph and trip permutation can be found in Section 2.

We associate with a Young diagram λ a permutation $\pi^{\swarrow}_{\lambda}$: in list notation, this permutation is obtained by first reading the labels of the horizontal steps of $L^{\swarrow}_{\lambda}$, and then reading the labels of the vertical steps of $L^{\swarrow}_{\lambda}$. (Moreover any fixed points in positions $1, 2, \dots, n-k$ are "black" and any fixed points in positions $n-k+1, \dots, n$ are "white.")

Theorem 1.3. *Consider the open Schubert variety X°_λ of $Gr_{k,n}$. Let G be a reduced plabic graph (with boundary vertices labeled clockwise from 1 to n) with trip permutation $\pi^{\swarrow}_{\lambda}$. Construct the dual quiver of G and label its vertices by the Plücker coordinates given by the target labeling of G, see Definition 2.12 and Figure 1. The coordinate ring of (the affine cone over) X°_λ is a cluster algebra, and this labeled quiver gives a seed for this cluster algebra.*

We actually prove something a bit more general than Theorem 1.3.

Theorem 1.4. *Consider the open skew Schubert variety $\pi_k(\mathcal{R}_{v,w})$, where $v \in W^K_{max}$ and w has a length-additive factorization $w = xv$. Let G be a reduced plabic graph (with boundary vertices labeled clockwise from 1 to n) with trip permutation $vw^{-1} = x^{-1}$, such that boundary lollipops are white if and only if they are in $[k]$. Then the coordinate ring of (the affine cone over) the open skew Schubert variety $\pi_k(\mathcal{R}_{v,w})$ is a cluster algebra, and G gives rise to a seed as follows: apply v^{-1} to the boundary vertices of G and then label the dual quiver using the target labeling.*

In the case of Schubert varieties, Theorem 1.3 resolves Conjecture 1.1. Note that there is another version of the conjecture which uses the *source labeling* of G instead of the target labeling [10, Remark 3.5]. Both conjectures make sense more generally for

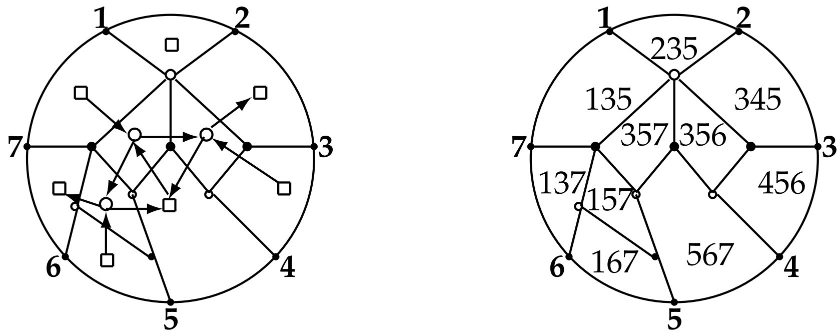

Figure 1: A plabic graph G for $Gr_{3,7}$ with trip permutation $\pi_\lambda^{\swarrow} = (2,4,6,7,1,3,5)$, for $\lambda = (4,3,2)$, together with the dual quiver of G and the target face labeling.

positroid varieties and arbitrary reduced plabic graphs (whose trip permutations can be arbitrary decorated permutations). However, the cluster structure that we give in Theorem 1.4 is different from either of the cluster structures proposed in [10].

Remark 1.5. When $\pi_k(\mathcal{R}_{v,w})$ is not an open skew Schubert variety, the seeds in Leclerc's cluster subalgebra in general do not come from generalized plabic graphs. Indeed, for $v = (2,5,1,4,3)$ and $w = (5,3,4,2,1)$, the unique seed in Leclerc's cluster subalgebra for $\pi_k(\mathcal{R}_{v,w})$ consists of frozen variables and has extended cluster $\{\Delta_{13}, \Delta_{23}, \Delta_{14}, \Delta_{15}, \Delta_{45}\}$. These Plücker coordinates cannot be obtained as the set of face labels of any (generalized) plabic graph (note that the label 2 occurs only once in the set of Plücker coordinates).

1.4 Outline of the proof

While each open skew Schubert variety $\pi_k(\mathcal{R}_{v,w})$ (where $v = w_K v' \in W^K_{max}$ and $w \in W$ has a length-additive factorization $\mathbf{w} = \mathbf{x}\mathbf{v} = \mathbf{x}\mathbf{w_K}\mathbf{v}'$ into reduced expressions for x, w_K, and v') corresponds to an equivalence class of plabic graphs, there is one among them which is particularly nice, which we call the *rectangles seed*. The first step of our proof is to give an explicit description of the rectangles seed for each open skew Schubert variety $\pi_k(\mathcal{R}_{v,w})$, in terms of v and a Young diagram associated to (v,w).

A construction of Karpman [4] produces a bridge-decomposable plabic graph associated to a pair $(y, \mathbf{z})$, where $y^{-1} \in W^K_{\max}$, $\mathbf{z}$ is a reduced decomposition for z, and $y \leq z$. The second step of our proof is to show that if we perform her construction for the pair $(w_K, \mathbf{x}\mathbf{w_K})$ and then relabel boundary vertices of the resulting plabic graph by v^{-1} to obtain a graph G, the target labeling of the dual quiver of G gives rise to the rectangles seed for $\pi_k(\mathcal{R}_{v,w})$.

In [7], Leclerc produces a cluster seed associated to each pair $(v, \mathbf{w})$, where $v \in W^K_{max}$ and $v \leq w$. The third step of our proof is to verify that for the choice $(v, \mathbf{w} = \mathbf{x}\mathbf{w_K}\mathbf{v}')$, Leclerc's construction gives rise to the rectangles seed. To prove Theorem 1.4 from the previous steps, we show that mutations of the plabic graph G (known as "square moves") coincide with certain mutations of the rectangles seed. Theorem 1.3 is then deduced from Theorem 1.4.

1.5 Applications of the main result

In this section we sketch some applications of the main result, including ring-theoretic properties for the coordinate rings of open skew Schubert varieties, canonical bases, and a classification of the finite type cluster structures we obtain.

Combining Theorem 1.3 and Theorem 1.4 with results of Muller and Speyer [9], [10, Theorem 3.3], we find that the coordinate rings of open Schubert and skew Schubert varieties (viewed as cluster algebras) are *locally acyclic*, which implies that they are finitely generated, normal, locally a complete intersection, and equal to their own upper cluster algebras.

Combining our results with results of Ford-Serhiyenko [2, Theorem 1.2], we find that the quivers giving rise to the cluster structures for open Schubert and skew Schubert varieties admit *green-to-red sequences*, which by Gross-Hacking-Keel-Kontsevich [3] implies that the cluster algebras have *Enough Global Monomials* and hence each coordinate ring has a canonical basis of *theta functions*, parameterized by the lattice of g-vectors.

In [15], Scott classified the Grassmannians whose coordinate rings have a cluster algebra of finite type. He showed that in general the cluster algebras have infinite type, except in the following cases: the coordinate ring of $Gr_{2,n}$ is a cluster algebra of type A_{n-3}, while the coordinate rings of $Gr_{3,6}$, $Gr_{3,7}$, and $Gr_{3,8}$ are cluster algebras of types D_4, E_6, and E_8, respectively.

It is straightforward to classify for which open skew Schubert varieties $\pi_k(\mathcal{R}_{v,w})$ the cluster structure described here is finite type. It depends only on wv^{-1}.

Proposition 1.6. *Let $v \leq w$, where $v \in W^K_{\max}$ and $w = xv$ is length-additive. Let $\lambda = \lambda^{\nearrow}(x[k])$ and let λ' be the Young diagram obtained from λ by removing all boxes that touch the southeast boundary of λ either along an edge or at the southeast corner. Then the cluster structure on the coordinate ring of $\pi_k(\mathcal{R}_{v,w})$ given in Theorem 1.4 is*

1. *type A if and only if λ' does not contain a 2×2 rectangle;*
2. *type D if and only if $\lambda' = (i, 2)$ or its transpose for $i \geq 2$;*
3. *type E_6, E_7, or E_8 if and only if λ' or its transpose is one of $(3,3)$, $(3,2,1)$, $(4,3)$, $(4,2,1)$, $(3,3,1)$, $(5,3)$, $(5,2,1)$, $(4,4)$, $(4,2,2)$.*

In particular, the open Schubert variety X°_λ is of finite type if and only if λ' is in the above list.

2 Background on cluster structures and plabic graphs

2.1 Background on cluster structures

Cluster algebras are a class of rings with a particular combinatorial structure; they were introduced by Fomin and Zelevinsky in [1].

Definition 2.1. A *quiver* Q is a directed graph; we will assume that Q has no loops or 2-cycles. Each vertex is designated either *mutable* or *frozen*.

Definition 2.2. Let q be a mutable vertex of quiver Q. The quiver mutation μ_q transforms Q into a new quiver $Q' = \mu_q(Q)$ via a sequence of three steps:

1. For each oriented two path $r \to q \to s$, add a new arrow $r \to s$ (unless r and s are both frozen, in which case do nothing).
2. Reverse the direction of all arrows incident to the vertex q.
3. Repeatedly remove oriented 2-cycles until unable to do so.

Definition 2.3. Choose $M \geq N$ positive integers. Let $\mathcal{F}$ be an *ambient field* of rational functions in N independent variables over $\mathbb{C}(x_{N+1}, \ldots, x_M)$. A *labeled seed* in $\mathcal{F}$ is a pair $(\widetilde{\mathbf{x}}, Q)$, where $\widetilde{\mathbf{x}} = (x_1, \ldots, x_M)$ forms a free generating set for $\mathcal{F}$, and Q is a quiver on vertices $1, 2, \ldots, N, N+1, \ldots, M$, whose vertices $1, 2, \ldots, N$ are *mutable*, and whose vertices $N+1, \ldots, M$ are *frozen*.

We refer to $\widetilde{\mathbf{x}}$ as the (labeled) *extended cluster* of a labeled seed $(\widetilde{\mathbf{x}}, Q)$. The variables $\{x_1, \ldots, x_N\}$ are called *cluster variables*, and the variables $c = \{x_{N+1}, \ldots, x_M\}$ are called *frozen* or *coefficient variables*. We often view the labeled seed as a quiver Q where each vertex i is labeled by the corresponding variable x_i.

Definition 2.4. Let $(\widetilde{\mathbf{x}}, Q)$ be a labeled seed in $\mathcal{F}$, and let $q \in \{1, \ldots, N\}$. The *seed mutation* μ_q in direction q transforms $(\widetilde{\mathbf{x}}, Q)$ into the labeled seed $\mu_q(\widetilde{\mathbf{x}}, Q) = (\widetilde{\mathbf{x}}', \mu_q(Q))$, where the cluster $\widetilde{\mathbf{x}}' = (x'_1, \ldots, x'_M)$ is defined as follows: $x'_j = x_j$ for $j \neq q$, whereas $x'_q \in \mathcal{F}$ is determined by the *exchange relation*

$$x'_q\, x_q = \prod_{q \to r} x_r + \prod_{s \to q} x_s,$$

where the first product is over all arrows $q \to r$ in Q which start at q, and the second product is over all arrows $s \to q$ which end at q.

It is not hard to check that seed mutation is an involution. Note that arrows between two frozen vertices of a quiver do not affect seed mutation (they do not affect the mutated quiver or the exchange relation). For that reason, one may omit arrows between two frozen vertices.

Definition 2.5. Consider the *N-regular tree* $\mathbb{T}_N$ whose edges are labeled by the numbers $1, \ldots, N$, so that the N edges emanating from each vertex receive different labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\widetilde{\mathbf{x}}_t, Q_t)$ to every vertex $t \in \mathbb{T}_N$, such that the seeds assigned to the endpoints of any edge $t \overset{q}{—} t'$ are obtained from each other by the seed mutation in direction q. The components of $\widetilde{\mathbf{x}}_t$ are written as $\widetilde{\mathbf{x}}_t = (x_{1;t}, \ldots, x_{N;t})$.

Clearly, a cluster pattern is uniquely determined by an arbitrary seed.

Definition 2.6. Given a cluster pattern, we denote

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_N} \widetilde{\mathbf{x}}_t = \{x_{i,t} : t \in \mathbb{T}_N , 1 \le i \le N\} ,$$

the union of clusters of all the seeds in the pattern. The elements $x_{i,t} \in \mathcal{X}$ are called *cluster variables*. The *cluster algebra* $\mathcal{A}$ associated with a given pattern is the $\mathbb{C}[x_{N+1}^{\pm 1}, \ldots, x_M^{\pm 1}]$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\widetilde{\mathbf{x}}, Q)$, where $(\widetilde{\mathbf{x}}, Q)$ is any seed in the underlying cluster pattern. In this generality, $\mathcal{A}$ is called a *cluster algebra from a quiver*, or a *skew-symmetric cluster algebra of geometric type.*

2.2 Background on plabic graphs

In this section we review Postnikov's notion of *plabic graphs* [12].

Definition 2.7. A *plabic (or planar bicolored) graph* is an undirected graph G drawn inside a disk (considered modulo homotopy) with *n boundary vertices* on the boundary of the disk, labeled $1, \ldots, n$ in clockwise order, as well as some colored *internal vertices*. These internal vertices are strictly inside the disk and are colored in black and white. An internal vertex of degree one adjacent to a boundary vertex is a *lollipop*. We will always assume that no vertices of the same color are adjacent, and that each boundary vertex i is adjacent to a single internal vertex.

See Figure 1 for an example of a plabic graph.

There is a natural set of local transformations (moves) of plabic graphs, which we now describe. Note that we will always assume that a plabic graph G has no isolated components (i.e. every connected component contains at least one boundary vertex). We will also assume that G is *leafless*, i.e. if G has an internal vertex of degree 1, then that vertex must be adjacent to a boundary vertex.

(M1) SQUARE MOVE (Urban renewal). If a plabic graph has a square formed by four trivalent vertices whose colors alternate, then we can switch the colors of these four vertices (and add some degree 2 vertices to preserve the property that no vertices of the same color are adjacent).

(M2) CONTRACTING/EXPANDING A VERTEX. Any degree 2 internal vertex not adjacent to the boundary can be deleted, and the two adjacent vertices merged. This operation can also be reversed. Note that this operation can always be used to change an arbitrary square face of G into a square face whose four vertices are all trivalent.

(M3) MIDDLE VERTEX INSERTION/REMOVAL. We can always remove or add degree 2 vertices at will, subject to the condition that the graph remains bipartite.

See Figure 2 for depictions of these three moves.

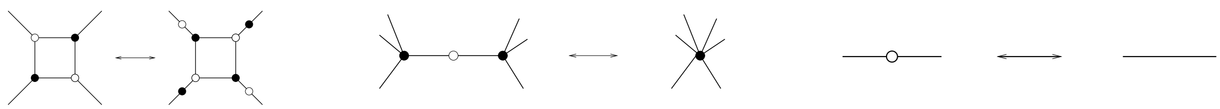

Figure 2: Moves (M1), (M2), (M3).

(R1) PARALLEL EDGE REDUCTION. If a plabic graph contains two trivalent vertices of different colors connected by a pair of parallel edges, then we can remove these vertices and edges, and glue the remaining pair of edges together.

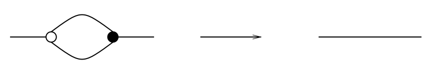

Figure 3: Parallel edge reduction

Definition 2.8. Two plabic graphs are called *move-equivalent* if they can be obtained from each other by moves (M1)-(M3). The *move-equivalence class* of a given plabic graph G is the set of all plabic graphs which are move-equivalent to G. A leafless plabic graph without isolated components is called *reduced* if there is no graph in its move-equivalence class to which we can apply (R1).

Definition 2.9. A *decorated permutation* $\pi^{:}$ is a permutation $\pi \in S_n$ together with a coloring $\{i \mid \pi(i) = i\} \to \{\text{black, white}\}$.

Definition 2.10. Given a reduced plabic graph G, a *trip* T is a directed path which starts at some boundary vertex i, and follows the "rules of the road": it turns (maximally) right at a black vertex, and (maximally) left at a white vertex. Note that T will also end at a boundary vertex j; we then refer to this trip as $T_{i\to j}$. Setting $\pi(i) = j$ for each such trip, we associate a (decorated) *trip permutation* $\pi_G = (\pi(1), \ldots, \pi(n))$ to each reduced plabic graph G, where a fixed point $\pi(i) = i$ is colored white (black) if there is a white (black) lollipop at boundary vertex i.

For example, the trip permutation of the reduced plabic graph in Figure 1 is $(2, 4, 6, 7, 1, 3, 5)$.

Remark 2.11. Note that the trip permutation of a plabic graph is preserved by the local moves (M1)-(M3), but not by (R1). For reduced plabic graphs the converse holds, namely it follows from [12, Theorem 13.4] that any two reduced plabic graphs with the same trip permutation are move-equivalent.

Now we use the notion of trips to label each face of G by a Plücker coordinate. Towards this end, note that every trip will partition the faces of a plabic graph into two parts: those on the left of the trip, and those on the right of the trip.

Definition 2.12. Let G be a reduced plabic graph with b boundary vertices. For each one-way trip $T_{i \to j}$ with $i \neq j$, we place the label j in every face which is to the left of $T_{i \to j}$. If $i = j$ (that is, i is adjacent to a lollipop), we place the label i in all faces if the lollipop is white and in no faces if the lollipop is black. We then obtain a labeling $\mathcal{F}(G)$ of faces of G by subsets of $[b]$ which we call the *target labeling* of G. We identify each a-element subset of $[b]$ with the corresponding Plücker coordinate.

The following statements relate quivers to plabic graphs.

Definition 2.13. Let G be a reduced plabic graph. The dual quiver $Q(G)$ of G is defined as follows. The vertices of $Q(G)$ are labeled by the faces of G. We say that a vertex of $Q(G)$ is *frozen* if the corresponding face is incident to the boundary of the disk, and is *mutable* otherwise. For each edge e in G which separates two faces, at least one of which is mutable, we introduce an arrow connecting the faces; this arrow is oriented so that it "sees the white endpoint of e to the left and the black endpoint to the right" as it crosses over e. We then remove oriented 2-cycles from the resulting quiver, one by one, to get $Q(G)$. See Figure 1.

Lemma 2.14. *If G and G' are related via a square move at a face, then $Q(G)$ and $Q(G')$ are related via mutation at the corresponding vertex.*

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