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# The lattice permutation condition for Kronecker tableaux

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**Abstract.** We recently generalised the lattice permutation condition for Young tableaux to Kronecker tableaux and hence calculate a large new class of stable Kronecker coefficients labelled by co-Pieri triples. In this extended abstract we discuss important families of co-Pieri triples for which our combinatorics simplifies drastically.

**Keywords:** Kronecker coefficients, tensor product, partition algebra, symmetric group

## 1 Introduction

Perhaps the last major open problem in the complex representation theory of symmetric groups is to describe the decomposition of a tensor product of two simple representations. The coefficients describing the decomposition of these tensor products are known as the *Kronecker coefficients* and they have been described as 'perhaps the most challenging, deep and mysterious objects in algebraic combinatorics'. Much recent progress has focused on the stability properties enjoyed by Kronecker coefficients.

Whilst a complete understanding of the Kronecker coefficients seems out of reach, the purpose of this work is to attempt to understand the *stable* Kronecker coefficients in terms of oscillating tableaux. Oscillating tableaux hold a distinguished position in the study of tensor product decompositions [4, 10, 11] but surprisingly they have never before been used to calculate Kronecker coefficients of symmetric groups. In this work, we see that the oscillating tableaux defined as paths on the graph given in Figure 1 (which we call Kronecker tableaux) provide bases of certain modules for the partition algebra, $P_s(n)$, which is closely related to the symmetric group. We hence add a new level of structure to the classical picture — this extra structure is the key to our main result: the co-Pieri rule for stable Kronecker coefficients.

A momentary glance at the graph given in Figure 1 reveals a very familiar subgraph: namely Young's graph (with each level doubled up). The stable Kronecker coefficients labelled by triples from this subgraph are well-understood — the values of these coefficients can be calculated via a tableaux counting algorithm known as the Littlewood–Richardson rule [6]. This rule has long served as the hallmark for our understanding of

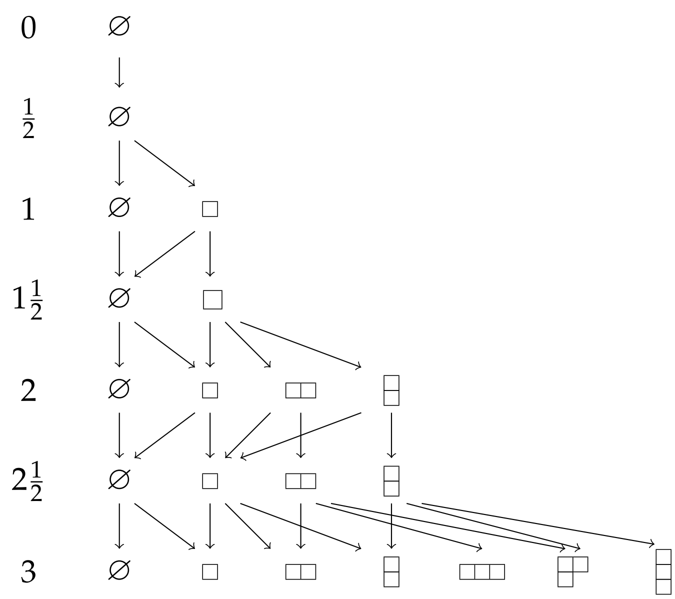


**Figure 1:** The first three layers of the branching graph $\mathcal{Y}$

Kronecker coefficients. The Littlewood–Richardson rule was discovered as a rule of two halves (as we explain below). In [1] we succeed in generalising one half of this rule to all Kronecker tableaux, and thus solve one half of the stable Kronecker problem. Our main result unifies and vastly generalises the work of Littlewood–Richardson [6] and many other authors [2, 3, 7, 8, 9]. Most promisingly, our result counts explicit homomorphisms and thus works on a structural level above any description of a family of Kronecker coefficients since those first considered by Littlewood–Richardson [6].

In more detail, given a triple of partitions $(\lambda, \nu, \mu)$ and with $|\mu| = s$, we have an associated skew $P_s(n)$-module spanned by the Kronecker tableaux from $\lambda$ to $\nu$ of length $s$, which we denote by $\Delta_s(\nu \setminus \lambda)$. For $\lambda = \varnothing$ and $n \geq 2s$ these modules provide a complete set of non-isomorphic $P_s(n)$-modules (and we drop the partition $\varnothing$ from the notation). The stable Kronecker coefficients are then interpreted as the dimensions,

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda))) \qquad (\dagger)$$

for $n \geq 2s$. Restricting to the Young subgraph, or equivalently to a triple $(\lambda, \nu, \mu)$ of so-called *maximal depth* such that $|\lambda| + |\mu| = |\nu|$, these modules specialise to the usual simple and skew modules for symmetric groups; hence the multiplicities $\overline{g}(\lambda, \nu, \mu)$ are the Littlewood–Richardson coefficients. We hence recover the well-known fact that the Littlewood–Richardson coefficients appear as the subfamily of stable Kronecker coefficients labelled by triples of maximal depth. The tableaux counted by the Littlewood–Richardson rule satisfy 2 conditions: the *semistandard* and *lattice permutation* conditions. In [1] we generalise the lattice permutation condition to Kronecker tableaux.

**Theorem** ([1, Main Theorem]). *Let $(\lambda, \nu, \mu)$ be a co-Pieri triple or a triple of maximal depth. Then the stable Kronecker coefficient $\overline{g}(\lambda, \nu, \mu)$ is given by the number of semistandard Kronecker tableaux of shape $\nu \setminus \lambda$ and weight $\mu$ whose reverse reading word is a lattice permutation.*

The observant reader will notice that the statement above describes the Littlewood–Richardson coefficients uniformly as part of a far broader family of stable Kronecker coefficients (and is the first result in the literature to do so). Whilst the classical Pieri rule (describing the semistandardness condition for Littlewood–Richardson tableaux) is elementary, it served as a first step towards understanding the full Littlewood–Richardson rule; indeed Knutson–Tao–Woodward have shown that the Littlewood–Richardson rule follows from the Pieri rule by associativity [5]. We hope that our generalisation of the co-Pieri rule (the lattice permutation condition for Kronecker tableaux) will prove equally useful in the study of stable Kronecker coefficients.

**Special cases of co-Pieri triples.** The definition of *co-Pieri triples* is given in [1, Theorem 4.12] and can appear quite technical at first reading; we present a few special cases here.

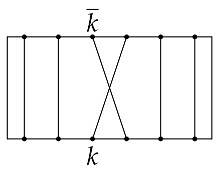
- $(i)$ $\lambda$ and $\mu$ are one-row partitions and $\mu$ is arbitrary. This family has been extensively studied over the past thirty years and there are many distinct combinatorial descriptions of some or all of these coefficients [2, 3, 7, 8, 9], none of which generalises.
- $(ii)$ the two skew partitions $\lambda \ominus (\lambda \cap \nu)$ and $\nu \ominus (\lambda \cap \nu)$ have no two boxes in the same column and $|\mu| = \max\{|\lambda \ominus (\lambda \cap \nu)|, |\nu \ominus (\lambda \cap \nu)|\}$. It is easy to see that if, in addition, $(\lambda, \nu, \mu)$ is a triple of maximal depth, then this case specialises to the classical co-Pieri triples.
- $(iii)$ $\lambda = \nu = (dl, d(l-1), \dots, 2d, d)$ for any $l, d \geq 1$ and $|\mu| \leq d$.

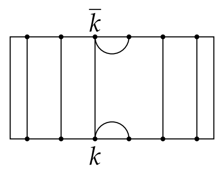
In this extended abstract we have chosen to focus primarily on case $(i)$ as these triples carry many of the tropes of general co-Pieri triples (but with significant simplifications which serve to make this abstract more approachable) and because case $(i)$ should be familiar to many readers due to its many appearances in the literature.

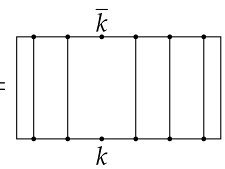
# 2 The partition algebra and Kronecker tableaux

The combinatorics underlying the representation theory of the partition algebras and symmetric groups is based on partitions. A *partition* $\lambda$ of $n$, denoted $\lambda \vdash n$, is defined to be a sequence of weakly decreasing non-negative integers which sum to $n$. We let $\varnothing$ denote the unique partition of 0. Given a partition, $\lambda = (\lambda_1, \lambda_2, \dots)$, the associated *Young diagram* is the set of nodes $[\lambda] = \left\{(i,j) \in \mathbb{Z}^2_{>0} \mid j \leq \lambda_i\right\}$. We define the length, $\ell(\lambda)$, of a partition $\lambda$, to be the number of non-zero parts. Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ a partition and $n$ an integer, define $\lambda_{[n]} = (n - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Given $\lambda_{[n]}$ a partition of $n$, we say that the partition has *depth* equal to $|\lambda|$.

The partition algebra is generated as an algebra by the elements $s_{k,k+1}$, $p_{k+1/2}$ $(1 \leq k \leq r-1)$ and $p_k$ $(1 \leq k \leq r)$ pictured below modulo a long list of relations. One can visualise any product in this algebra as simply being given by concatenation of diagrams, modulo some surgery to remove closed loops [1].

$s_{k,k+1} =$ 

$p_{k+1/2} =$ 

$p_k =$ 


Define the branching graph $\mathcal{Y}$ as follows. For $k \in \mathbb{Z}_{\geq 0}$, we denote by $\mathscr{P}_{\leq k}$ the set of partitions of degree less or equal to $k$. Now the set of vertices on the $k$th and $(k+1/2)$th levels of $\mathcal{Y}$ are given by

$$\mathcal{Y}_k = \{(\lambda, k - |\lambda|) \mid \lambda \in \mathscr{P}_{\leq k}\} \qquad \mathcal{Y}_{k+1/2} = \{(\lambda, k - |\lambda|) \mid \lambda \in \mathscr{P}_{\leq k}\}.$$

The edges of $\mathcal{Y}$ are as follows,

- for $(\lambda, l) \in \mathcal{Y}_k$ and $(\mu, m) \in \mathcal{Y}_{k+1/2}$ there is an edge $(\lambda, l) \to (\mu, m)$ if $\mu = \lambda$, or if $\mu$ is obtained from $\lambda$ by removing a box in the $i$th row for some $i \geq 1$; we write $\mu = \lambda - \varepsilon_0$ or $\mu = \lambda - \varepsilon_i$, respectively.

- for $(\lambda, l) \in \mathcal{Y}_{k+1/2}$ and $(\mu, m) \in \mathcal{Y}_{k+1}$ there is an edge $(\lambda, l) \to (\mu, m)$ if $\mu = \lambda$, or if $\mu$ is obtained from $\lambda$ by adding a box in the $i$th row for some $i \geq 1$; we write $\mu = \lambda + \varepsilon_0$ or $\mu = \lambda + \varepsilon_i$, respectively.

When it is convenient, we decorate each edge with the index of the node that is added or removed when reading down the diagram. The first few levels of $\mathcal{Y}$ are given in Figure 1. When no confusion is possible, we identify $(\lambda, l) \in \mathcal{Y}_k$ with the partition $\lambda$.

**Definition 1.** *Given $\lambda \in \mathscr{P}_{r-s} \subseteq \mathcal{Y}_{r-s}$ and $\nu \in \mathscr{P}_{\leq r} \subseteq \mathcal{Y}_r$, we define a* standard Kronecker tableau *of shape $\nu \setminus \lambda$ and degree $s$ to be a path $\mathsf{t}$ of the form*

$$\lambda = \mathsf{t}(0) \to \mathsf{t}(\tfrac{1}{2}) \to \mathsf{t}(1) \to \cdots \to \mathsf{t}(s - \tfrac{1}{2}) \to \mathsf{t}(s) = \nu, \tag{2.1}$$

*in other words $\mathsf{t}$ is a path in $\mathcal{Y}$ which begins at $\lambda$ and terminates at $\nu$. We let $\mathrm{Std}_s(\nu \setminus \lambda)$ denote the set of all such paths. If $\lambda = \varnothing \in \mathcal{Y}_0$ then we write $\mathrm{Std}_r(\nu)$ instead of $\mathrm{Std}_r(\nu \setminus \varnothing)$. Given $\mathsf{s}, \mathsf{t}$ two standard Kronecker tableaux of degree $s$, we write $\mathsf{s} \trianglerighteq \mathsf{t}$ if $\mathsf{s}(k) \trianglerighteq \mathsf{t}(k)$ for all $0 \leq k \leq s$.*

We can think of a path as either the sequence of partitions or the sequence of boxes removed and added. We usually prefer the latter case and record these boxes removed and added pairwise. For a pair $(-\varepsilon_p, +\varepsilon_q)$ we call this an add or remove step if $p = 0$ or

$q = 0$ respectively (because the effect of this step is to add or remove a box) and we call this a dummy step if $p = q$ (as we end up at the same partition as we started); we write $a(q)$ or $r(p)$ for an add or remove step and $d(p)$ for a dummy step. Many examples are given below, in particular the reader should compare the paths of Example 3 with those depicted in the central diagram in Figure 2. We let $\mathsf{t}^\lambda$ denote the most dominant element of $\mathrm{Std}_s(\lambda)$, namely that of the form:

$$\underbrace{d(0) \circ d(0) \circ \cdots \circ d(0)}_{r-|\lambda|} \circ \underbrace{a(1) \circ \cdots \circ a(1)}_{\lambda_1} \circ \underbrace{a(2) \circ \cdots \circ a(2)}_{\lambda_2} \circ \cdots$$

Given $\lambda \in \mathscr{P}_{r-s} \subseteq \mathcal{Y}_{r-s}$ and $\nu \in \mathscr{P}_{\leq r} \subseteq \mathcal{Y}_r$, define the *skew cell module*

$$\Delta_s(\nu \setminus \lambda) = \mathrm{Span}\{\mathsf{t}^\lambda \circ \mathsf{s} \mid \mathsf{s} \in \mathrm{Std}_s(\nu \setminus \lambda)\}$$

with the action of $P_s(n) \hookrightarrow P_{r-s}(n) \otimes P_s(n) \hookrightarrow P_r(n)$ given as in [1, Section 2.3]. If $\lambda = \varnothing$, then we simply denote this module by $\Delta_s(\nu)$. Let $\lambda \in \mathscr{P}_{r-s}$, $\mu \in \mathscr{P}_s$ and $\nu \in \mathscr{P}_{\leq r}$. Then we are able to define the stable Kronecker coefficients (even if this is not their usual definition) to be the multiplicities

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda)))$$

for all $n \geq 2s$. When $s = |\nu| - |\lambda|$, the (skew) cell modules for partition algebras specialise to the usual Specht modules of the symmetric groups and we hence easily see that these stable coefficients coincide with the classical Littlewood–Richardson coefficients.

# 3 The action of the partition algebra

Understanding the action of the partition algebra on skew modules is difficult in general. In this section, we show that this can be done to some extent in the cases of interest to us. We have assumed that $|\mu| = s$, therefore the ideal $P_s(n)p_rP_s(n) \subset P_s(n)$ annihilates $\Delta_s(\mu)$ and this motivates the following definition.

**Definition 2.** *We define the* Dvir radical *of the skew module* $\Delta_s(\nu \setminus \lambda)$ *by*

$$\mathrm{DR}_s(\nu \setminus \lambda) = \Delta_s(\nu \setminus \lambda)P_s(n)p_rP_s(n) \subseteq \Delta_s(\nu \setminus \lambda)$$

*and set*

$$\Delta^0_s(\nu \setminus \lambda) = \Delta_s(\nu \setminus \lambda)/\mathrm{DR}_s(\nu \setminus \lambda).$$

*If* $s = |\nu| - |\lambda|$, *then set* $\mathrm{Std}^0_s(\nu \setminus \lambda) = \mathrm{Std}_s(\nu \setminus \lambda)$. *If* $\lambda$ *and* $\nu$ *are one-row partitions, then set* $\mathrm{Std}^0_s(\nu \setminus \lambda) \subseteq \mathrm{Std}_s(\nu \setminus \lambda)$ *to be the subset of paths,* $\mathsf{s}$, *whose steps are of the form*

$$r(1) = (-1, +0) \qquad d(1) = (-1, +1) \qquad a(1) = (-0, +1)$$

*and such that the total number of boxes removed in* $\mathsf{s}$ *is less than or equal to* $|\lambda|$.

Fix $\mathsf{t} \in \mathrm{Std}_r(\nu)$ and $1 \le k \le r$ and suppose that

$$\mathsf{t}(k-1) \xrightarrow{-t} \mathsf{t}(k-\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}(k+1) \xrightarrow{-v} \mathsf{t}(k+\tfrac{1}{2}) \xrightarrow{+w} \mathsf{t}(k+1).$$

We define $\mathsf{t}_{k\leftrightarrow k+1} \in \mathrm{Std}_r(\nu)$ to be the tableau, if it exists, determined by $\mathsf{t}_{k\leftrightarrow k+1}(l) = \mathsf{t}(l)$ for $l \neq k, k \pm \frac{1}{2}$ and

$$\mathsf{t}_{k\leftrightarrow k+1}(k-1) \xrightarrow{-v} \mathsf{t}_{k\leftrightarrow k+1}(k-\tfrac{1}{2}) \xrightarrow{+w} \mathsf{t}_{k\leftrightarrow k+1}(k) \xrightarrow{-t} \mathsf{t}_{k\leftrightarrow k+1}(k+\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}_{k\leftrightarrow k+1}(k+1).$$

Let $(\lambda, \nu, s)$ be such that $s = |\nu| - |\lambda|$, or $\lambda$ and $\nu$ are both one-row partitions, then $\Delta^0_s(\nu \setminus \lambda)$ is free as a $\mathbb{Z}$-module with basis

$$\{\mathsf{t} \mid \mathsf{t} \in \mathrm{Std}^0_s(\nu \setminus \lambda)\}$$

and the $P_s(n)$-action on $\Delta^0_s(\nu \setminus \lambda)$ is as follows:

$$(\mathsf{t} + \mathsf{DR_s}(\,\mathring{}\setminus\breve{}\,))s_{k,k+1} = \begin{cases} \mathsf{t}_{k\leftrightarrow k+1} + \mathsf{DR_s}(\,\mathring{}\setminus\breve{}\,) & \text{if } \mathsf{t}_{k\leftrightarrow k+1} \text{ exists} \\ -\mathsf{t} + \sum_{\mathsf{s}\rhd\mathsf{t}} r_{\mathsf{st}}\mathsf{s} + \mathsf{DR_s}(\,\mathring{}\setminus\breve{}\,) & \text{otherwise} \end{cases} \tag{3.1}$$

for $1 \le k < s$ and $(\mathsf{t} + \mathsf{DR_s}(\,\mathring{}\setminus\breve{}\,))p_{k,k+1} = 0$ and $(\mathsf{t} + \mathsf{DR_s}(\,\mathring{}\setminus\breve{}\,))p_k = 0$ for $1 \le k \le s$. The coefficients $r_{\mathsf{st}} \in \mathbb{Q}$ are given in [1, Theorem 2.9].

**Example 3.** *The set* $\mathrm{Std}^0_3((4) \setminus (4))$ *consists of the 7 oscillating tableaux*

$$\begin{array}{lll} \mathsf{s}_1 = r(1) \circ d(1) \circ a(1) & \mathsf{s}_2 = d(1) \circ r(1) \circ a(1) & \mathsf{s}_3 = r(1) \circ a(1) \circ d(1) \\ \mathsf{s}_4 = a(1) \circ r(1) \circ d(1) & \mathsf{s}_5 = d(1) \circ a(1) \circ r(1) & \mathsf{s}_6 = a(1) \circ d(1) \circ r(1) \\ & \mathsf{s}_7 = d(1) \circ d(1) \circ d(1) & \end{array}$$

*We have that*

$$s_{1,2} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \qquad s_{2,3} = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

*It is not difficult to see that this module decomposes as follows*

$$\Delta^0_3((4) \setminus (4)) = 2\Delta^0_3((3)) \oplus 2\Delta^0_3((2,1)) \oplus \Delta^0_3((1^3)).$$

# 4 Semistandard Kronecker tableaux

For any $(\lambda, \nu, s) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leq r} \times \mathbb{Z}_{>0}$ and any $\mu \vdash s$ we have

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}} \operatorname{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s^0(\nu \setminus \lambda)) = \dim_{\mathbb{Q}} \operatorname{Hom}_{\mathbb{Q}\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\nu \setminus \lambda)),$$

where $\mathbb{Q}\mathfrak{S}_s$ is viewed as the quotient of $P_s(n)$ by the ideal generated by $p_r$. Now for each $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash s$ we have an associated Young permutation module $\mathsf{M}(\mu) = \mathbb{Q} \otimes_{\mathfrak{S}_\mu} \mathbb{Q}\mathfrak{S}_s$ where $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_l} \subseteq \mathfrak{S}_s$. As a first step towards understanding the stable Kronecker coefficients, it is natural to consider

$$\dim_{\mathbb{Q}} \operatorname{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta_s^0(\nu \setminus \lambda))$$

and to attempt to construct a basis in terms of semistandard (Kronecker) tableaux.

**Definition 4.** *Let $(\lambda, \nu, s) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leq r} \times \mathbb{N}$ be a pair of one-row partitions or a triple of maximal depth. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash s$ and let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s^0(\nu \setminus \lambda)$.*

1. *For $1 \leq k < s$ we write $\mathsf{s} \overset{k}{\sim} \mathsf{t}$ if $\mathsf{s} = \mathsf{t}_{k \leftrightarrow k+1}$.*

2. *We write $\mathsf{s} \overset{\mu}{\sim} \mathsf{t}$ if there exists a sequence of standard Kronecker tableaux $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_d \in \mathrm{Std}_s^0(\nu \setminus \lambda)$ such that*
$$\mathsf{s} = \mathsf{t}_1 \overset{k_1}{\sim} \mathsf{t}_2,\ \mathsf{t}_2 \overset{k_2}{\sim} \mathsf{t}_3,\ \ldots\ , \mathsf{t}_{d-1} \overset{k_{d-1}}{\sim} \mathsf{t}_d = \mathsf{t}$$
*for some $k_1, \ldots, k_{d-1} \in \{1, \ldots, s-1\} \setminus \{[\mu]_c \mid c = 1, \ldots, l-1\}$. We define a* tableau of weight *$\mu$ to be an equivalence class of tableaux under $\overset{\mu}{\sim}$, denoted $[\mathsf{t}]_\mu = \{\mathsf{s} \in \mathrm{Std}_s^0(\nu \setminus \lambda) \,|\, \mathsf{s} \overset{\mu}{\sim} \mathsf{t}\}$.*

3. *We say that a Kronecker tableau, $[\mathsf{t}]_\mu$, of shape $\nu \setminus \lambda$ and weight $\mu$ is* semistandard *if for any $\mathsf{s} \in [\mathsf{t}]_\mu$ and any $k \notin \{[\mu_c] \mid c = 1, \ldots, l-1\}$ the tableau $\mathsf{s}_{k \leftrightarrow k+1}$ exists. Let $\mathrm{SStd}_s^0(\nu \setminus \lambda, \mu)$ denote the set of semistandard Kronecker tableaux of shape $\nu \setminus \lambda$ and weight $\mu$.*

To represent these semistandard Kronecker tableaux graphically, we will add 'frames' corresponding to the composition $\mu$ on the set of paths $\mathrm{Std}_s^0(\nu \setminus \lambda)$ in $\mathcal{Y}$. For $\mathsf{t} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, \ldots, -\varepsilon_{i_s}, +\varepsilon_{j_s})$ we say that the integral step $(-\varepsilon_{i_k}, +\varepsilon_{j_k})$ belongs to the $c$th frame if $[\mu]_{c-1} < k \leq [\mu]_c$. Thus for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s^0(\nu \setminus \lambda)$ we have that $\mathsf{s} \overset{\mu}{\sim} \mathsf{t}$ if and only if $\mathsf{s}$ is obtained from $\mathsf{t}$ by permuting integral steps within each frame (as in Figure 2).

**Theorem 5.** *Let $(\lambda, \nu, s)$ be a co-Pieri triple and $\mu \vdash s$. We define $\varphi_{\mathrm{T}}(\mathsf{t}^\mu) = \sum_{\mathsf{s} \in \mathrm{T}} \mathsf{s}$ for $\mathrm{T} \in \mathrm{SStd}_s^0(\nu \setminus \lambda, \mu)$. Then $\operatorname{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta_s^0(\nu \setminus \lambda))$ has $\mathbb{Z}$-basis $\{\varphi_{\mathrm{T}} \mid \mathrm{T} \in \mathrm{SStd}_s^0(\nu \setminus \lambda, \mu)\}$.*

**Example 6.** *Let $\lambda = (4)$, $\nu = (4)$ and $s = 5$ and $\mu = (2,2,1) \vdash 5$. An example of a semistandard tableau, $\mathrm{V}$, of shape $\nu \setminus \lambda$ and weight $\mu$ is given by the rightmost diagram in*

*Figure 2. The semistandard tableau* V *is an orbit consisting of the following four standard tableaux*

$$\mathrm{v}_1 = r(1) \circ d(1) \circ d(1) \circ a(1) \circ a(1) \qquad \mathrm{v}_2 = d(1) \circ r(1) \circ d(1) \circ a(1) \circ a(1)$$
$$\mathrm{v}_3 = r(1) \circ d(1) \circ a(1) \circ d(1) \circ a(1) \qquad \mathrm{v}_4 = d(1) \circ r(1) \circ a(1) \circ d(1) \circ a(1)$$

*We have a corresponding homomorphism* $\varphi_{\mathrm{V}} \in \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(2,2,1), \Delta_s((4) \setminus (4))$ *given by*

$$\varphi_{\mathrm{T}}(\mathsf{t}^{(2,2,1)}) = \mathrm{v}_1 + \mathrm{v}_2 + \mathrm{v}_3 + \mathrm{v}_4.$$

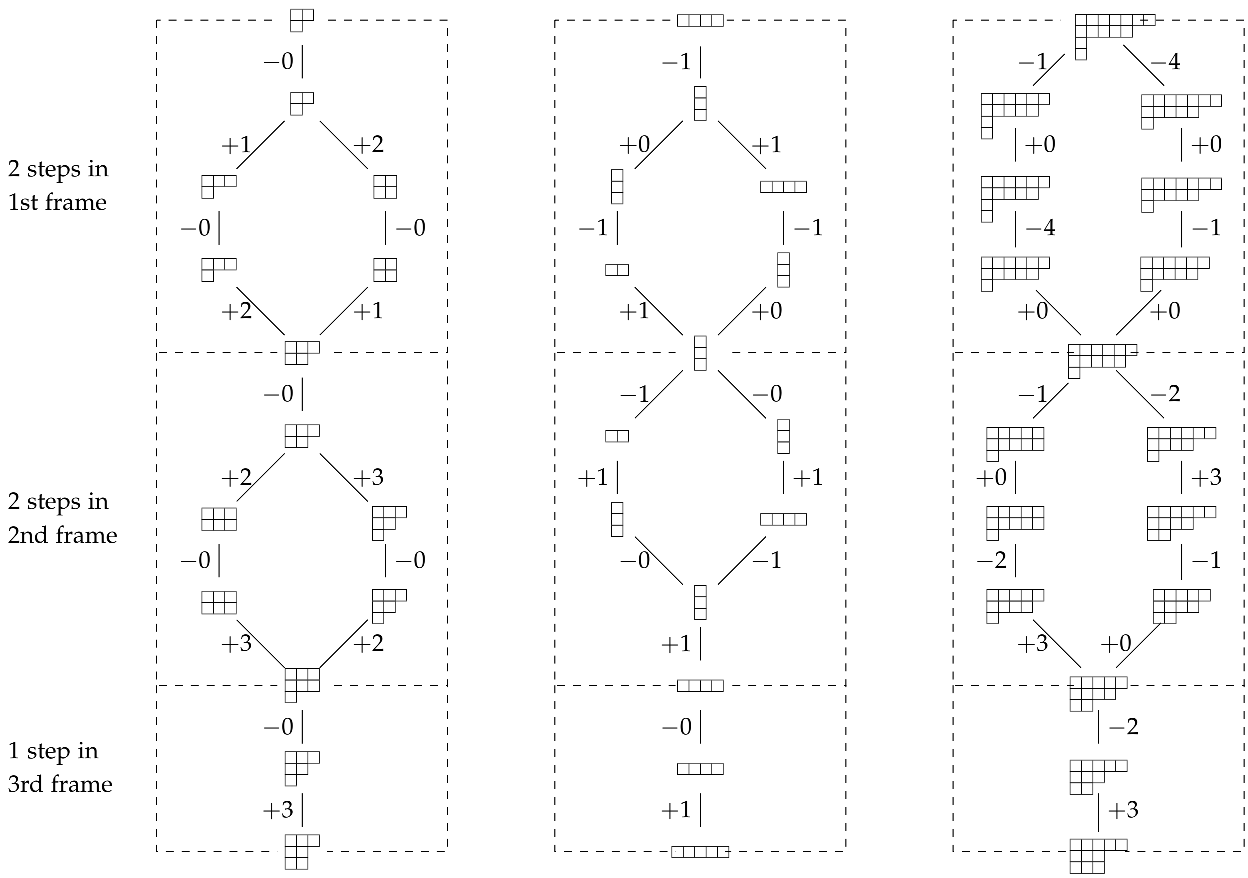



**Figure 2:** Three examples of semistandard Kronecker tableaux of weight $\mu = (2,2,1)$. The number of steps in the $i$th frame is $\mu_i$. The first is a triple of maximal depth, the latter two are co-Pieri triples. Compare the leftmost picture with Example 7 below.

## 4.1 The classical picture for semistandard Young tableaux

We now wish to illustrate how our Definition 4 and the familiar visualisation of a semistandard Young tableaux coincide for triples of maximal depth. Given $\lambda \vdash r - s, \nu \vdash$

$r, \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \vdash s$ such that $\lambda \subseteq \nu$ a Young tableau of shape $\nu \ominus \lambda$ and weight $\mu$ in the classical picture is visualised as a filling of the boxes of $[\nu \ominus \lambda]$ with the entries

$$\underbrace{1, \ldots, 1}_{\mu_1}, \underbrace{2, \ldots, 2}_{\mu_2}, \ldots, \underbrace{\ell, \ldots, \ell}_{\mu_\ell}$$

so that they are weakly increasing along the rows and columns. One should think of this classical picture of a Young tableau of weight $\mu$ simply as a diagrammatic way of encoding an $\mathfrak{S}_\mu$-orbit of standard Young tableaux as follows. Let $\mathsf{s}$ be a standard Young tableau of shape $\nu \ominus \lambda$ and let $\mu$ be a partition. Then define $\mu(\mathsf{s})$ to be the Young tableau of weight $\mu$ obtained from $\mathsf{s}$ by replacing each of the entries $[\mu]_{c-1} < i \leq [\mu]_c$ in $\mathsf{s}$ by the entry $c$ for $c \geq 1$. We identify a Young tableau, $\mathsf{S}$, of weight $\mu$ with the set of standard Young tableaux, $\mu^{-1}(\mathsf{S}) = \{\mathsf{s} \mid \mu(\mathsf{s}) = \mathsf{S}\}$.

In either picture, a Young tableau of weight $\mu$ is merely a picture which encodes an $\mathfrak{S}_\mu$-orbit of standard Young tableaux. We picture a Young tableau, $\mathsf{S}$, of weight $\mu$ as the orbit of paths $\mu^{-1}(\mathsf{S})$ in the branching graph with a frame to record the partition $\mu$.

A tableau of weight $\mu$ in the classical picture would be said to be semistandard if and only if the entries are strictly increasing along the columns. In our picture, this is equivalent to condition 3 of Definition 4.

**Example 7.** *Let $\lambda = (2, 1)$, $\nu = (3, 3, 2)$ and $s = 5$. Then $(\lambda, \nu, s)$ is a triple of maximal depth. Take $\mu = (2, 2, 1) \vdash 5$. The semistandard tableau $\mathsf{U}$ is an orbit consisting of the following four standard tableaux*

$$\mathsf{u}_1 = a(1) \circ a(2) \circ a(2) \circ a(3) \circ a(3) \qquad \mathsf{u}_2 = a(2) \circ a(1) \circ a(2) \circ a(3) \circ a(3)$$
$$\mathsf{u}_3 = a(1) \circ a(2) \circ a(3) \circ a(2) \circ a(3) \qquad \mathsf{u}_4 = a(2) \circ a(1) \circ a(3) \circ a(2) \circ a(3)$$

*pictured as follows*

$$\mu^{-1}\left(\begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}\right) = \left\{ \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\ , \ \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\ , \ \begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline \ & 1 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\ , \ \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \right\}.$$

*We have a corresponding homomorphism $\varphi_{\mathsf{U}} \in \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(2, 2, 1), \Delta_s((3, 3, 2) \setminus (2, 1))$ given by*

$$\varphi_{\mathsf{T}}(\mathsf{t}^{(2,2,1)}) = \mathsf{u}_1 + \mathsf{u}_2 + \mathsf{u}_3 + \mathsf{u}_4.$$

*Compare this orbit sum over 4 tableaux with Figure 2 (left) and the statement of Theorem 5.*

## 5 Latticed Kronecker tableaux

We now provide the main result of the paper, namely we combinatorially describe

$$\overline{g}(\lambda, \nu, \mu) = \dim \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\nu \setminus \lambda))$$

for $(\lambda, \nu, \mu)$ a triple of maximal depth or such that $\lambda$ and $\nu$ are both one-row partitions. One can think of a path $\mathsf{t} \in \mathrm{Std}_s(\nu \setminus \lambda)$ as a sequence of partitions; or equivalently, as the sequence of boxes added and removed. We shall refer to a pair of steps, $(-\varepsilon_a, +\varepsilon_b)$, between consecutive integral levels of the branching graph as an *integral step* in the branching graph. We define *types* of integral step (move-up, dummy, move-down) in the branching graph of $P_r(n)$ and order them as follows,

$$\begin{array}{ccccc} \text{move-up} & & \text{dummy} & & \text{move-down} \\ (-\varepsilon_p, +\varepsilon_q) & < & (-\varepsilon_t, +\varepsilon_t) & < & (-\varepsilon_u, +\varepsilon_v) \end{array}$$

for $p > q$ and $u < v$; we refine this to a total order as follows,

$(m\uparrow)$ we order $(-\varepsilon_p, +\varepsilon_q) < (-\varepsilon_{p'}, +\varepsilon_{q'})$ if $q < q'$ or $q = q'$ and $p > p'$;

$(d)$ we order $(-\varepsilon_t, +\varepsilon_t) < (-\varepsilon_{t'}, +\varepsilon_{t'})$ if $t > t'$;

$(m\downarrow)$ we order $(-\varepsilon_u, +\varepsilon_v) < (-\varepsilon_{u'}, +\varepsilon_{v'})$ if $u > u'$ or $u = u'$ and $v < v'$.

We sometimes let $a(i) := m{\downarrow}(0, i)$ (respectively $r(i) := m{\uparrow}(i, 0)$) and think of this as *adding* (respectively *removing*) a box. We start with any standard tableau $\mathsf{s} \in \mathrm{Std}^0_s(\nu \setminus \lambda)$ and any $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash s$. Write

$$\mathsf{s} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, -\varepsilon_{i_2}, +\varepsilon_{j_2}, \ldots, -\varepsilon_{i_s}, +\varepsilon_{j_s}).$$

Recall from the previous section that, to each integral step $(-\varepsilon_{i_k}, +\varepsilon_{j_k})$ in $\mathsf{s}$, we associate its frame $c$, that is the unique positive integer such that $[\mu]_{c-1} < k \leq [\mu]_c$.

**Definition 8.** *We encode the integral steps of* $\mathsf{s}$ *and their frames in a* $2 \times s$ *array, denoted by* $\omega_\mu(\mathsf{s})$ *(called the* $\mu$*-reverse reading word of* $\mathsf{s}$*) as follows. The first row of* $\omega_\mu(\mathsf{s})$ *contains all the integral steps of* $\mathsf{s}$ *and the second row contains their corresponding frames. We order the columns of* $\omega_\mu(\mathsf{s})$ *increasingly using the ordering on integral steps given in Definition 2.5. For two equal integral steps we order the columns so that the frame numbers are weakly decreasing. Given* $\mathsf{S} \in \mathrm{SStd}^0_s(\nu \setminus \lambda, \mu)$*, it is easy to see that* $\omega_\mu(\mathsf{s}) = \omega_\mu(\mathsf{t})$ *for any pair* $\mathsf{s}, \mathsf{t} \in \mathsf{S}$ *and so we define the* $\mu$*-reverse reading word,* $\omega(\mathsf{S})$*, of* $\mathsf{S}$ *in the obvious fashion. For* $\mathsf{S} \in \mathrm{SStd}^0_s(\nu \setminus \lambda, \mu)$ *we write*

$$\omega(\mathsf{S}) = (\omega_1(\mathsf{S}), \omega_2(\mathsf{S}))$$

*where* $\omega_1(\mathsf{S})$ *(respectively* $\omega_2(\mathsf{S})$*) is the first (respectively second) row of* $\omega(\mathsf{S})$*. Note that* $\omega_2(\mathsf{S})$ *is a sequence of positive integers such that* $i$ *appears precisely* $\mu_i$ *times, for* $i \geq 1$*.*

**Example 9.** *For* $\lambda = (2, 1)$ *and* $\nu = (3, 3, 2)$*, the steps taken in the semistandard tableau* $\mathsf{U}$ *on the left of Figure 2 are*

$$a(1), a(2), a(2), a(3), a(3).$$

*We record the steps according to the dominance ordering for the partition algebra ($a(1) < a(2) < a(3)$) and refine this by recording the frame in which these steps occur backwards, as follows*

$$\omega(\mathrm{U}) = \begin{pmatrix} a(1) & a(2) & a(2) & a(3) & a(3) \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix}.$$

*For $\lambda = (4)$ and $\nu = (5)$, the steps taken in the semistandard tableau* V *in Figure 2 (right) are*

$$r(1), d(1), d(1), a(1), a(1).$$

*We record the steps according to the dominance ordering for the partition algebra ($r(1) < d(1) < a(1)$) and we refine this by recording the frame in which these steps occur backwards, as follows*

$$\omega(\mathrm{V}) = \begin{pmatrix} r(1) & d(1) & d(1) & a(1) & a(1) \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix}$$

*and notice that $\omega_2(\mathrm{U}) = \omega_2(\mathrm{V})$. We leave it as an exercise for the reader to verify that the final tableau depicted in Figure 2 has reading word*

$$\begin{pmatrix} r(4) & r(1) & r(1) & m{\downarrow}(2,3) & m{\downarrow}(2,3) \\ 1 & 2 & 1 & 2 & 3 \end{pmatrix}.$$

**Theorem 10.** *For $\mathsf{S} \in \mathrm{SStd}_s^0(\nu \setminus \lambda, \mu)$ we say that its reverse reading word $\omega(\mathsf{S})$ is a lattice permutation if $\omega_2(\mathsf{S})$ is a string composed of positive integers, in which every prefix contains at least as many positive integers $i$ as integers $i+1$ for $i \geq 1$. We define $\mathrm{Latt}_s^0(\nu \setminus \lambda, \mu)$ to be the set of all $\mathsf{S} \in \mathrm{SStd}_s^0(\nu \setminus \lambda, \mu)$ such that $\omega(\mathsf{S})$ is a lattice permutation. For any co-Pieri triple $(\lambda, \nu, s)$ and any $\mu \vdash s$ we have*

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\nu \setminus \lambda)) = |\mathrm{Latt}_s^0(\nu \setminus \lambda, \mu)|.$$

**Example 11.** *For example, we have that*

$$\overline{g}((2,1),(3,3,2),(2,2,1)) = 1 = \overline{g}((4),(4),(2,2,1))$$

*and that the corresponding homomorphisms are constructed in Examples 6 and 7. That these semistandard tableaux satisfy the lattice permutation property is checked in Example 9. Verifying that these are the only semistandard tableaux satisfying the lattice permutation property is left as an exercise for the reader. Similarly, one can check that $\overline{g}((7,5,1^2),(6,3,3),(2,2,1)) = 1$.*

**Remark 12.** *The (non-stable) Kronecker coefficients are also indexed by partitions. As we increase the size of the first row of each of the indexing partitions of the Kronecker coefficients, we obtain a weakly increasing sequence of coefficients; the limiting values of these sequences are the stable Kronecker coefficients which have been the focus of this paper. The non-stable Kronecker coefficients labelled by two 2-line partitions can be written as an alternating sum of at most 4* *stable Kronecker coefficients labelled by two 1-line partitions [1, Proposition 7.6]. (In fact, any non-stable Kronecker coefficient can be written as an alternating sum of stable Kronecker coefficients.) This should be compared with the existing descriptions of Kronecker coefficients labelled by two 2-line partitions [8, 9] which also involve alternating sums with at most 4 terms.*

*The advantages of our description are that* (1) *ours is the first description that generalises to other stable Kronecker coefficients (and in particular the first description of any family of Kronecker coefficients subsuming the Littlewood–Richardson coefficients) and* (2) *it counts explicit homomorphisms and therefore works on a higher structural level than all other descriptions of stable Kronecker coefficients since those first considered by Littlewood and Richardson [6].*

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