Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #7, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Ballot-Noncrossing Partitions

Henri Mühle*

Institut für Algebra, Technische Universität Dresden, Zellescher Weg 12–14, 01069 Dresden, Germany

Abstract. Noncrossing partitions, Dyck paths, and 231-avoiding permutations are classical examples of Catalan objects, and they may be defined in terms of the symmetric group. Moreover, when we consider noncrossing partitions ordered by refinement and 231-avoiding permutations ordered by inclusion of inversion sets, then there is a close structural relationship between the two resulting posets. In this abstract we show that this connection, together with some other properties of these two posets, still holds in the generalized setting of (certain special) parabolic quotients of the symmetric group.

Résumé. Les partitions non-croisées, les chemins de Dyck et les permutations évitant 231 sont des exemples classiques d'objets de Catalan, et peuvent être définis à partir du groupe symétrique. De plus, lorsque l'on considère les partitions non-croisées ordonnées par raffinement, et les permutations évitant 231 ordonnées par inclusion de leurs ensembles d'inversion, les posets obtenus ont des structures proches. Dans ce résumé nous montrons que cette connexion, ainsi que d'autres propriétés de ces deux posets, s'étend au cadre plus général de certains quotients paraboliques du groupe symétrique.

Keywords: noncrossing partitions, ballot paths, 231-avoiding permutations, Tamari lattice, Galois graph, alternate order

1 Introduction

Noncrossing set partitions, Dyck paths and 231-avoiding permutations are well-studied members of the huge class of Catalan objects, i.e. combinatorial objects enumerated by the Catalan numbers.

These three families have another remarkable property: they can be defined in terms of the symmetric group. In this guise, noncrossing set partitions become the members of the order ideal under absolute order generated by a long cycle, Dyck paths become order ideals in a particular order on all transpositions, and 231-avoiding permutations become the elements sortable with respect to a particular long cycle.

In [12], a generalization of these three families to parabolic quotients of the symmetric group was proposed, and it was shown that they are still equinumerous by exhibiting explicit bijections. Subsequently, in [10] these *parabolic* noncrossing partitions and

*henri.muehle@tu-dresden.de.

parabolic 231-avoiding permutations were studied more closely, and it was shown that many structural properties and connections that were previously known for the classical noncrossing partitions and classical 231-avoiding permutations are still true in the parabolic setting.

In this abstract we focus on particular parabolic quotients of the symmetric group: those generated by an initial segment of the lexicographic order on the set of simple transpositions. In this case, the parabolic Dyck paths are essentially ballot paths, and we therefore refer to the corresponding parabolic noncrossing partitions as *ballot-noncrossing partitions*.

The purpose of this abstract is to illustrate some of the main results of [10] in the setting of ballot-noncrossing partitions. We give some of the proofs here (in a shortened form) for the convenience of the reader, but skip most of the details. They can be found in the appropriate places of [10]; as is the case for any undefined terminology.

The main results presented here are Theorems 3.2, 4.1 and 4.2, as well as Conjecture 5.2 stated in Sections 3, 4.1, 4.2 and 5, respectively. In Section 2 we define the basic concepts needed here.

2 Preliminaries

We assume the reader to be familiar with the basic notions of poset and lattice theory, and we refer to [10, Section 2.1] for any undefined terminology.

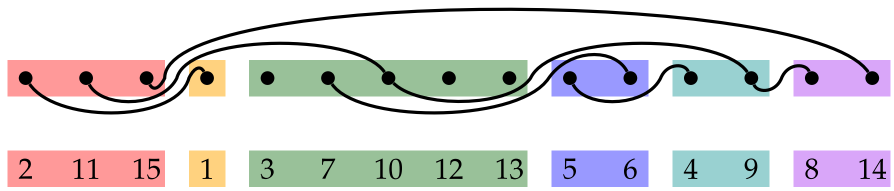
For any two natural numbers a, b we define $[a, b] \stackrel{\text{def}}{=} \{a, a+1, \ldots, b\}$. If $a > b$, then this set is empty, and if $a = 1$, then we usually write $[b]$ instead of $[1, b]$.

For the rest of the article, fix a positive integer $n > 0$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ be a composition of n. We define $s_k \stackrel{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots + \alpha_k$ for $k \in [0, r]$, and for $k \in [r]$ we call the set $\{s_{k-1}+1, s_{k-1}+2, \ldots, s_k\}$ the k-th *α-region*.

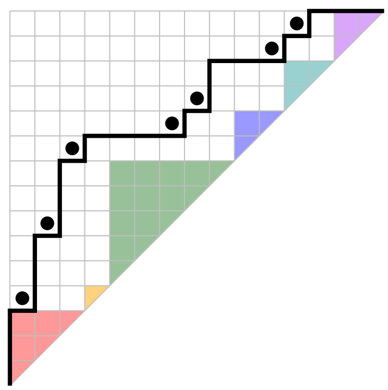
2.1 Parabolic Noncrossing Partitions

Let us introduce parabolic noncrossing partitions graphically. To that end we write n dots labeled from 1 through n on a horizontal line, and we highlight the elements of the k-th α-region by color k. For $a, b \in [n]$ in different α-regions, the *bump* connecting a and b is a curve that leaves the a-th dot to the bottom, passes below the α-region containing a, passes above every subsequent α-region and enters b from above. Two bumps *cross* if there is no way to draw them according to those rules without crossing. A *noncrossing α-partition* is a collection of pairwise noncrossing bumps. Let us denote the set of all noncrossing α-partitions by NC_α.

We may view $\mathbf{P} \in NC_\alpha$ as a set partition of $[n]$, whose parts are given by the connected components of $\mathbf{P}$. Therefore, we may order NC_α by dual refinement, and obtain the



(a) A noncrossing α-partition, and its corresponding $(\alpha, 231)$-avoiding permutation for $\alpha = (3, 1, 5, 2, 2, 2)$.



(b) The α-Dyck path corresponding to the noncrossing α-partition from Figure 1a. The region between ν_α and the main diagonal is highlighted, and the marked valleys correspond to the bumps of the partition.

Figure 1: Examples of noncrossing α-partitions, $(\alpha, 231)$-avoiding permutations and α-Dyck paths.

partially ordered set $\mathcal{NC}_\alpha$. This is in general a ranked meet-semilattice, where the rank of $\mathbf{P} \in \mathcal{NC}_\alpha$ is given by the number of bumps of $\mathbf{P}$ [10, Proposition 3.6].

If $\alpha = (1, 1, \ldots, 1)$, then we recover the classical noncrossing set partitions introduced in [8]. Figure 1a shows an example.

2.2 Parabolic Dyck Paths

Recall that a *Dyck path* is a lattice path in $\mathbb{N} \times \mathbb{N}$ that starts at the origin and consists only of *north-steps* $N \stackrel{\text{def}}{=} (0, 1)$ and *east-steps* $E \stackrel{\text{def}}{=} (1, 0)$ and ends on the main diagonal. The *α-bounce path* ν_α is the Dyck path given by the following sequence of $2n$ steps:

$$N^{\alpha_1} E^{\alpha_1} N^{\alpha_2} E^{\alpha_2} \cdots N^{\alpha_r} E^{\alpha_r}.$$

An *α-Dyck path* is any Dyck path consisting of $2n$ steps, which stays weakly above ν_α. Let us denote the set of all α-Dyck paths by $\mathcal{D}_\alpha$.

If $\alpha = (1, 1, \ldots, 1)$, then we recover the classical Dyck paths. Figure 1b shows an example.

2.3 Parabolic 231-Avoiding Permutations

The symmetric group $\mathfrak{S}_n$ is the group of all permutations of $[n]$. For $w \in \mathfrak{S}_n$, an *inversion* of w is a pair (i, j) with $i < j$ and $w(i) > w(j)$. If an inversion (i, j) satisfies $w(i) = w(j) + 1$, then it is a *descent*. The parabolic quotient of $\mathfrak{S}_n$ with respect to α is

$$\mathfrak{S}_\alpha \overset{\text{def}}{=} \big\{ w \in \mathfrak{S}_n \mid w(i) < w(i+1) \text{ for all } i \notin \{s_1, s_2, \ldots, s_{r-1}\} \big\}.$$

We normally represent the elements of $\mathfrak{S}_\alpha$ in one-line notation, where we highlight the values in the k-th α region by color k.

A permutation $w \in \mathfrak{S}_\alpha$ has an *$(\alpha, 231)$-pattern* if there exist three integers $i < j < k$ in different α-regions such that $w(k) < w(i) < w(j)$ and (i, k) is a descent of w. A permutation is *$(\alpha, 231)$-avoiding* if it does not have an $(\alpha, 231)$-pattern. Let $\mathfrak{S}_\alpha(231)$ denote the set of all $(\alpha, 231)$-avoiding permutations in $\mathfrak{S}_\alpha$. If $\alpha = (1, 1, \ldots, 1)$, then it is an easy exercise to prove that we recover the classical 231-avoiding permutations. Figure 1a shows an example.

It is classical to order permutations by inclusion of inversion sets; the resulting order is usually called the *(left) weak order*. We write $\mathrm{Weak}(\mathfrak{S}_n)$ for the corresponding poset. Since $\mathfrak{S}_\alpha \subseteq \mathfrak{S}_n$ it makes sense to consider the *parabolic Tamari lattice* $\mathcal{T}_\alpha \overset{\text{def}}{=} \mathrm{Weak}\big(\mathfrak{S}_\alpha(231)\big)$.

Theorem 2.1 ([10, Theorem 1.3])**.** *For every integer composition α, the poset $\mathcal{T}_\alpha$ is a trim, congruence-uniform lattice.*

We do not want to explain trimness and congruence-uniformity in detail here. See for instance [10, Sections 2.2 and 2.4] for exact definitions and examples. Let us instead just mention that trimness is a generalization of distributivity to non-graded lattices [18], and that congruence-uniform lattices can be constructed from the singleton lattice by iterated doublings of intervals [5].

Remark 2.2. *L.-F. Préville-Ratelle and X. Viennot have introduced a partial order on the set of Dyck paths that stay weakly above a fixed Dyck path ν [13]. This poset is in general a lattice; the* ν-Tamari lattice. *In the case, where $\nu = \nu_\alpha$ is the α-bounce path, the author has shown together with C. Ceballos and W. Fang in [2, Theorem II] that the ν_α-Tamari lattice is isomorphic to $\mathcal{T}_\alpha$.*

We conclude this section by stating that all three parabolic families that we have introduced so far are equinumerous.

Theorem 2.3 ([12, Theorem 1.2])**.** *For every integer composition α, the sets NC_α, $\mathfrak{S}_\alpha(231)$, and $\mathcal{D}_\alpha$ are in bijection.*

Remark 2.4. *In fact, instead of α-Dyck paths, the article [12] considers nonnesting α-partitions, which are certain order ideals in certain subposets of a triangular poset. However, it is quickly verified that the set of α-Dyck paths and the set of nonnesting α-partitions are in bijection.*

3 Ballot-Noncrossing Partitions

In this abstract we want to restrict our attention to the case where only the first entry of α may exceed 1. From now on, we fix $t \in [n]$ and consider the composition $\alpha_{(n;t)} \stackrel{\text{def}}{=} (t, 1, 1, \ldots, 1)$ of n.

The main reason for this restriction is the fact that noncrossing $\alpha_{(n;t)}$-partitions and $(\alpha_{(n;t)}, 231)$-avoiding permutations possess many of the nice properties that are known to hold for classical noncrossing partitions and 231-avoiding permutations. Some of these properties do not generalize to the case of arbitrary α. We point this out at the appropriate moment.

We first observe that $\alpha_{(n;t)}$-Dyck paths are in bijection with *ballot paths*, i.e. lattice paths from $(0, 0)$ to $(n-t, n)$ which stay weakly above the main diagonal and use only north- and east-steps.

Proposition 3.1. *For $n > 0$ and $t \in [n]$ we have*

$$\left|\mathcal{D}_{\alpha_{(n;t)}}\right| = \frac{t+1}{n+1}\binom{2n-t}{n-t}.$$

Proof. By definition an $\alpha_{(n;t)}$-Dyck path is a Dyck path that stays weakly above $\nu_{\alpha_{(n;t)}} = N^t E^t (NE)^{n-t}$. In particular it must start with t north-steps. If we flip the path across the line $x = n - x$, then such a path ends with t east-steps, and may therefore be viewed as a ballot path from $(0, 0)$ to $(n-t, n)$. The enumeration of these ballot paths is well known, see for instance [6, Corollary 10.3.2], and we obtain the formula in the statement. □

We may thus conclude our first main result.

Theorem 3.2. *Let $n > 0$ and $t \in [n]$. For $k \in \{0, 1, \ldots, n-t\}$, the number of noncrossing $\alpha_{(n;t)}$-partitions with exactly k bumps is given by*

$$R(n, t, k) \stackrel{\text{def}}{=} \binom{n}{k}\binom{n-t}{k} - \binom{n-1}{k-1}\binom{n-t+1}{k+1}. \tag{3.1}$$

Moreover, the cardinality of $NC_{\alpha_{(n;t)}}$ is given by

$$C(n, t) \stackrel{\text{def}}{=} \frac{t+1}{n+1}\binom{2n-t}{n-t}. \tag{3.2}$$

Proof. Theorem 2.3 and Proposition 3.1 imply the claim for the cardinality of $NC_{\alpha_{(n;t)}}$.

Moreover, the rank of $\mathbf{P} \in NC_{(\alpha_{(n;t)}}$ is given by the number of bumps, and the bijection from $NC_{\alpha_{(n;t)}}$ to $\mathcal{D}_{\alpha_{(n;t)}}$ described in [12, Theorem 5.2] sends bumps to *valleys*, i.e. subpaths of the form EN. The enumeration of ballot paths with respect to the number of valleys is also well known, see for instance [6, Theorem 10.14.1], and yields (3.1). □

We end this section with some further observations on the poset $\mathcal{NC}_{\alpha_{(n;t)}}$. Recall that the *zeta polynomial* of a poset is the polynomial $\mathcal{Z}$ whose evaluation at an integer q yields the number of $(q-1)$-multichains in the poset.

Theorem 3.3 ([7])**.** *For $n > 0$ and $t \in [n]$, the zeta polynomial of $\mathcal{NC}_{\alpha_{(n;t)}}$ is*

$$\mathcal{Z}_{\mathcal{NC}_{\alpha_{(n;t)}}}(q) = \frac{t(q-1)+1}{n(q-1)+1}\binom{nq-t}{n-t}.$$

In the first draft of this extended abstract, Theorem 3.3 was still a conjecture, but in the meantime C. Krattenthaler has found a beautiful bijective proof of this result. As a consequence, the number of intervals of $\mathcal{NC}_{\alpha_{(n;t)}}$ is given by $\mathcal{Z}_{\mathcal{NC}_{\alpha_{(n;t)}}}(3) = \frac{2t+1}{2n+1}\binom{3n-t}{n-t}$. Another consequence of Theorem 3.3 is the following enumeration of maximal chains in $\mathcal{NC}_{\alpha_{(n;t)}}$.

Corollary 3.4. *For $n > 0$ and $t \in [n]$, the number of maximal chains in $\mathcal{NC}_{\alpha_{(n;t)}}$ is tn^{n-t-1}.*

In fact, the formula in Corollary 3.4 recovers [16, A058127], and counts acyclic functions from $[n-t]$ to $[n]$. (Here, a function $f\colon [n-t] \to [n]$ is *acyclic* if it does not fix any non-empty subset of $[n-t]$.) A nice follow-up would be an explicit bijection from the set of maximal chains of $\mathcal{NC}_{\alpha_{(n;t)}}$ to the set of acyclic functions from $[n-t]$ to $[n]$. Figure 2 shows the poset $\mathcal{NC}_{(3,1,1)}$.

In the case of arbitrary integer compositions α, the zeta polynomial of $\mathcal{NC}_\alpha$ may have non-integral roots. For instance, the zeta polynomial of $\mathcal{NC}_{(2,2)}$ is q^2+2q-2, which does not factor (nontrivially) over $\mathbb{Z}$. We suspect, however, that the cases $\alpha = (t,1,1,\ldots,1)$ and $\alpha = (1,1,\ldots,1,t)$ for arbitrary t are the only cases in which the zeta polynomial of $\mathcal{NC}_\alpha$ has only integral roots.

Moreover, a ranked poset is *Sperner* if the size of the largest antichain does not exceed the size of the largest rank.

Conjecture 3.5. *For $n > 0$ and $t \in [n]$, the poset $\mathcal{NC}_{\alpha_{(n;t)}}$ is Sperner.*

Conjecture 3.5 is known to hold for $t = 1$ [15, Theorem 2], it holds trivially for $t = n$ or $t = n-1$, and it has been verified for $n \leq 7$. However, the poset $\mathcal{NC}_\alpha$ is not Sperner for arbitrary α, as can be witnessed for instance in the case $\alpha = (3,3)$. We do not, however, currently have a suggestion for which other integer compositions α the poset $\mathcal{NC}_\alpha$ is Sperner.

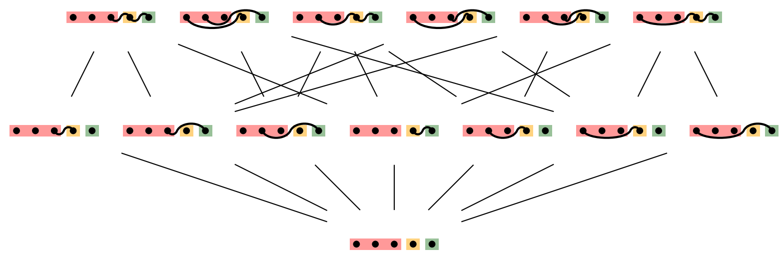


Figure 2: The poset $\mathcal{NC}_{(3,1,1)}$.

4 Ballot-Tamari Lattices

4.1 The Core Label Order of $\mathcal{T}_{\alpha_{(n;t)}}$

Let us now explain a nice connection between the posets $\mathcal{NC}_\alpha$ and $\mathcal{T}_\alpha$. It follows from Theorem 2.1 that $\mathcal{T}_\alpha$ is congruence uniform, which means that it can be constructed in a nice way from the singleton lattice by iterated doublings of intervals. This doubling procedure naturally assigns a label to each edge in the poset diagram, and in the present case, these labels can be identified with the *atomic* noncrossing α-partitions, i.e. noncrossing α-partitions with a unique bump; see Section 2.2 and Corollary 4.5 in [10]. Let us write $\mathbf{P}_{(a,b)}$ for the noncrossing α-partition whose only bump is (a, b).

For $w \in \mathfrak{S}_\alpha$ we define $w_\downarrow$ to be the meet of all lower covers of w in $\mathcal{T}_\alpha$, and we define the following set:

$$\Psi(w) \overset{\text{def}}{=} \Big\{(a,b) \mid \mathbf{P}_{(a,b)} \text{ appears as a label in the poset diagram of } \mathrm{Weak}\big([w_\downarrow, w]\big)\Big\}.$$

If we order $\{\Psi(w) \mid w \in \mathfrak{S}_\alpha(231)\}$ by containment, we obtain a new poset; the *core label order* of $\mathcal{T}_\alpha$ denoted by $\mathrm{CLO}(\mathcal{T}_\alpha)$. See [11] for more background. Figure 3 shows $\mathcal{T}_{(3,1,1)}$, and Figure 2 shows $\mathrm{CLO}(\mathcal{T}_{(3,1,1)})$, where we have identified the sets $\Psi(w)$ by the corresponding noncrossing $(3, 1, 1)$-partitions. We may now prove our second main result.

Theorem 4.1. *For $n > 0$ and $t \in [n]$, the core label order of $\mathcal{T}_{\alpha_{(n;t)}}$ is isomorphic to $\mathcal{NC}_{\alpha_{(n;t)}}$.*

Proof. Since $\alpha_{(n;t)}$ is such that only the first entry may exceed 1, we can show that $\Psi(w)$ contains precisely all the pairs (a, b), where a and b lie in the same block of the noncrossing $\alpha_{(n;t)}$-partition $\Phi(w)$ under the bijection $\Phi\colon \mathfrak{S}_{\alpha_{(n;t)}}(231) \to NC_{\alpha_{(n;t)}}$ described in [12, Theorem 4.2]; see [10, Propositions 5.2 and 5.3].

It remains to show that the dual refinement order on $NC_{\alpha_{(n;t)}}$ is equivalent to inclusion on $\{\Psi(w) \mid w \in \mathfrak{S}_{\alpha_{(n;t)}}(231)\}$, but this is straightforward to verify; see [10, Lemma 5.1]. $\square$

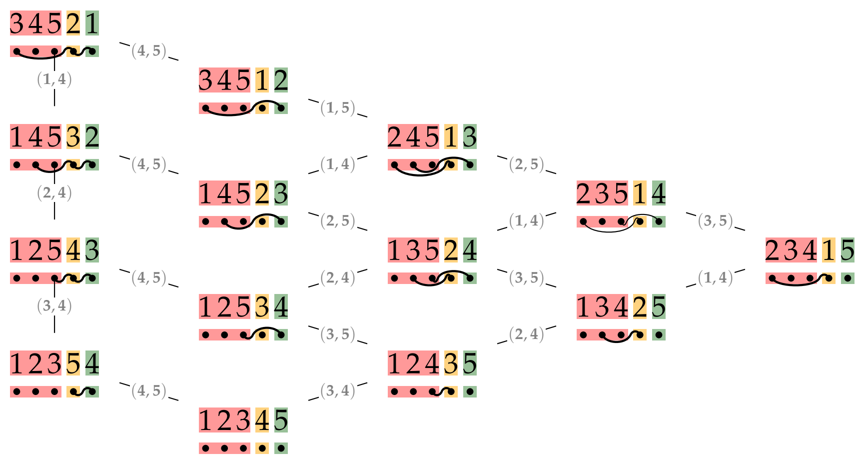


Figure 3: The lattice $\mathcal{T}_{(3,1,1)}$, where every element is additionally labeled by the corresponding noncrossing $(3,1,1)$-partition. The edges are labeled by the unique descent of the element on the top of the edge, which is not an inversion of the element on the bottom of the edge.

In the case $t = 1$, Theorem 4.1 is [14, Theorem 8.5]. In fact, [12, Theorem 1.5] states that this isomorphism holds precisely for the compositions $\alpha = (t_1, 1, 1, \ldots, 1, t_2)$.

4.2 The Galois Graph of $\mathcal{T}_{\alpha_{(n;t)}}$

Another consequence of Theorem 2.1 is that $\mathcal{T}_{\alpha_{(n;t)}}$ is a trim lattice. Trim lattices were introduced in [18], and should be regarded as non-graded analogues of distributive lattices. Such lattices are in particular *extremal* in the sense that they have the same number of join- and meet-irreducible elements, and this number equals the length of a longest maximal chain.

Extremal lattices can be uniquely represented by a particular directed graph on the vertex set $[k]$, where k denotes the number of join-irreducible elements of the lattice; the *Galois graph*, see [9, Theorem 11]. If a trim lattice is also congruence uniform, then this Galois graph may be viewed as a graph whose vertices *are* the join-irreducible elements of the lattice [10, Corollary 2.17].

Recall from [10, Corollary 4.5] that the join-irreducible elements of $\mathcal{T}_\alpha$ correspond bijectively to the atomic noncrossing α-partitions. Theorem 1.8 in [10] characterizes the Galois graph of $\mathcal{T}_\alpha$, and this result enables us to prove our third main result.

Theorem 4.2. *For $n > 0$ and $t \in [n]$, the ballot-Tamari lattice $\mathcal{T}_{\alpha_{(n;t)}}$ is trim, and its Galois graph is isomorphic to the directed graph with vertex set*

$$\big\{(a,b) \mid \text{either } a \in [t] \text{ and } b > t, \text{ or } t < a < b\big\},$$

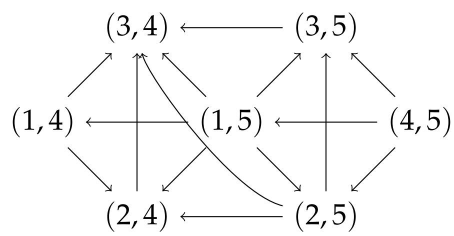


Figure 4: The Galois graph of $\mathcal{T}_{(3,1,1)}$.

where we have a directed edge

$$(a,b) \to (a',b') \quad \textit{if and only if} \quad \begin{cases} a' < b' \leq b, & \textit{if } a < a' \leq t, \\ a < b' \leq b, & \textit{if } a' < a \textit{ and } t < a, \\ a' < b' < b, & \textit{if } a = a'. \end{cases}$$

Proof. The fact that $\mathcal{T}_{\alpha_{(n;t)}}$ is trim follows from Theorem 2.1.

According to the paragraphs preceding this proof we may view the Galois graph of $\mathcal{T}_{\alpha_{(n;t)}}$ as a directed graph whose vertices are the bumps of the atomic noncrossing $\alpha_{(n;t)}$-partitions.

Let $\mathbf{P}_{(a,b)}$ and $\mathbf{P}_{(a',b')}$ be atomic noncrossing $\alpha_{(n;t)}$-partitions. From [10, Theorem 1.8] we conclude that there exists a directed edge $(a,b) \to (a',b')$ if and only if $(a,b) \neq (a',b')$ and either a and a' belong to the same $\alpha_{(n;t)}$-region and $a \leq a' < b' \leq b$ or they belong to different $\alpha_{(n;t)}$-regions and $a' < a < b' \leq b$, where a and b' belong to different $\alpha_{(n;t)}$-regions, too. By construction we conclude that there is an edge $(a,b) \to (a',b')$ if and only if the conditions in the statement are met. □

For $t = 1$, the Galois graph of $\mathcal{T}_{\alpha_{(n;t)}}$ has been described in [9, Corollary 1] and [19, Example 2.9] in a slightly different language. Figure 4 shows the Galois graph of $\mathcal{T}_{(3,1,1)}$.

5 Ballot-Chapoton Triangles

In this section we point out a strong enumerative connection between the sets NC_α and $\mathcal{D}_\alpha$. Recall that the *Möbius function* of a poset $\mathcal{P} = (P, \leq)$ is the function $\mu_{\mathcal{P}} \colon P \times P \to \mathbb{Z}$ which is recursively defined via the property that $\sum_{x \leq z \leq y} \mu(x,z) = \delta_{x,y}$ for all $x \leq y$. (Here $\delta_{x,y}$ is the indicator function that equals 1 precisely when $x = y$ and 0 otherwise.)

For $\mathbf{P} \in NC_\alpha$ let $\operatorname{bump}(\mathbf{P})$ denote its number of bumps. Let us define the *M_α-triangle* to be the following bivariate polynomial:

$$M_\alpha(p,q) \overset{\text{def}}{=} \sum_{\mathbf{P},\mathbf{P}' \in NC_\alpha} \mu_{\mathrm{CLO}(\mathcal{T}_\alpha)}\big(\mathbf{P},\mathbf{P}'\big) p^{\operatorname{bump}(\mathbf{P}')} q^{\operatorname{bump}(\mathbf{P})}. \tag{5.1}$$

For an α-Dyck path $\mu \in \mathcal{D}_\alpha$ we call a *peak* a subpath of the form NE. A peak of μ is a *bounce peak* if it is also a peak of ν_α. Now suppose that a peak consists of three lattice points (x_1, y_1), (x_2, y_2), and (x_3, y_3). (We thus have $x_1 = x_2$ and $y_2 = y_1 + 1$, as well as $y_2 = y_3$ and $x_3 = x_2 + 1$.) If (x_3, y_1) is a lattice point on ν_α, then we call such a peak a *base peak*. Let $\mathrm{peak}(\mu)$ denote the number of peaks of μ, let $\mathrm{bouncepeak}(\mu)$ denote the number of bounce peaks of μ, and let $\mathrm{basepeak}(\mu)$ denote the number of base peaks of μ. We define the *H_α-triangle* by

$$H_\alpha(p,q) \overset{\text{def}}{=} \sum_{\mu \in \mathcal{D}_\alpha} p^{\mathrm{peak}(\mu) - \mathrm{bouncepeak}(\mu)} q^{\mathrm{basepeak}(\mu)}. \tag{5.2}$$

Remark 5.1. *In [10, Section 6.2] we have defined the H_α-triangle in terms of antichains in a certain subposet of a triangular poset. If $\mu \in \mathcal{D}_\alpha$ and A_μ is the corresponding antichain, then the number* $\mathrm{peak}(\mu) - \mathrm{bouncepeak}(\mu)$ *corresponds to the size of A_μ, and the number* $\mathrm{basepeak}(\mu)$ *corresponds to the number of minimal elements of the triangular subposet contained in A_μ.*

Our main conjecture states that we can obtain $H_{\alpha_{(n;t)}}$ from $M_{\alpha_{(n;t)}}$ by appropriate substitution of variables, and it has been verified by computer for $n \leq 8$.

Conjecture 5.2. *For $n > 0$ and $t \in [n]$, we have*

$$H_{\alpha_{(n;t)}}(p,q) = \bigl(1 + p(q-1)\bigr)^{n-t} M_{\alpha_{(n;t)}}\left(\frac{p(q-1)}{p(q-1)+1}, \frac{q}{q-1}\right).$$

Computer evidence suggests that Conjecture 5.2 holds if and only if α has at most one part which may exceed 1. The reader is cordially invited to verify the failure of this conjecture in the case $\alpha = (2,2)$.

Moreover, it turns out that we can *define* yet another polynomial, the *$F_{\alpha_{(n;t)}}$-triangle* from the (conjectured) correspondence in Conjecture 5.2:

$$F_{\alpha_{(n;t)}}(p,q) \overset{\text{def}}{=} p^{n-t} H_{\alpha_{(n;t)}}\left(\frac{p+1}{p}, \frac{q+1}{p+1}\right). \tag{5.3}$$

A priori, this is only a rational function, but computer evidence suggests that it is indeed a polynomial with nonnegative integer coefficients.

Conjecture 5.3. *For $n > 0$ and $t \in [n]$, the rational function $F_{\alpha_{(n;t)}}(p,q)$ is in fact a polynomial with nonnegative integer coefficients.*

We expect Conjecture 5.3 to hold for the same compositions, for which Conjecture 5.2 is supposed to be true. It is intriguing to find a combinatorial definition of $F_{\alpha_{(n;t)}}$, i.e. a family $X_{\alpha_{(n;t)}}$ of combinatorial objects, and two statistics P and Q, such that

$$F_{\alpha_{(n;t)}}(p,q) = \sum_{x \in X_{\alpha_{(n;t)}}} p^{P(x)} q^{Q(x)}.$$

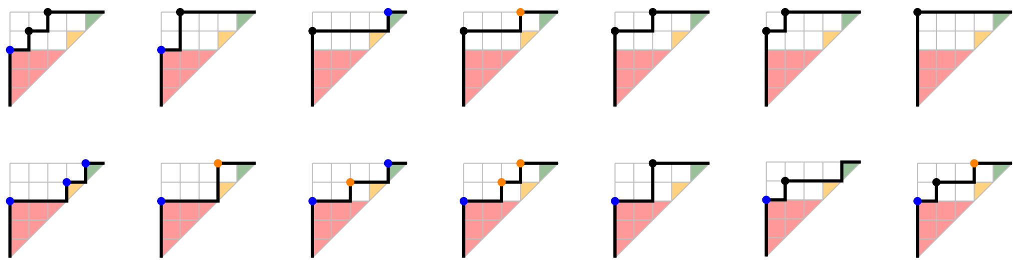


Figure 5: The fourteen $(3, 1, 1)$-Dyck paths. The base peaks are indicated by orange dots, the bounce peaks by blue dots, and the remaining peaks by black dots.

For $t = 1$, Conjecture 5.2 was first posed in [3, 4], and it was proven in [1, 17]. The desired family $X_{\alpha_{(n;1)}}$ is the set of triangulations of a convex $(n + 2)$-gon together with a canonical labeling of the diagonals using positive and negative labels. For any triangulation x, the statistic $P(x)$ counts the positively labeled diagonals and the statistic $Q(x)$ counts the negatively labeled diagonals. Can this be generalized to parabolic quotients?

Let us illustrate Conjecture 5.2 on the running example $\alpha = (3, 1, 1)$. By inspection of Figures 2 and 5 we obtain

$$M_{(3,1,1)}(p, q) = 6p^2q^2 - 15p^2q + 9p^2 + 7pq - 7p + 1,$$
$$H_{(3,1,1)}(p, q) = p^2q^2 + 2p^2q + 3p^2 + 2pq + 5p + 1.$$

It is now straightforward to see that Conjecture 5.2 works out, and that we obtain

$$F_{(3,1,1)}(p, q) = 9p^2 + 4pq + q^2 + 15p + 4q + 6.$$

Acknowledgements

I would like to express my gratitude to Christian Krattenthaler for pointing out that the formula in (3.2) counts certain ballot paths, and for explaining his bijective proof of Theorem 3.3 to me.

References

[1] C. A. Athanasiadis. “On some enumerative aspects of generalized associahedra”. *Europ. J. Combin.* **28**.4 (2007), pp. 1208–1215. Link.

[2] C. Ceballos, W. Fang, and H. Mühle. “The Steep-Bounce Zeta Map in Parabolic Cataland”. 2019. arXiv:1903.08515.

[3] F. Chapoton. "Enumerative Properties of Generalized Associahedra". *Sém. Loth. Combin.* **51** (2004), Art. #B51b, 16 pp. Link.

[4] F. Chapoton. "Sur le Nombre de Réflexions Pleines dans les Groupes de Coxeter Finis". *Bull. Belg. Math. Soc.* **13** (2006), pp. 585–596.

[5] A. Day. "Characterizations of Finite Lattices that are Bounded-Homomorphic Images or Sublattices of Free Lattices". *Canad. J. Math.* **31** (1979), pp. 69–78. Link.

[6] C. Krattenthaler. "Lattice Path Enumeration". *Handbook of Enumerative Combinatorics*. Ed. by M. Bóna. Discrete Math. Appl. 87. Boca Raton-London-New York: CRC Press, 2015, pp. 589–678.

[7] C. Krattenthaler. Personal communication. 2019.

[8] G. Kreweras. "Sur les partitions non croisées d'un cycle". *Discrete Math.* **1**.4 (1972), pp. 333–350. Link.

[9] G. Markowsky. "Primes, irreducibles and extremal lattices". *Order* **9**.3 (1992), pp. 265–290. Link.

[10] H. Mühle. "Noncrossing Partitions, Tamari Lattices, and Parabolic Quotients of the Symmetric Group". 2018. arXiv:1809.01405.

[11] H. Mühle. "The core label order of a congruence-uniform lattice". *Algebra Univ.* **80**.10 (2019). Link.

[12] H. Mühle and N. Williams. "Tamari Lattices for Parabolic Quotients of the Symmetric Group". 2018. arXiv:1804.02761.

[13] L.-F. Préville-Ratelle and X. Viennot. "The enumeration of generalized Tamari intervals". *Trans. Amer. Math. Soc.* **369**.7 (2017), pp. 5219–5239. Link.

[14] N. Reading. "Noncrossing partitions and the shard intersection order". *J. Algebraic Combin.* **33**.4 (2011), pp. 483–530. Link.

[15] R. Simion and D. Ullman. "On the structure of the lattice of noncrossing partitions". *Discrete Math.* **98**.3 (1991), pp. 193–206. Link.

[16] N. J. A. Sloane. "The Online Encyclopedia of Integer Sequences". Link.

[17] M. Thiel. "On the H-triangle of generalised nonnesting partitions". *Europ. J. Combin.* **39** (2014), pp. 244–255. Link.

[18] H. Thomas. "An Analogue of Distributivity for Ungraded Lattices". *Order* **23** (2006), pp. 249–269. Link.

[19] H. Thomas and N. Williams. "Rowmotion in slow motion". 2017. arXiv:1712.10123.