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A combinatorial formula for the Ehrhart h^*-vector of the hypersimplex

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Abstract. We give a combinatorial formula for the Ehrhart h^*-vector of the hypersimplex. In particular, we show that $h^*_d(\Delta_{k,n})$ is the number of hypersimplicial decorated ordered set partitions of type (k,n) with winding number d, thereby proving a conjecture of Nick Early. We do this by proving a more general conjecture of Nick Early on the Ehrhart h^*-vector of a generic cross-section of a hypercube.

Résumé. Nous donnons une formule combinatoire pour le vecteur Ehrhart h^*- de l'hypersimplex. En particulier, nous montrons que $h^*_d(\Delta_{k,n})$ est le nombre de partitions d'ensembles ordonnés décorés hypersimpliques de type (k,n) avec le numéro d'enroulement d, prouvant ainsi une conjecture de Nick Early. Nous faisons cela en prouvant une conjecture plus générale de Nick Early sur le vecteur Ehrhart h^*-d'une section transversale générique d'un hypercube.

1 Introduction

For two integers $0 < k < n$, the (k,n)-th hypersimplex is defined to be

$$\Delta_{k,n} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1,\, x_1 + \cdots + x_n = k\}.$$

It is an $(n-1)$-dimensional polytope inside $\mathbb{R}^n$ whose vertices are (0,1)-vectors with exactly k 1's. In particular it is an integral polytope. The hypersimplex can be found in several algebraic and geometric contexts, for example, as a moment polytope for the torus action on the Grassmannian, or as a weight polytope for the fundamental representation of GL_n.

For an n-dimensional integral polytope $\mathcal{P} \subset \mathbb{R}^N$, it's a standard fact from Ehrhart theory that the map $r \to |r\mathcal{P} \cap \mathbb{Z}^N|$ is a polynomial function in r of degree n. Now consider the Ehrhart series

$$\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r = \frac{h^*(t)}{(1-t)^{n+1}}.$$

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Knowing that $r \to |r\mathcal{P} \cap \mathbb{Z}^N|$ is a polynomial function in r of degree n, one can check that $h^*(t)$ is a polynomial of degree $\leq n$. Define h^*_d to be the coefficient of t^d in $h^*(t)$. The vector $(h^*_0, \cdots, h^*_n)$ is called the Ehrhart h^*-vector of $\mathcal{P}$ and $h^*(t)$ is called the h^*-polynomial of $\mathcal{P}$. A standard result from Ehrhart theory is that $\sum_{i=0}^{n} h^*_i$ equals the normalized volume of $\mathcal{P}$.

For a permutation $w \in S_n$, we say $i \in [n-1]$ is a *descent* of w if $w(i) > w(i+1)$ and define $des(w)$ to be the number of descents of w. The *Eulerian* number $A_{k,n-1}$ is the number of $w \in S_{n-1}$ with $des(w) = k-1$. A well-known fact about the hypersimplex $\Delta_{k,n}$ is that its normalized volume is $A_{k,n-1}$. So we have

$$\sum_{d=0}^{n-1} h^*_d(\Delta_{k,n}) = A_{k,n-1}.$$

In general, the entries of the h^*-vector of an integral polytope are nonnegative integers (see [6]). It has been an open problem for some time to give a combinatorial interpretation of $h^*(\Delta_{k,n})$. In [4], Nan Li gave a combinatorial interpretation of $h^*_d(\Delta'_{k,n})$, where $\Delta'_{k,n}$ is the hypersimplex with the lowest facet removed, using permutations $w \in S_{n-1}$ and their descents, excedances, and covers. In [2], Nick Early conjectured a combinatorial interpretation for $h^*_d(\Delta_{k,n})$ using hypersimplicial decorated ordered set partitions of type (k,n).

In [3], Katzman computed the Hilbert series of algebras of Veronese type, which gives a formula for the Ehrhart series of the hypersimplex $\Delta_{k,n}$ as a special case. The formula is

$$\frac{\sum_{i\geq 0} \binom{n}{i} \left(\sum_{j\geq 0} \binom{i}{j} (t-1)^j \left(\sum_{l\geq 0} \binom{n-j}{l(k-i)}_{k-i} t^l \right) \right)}{(1-t)^n} \tag{1.1}$$

where the notation $\binom{n}{b}_a$ means the coefficient of t^b in $(1+t+\cdots+t^{a-1})^n$. For example, when $a=2$, it becomes an ordinary binomial coefficient. The numerator of (1.1) is the h^*-polynomial of the hypersimplex, thus giving an explicit formula for its h^*-vector. However, it doesn't give a combinatorial or manifestly positive formula for the h^*-vector.

In this paper, we prove Nick Early's conjecture by relating it to (1.1). We now explain the conjecture. A *decorated ordered set partition* $((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ of type (k,n) consists of an ordered partition $(L_1, \cdots, L_m)$ of $\{1, 2, ..., n\}$ and an m-tuple $(l_1, \cdots, l_m) \in \mathbb{Z}^m$ such that $l_1 + \cdots + l_m = k$ and $l_i \geq 1$. We call each L_i a *block* and we place them on a circle in a clockwise fashion then think of l_i as the clockwise distance between adjacent blocks L_i and L_{i+1} (indices are considered modulo m). So the total length of the circle is $l_1 + \cdots + l_m = k$. We usually regard decorated ordered set partitions up to cyclic rotation of blocks (together with corresponding l). For example, $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$ is same as $(\{3,5\}_3, \{4,6\}_1, \{1,2,7\}_2)$. A decorated ordered set partition is called *hypersimplicial* if it satisfies $1 \leq l_i \leq |L_i| - 1$ for all i. For the motivation and more background on decorated ordered set partitions, see [1].

Example 1.1. Consider a decorated ordered set partition $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$ of type (6,7) (see Figure 1). This is not hypersimplicial as $3 > |\{3,5\}| - 1$.

By inserting empty spots, we can encode the distance information. For example, the (clockwise) distance between $\{1,2,7\}$ and $\{3,5\}$ is 2 so we insert one empty spot on the circle between those blocks. The distance between $\{3,5\}$ and $\{4,6\}$ is 3 so we insert two empty spots. We obtain the figure on the right as a result. Including empty spots, there will be $k = 6$ spots total.

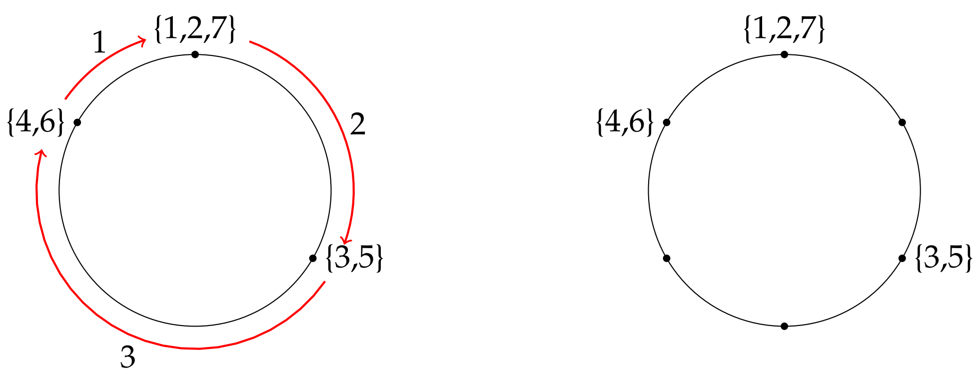

Figure 1: The figure on the left is the picture associated to the decorated ordered set partition $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$. The figure on the right is the picture obtained after inserting empty spots.

Given a decorated ordered set partition, we define the *winding vector* and the *winding number*. To define the winding vector, let w_i be the distance of the path starting from the block containing i to the block containing $(i+1)$ moving clockwise (i and $(i+1)$ are considered modulo n). If i and $(i+1)$ are in the same block then $w_i = 0$. In Figure 1, the winding vector is $w = (0,2,3,3,3,1,0)$.

The total length of the path is $(w_1 + \cdots + w_n)$, which should be a multiple of k as we started from 1 and came back to 1 moving clockwise. If $(w_1 + \cdots + w_n) = kd$, then we define the winding number to be d. In Figure 1, the winding number is 2.

It is known that hypersimplicial decorated ordered set partitions of type (k,n) are in bijection with $w \in S_{n-1}$ such that $des(w) = k - 1$ (see [5]).

Conjecture 1.2 ([2], Conjecture 1)**.** The number of hypersimplicial decorated ordered set partitions of type (k,n) with winding number d is $h^*_d(\Delta_{k,n})$.

Next we want to state a more general version of Conjecture 1.2 for a generic cross section of a hypercube.

Definition 1.3. For positive integers r, k, n, the generic cross section of a hypercube is

$$I^n_{r,k} = \left\{ (x_1, \cdots, x_n) \in [0, r]^n \mid \sum_{i=1}^{n} x_i = k \right\}.$$

When $r = 1$, $I^n_{1,k}$ is the hypersimplex $\Delta_{k,n}$.

Definition 1.4. A decorated ordered set partition $P = ((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ is *r-hypersimplicial* if $1 \le l_i \le r|L_i| - 1$ for all i.

Note that the notions of hypersimplicial and 1-hypersimplicial are equivalent. The decorated ordered set partition $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$ in Example 1.1 is not hypersimplicial, but it is r-hypersimplicial for $r \ge 2$.

Conjecture 1.5 ([2], Conjecture 6)**.** The number of r-hypersimplicial decorated ordered set partitions of type (k, n) with winding number d is $h^*_d(I^n_{r,k})$.

2 Proof of Conjecture 1.5

2.1 A simplification of Katzman's formula

Again using the formula for Hilbert series of algebras of Veronese type (see [3]), the Ehrhart series of $I^n_{r,k}$ is

$$\frac{\sum_{i\ge 0} (-1)^i \binom{n}{i} \left(\sum_{j \ge 0} \binom{i}{j} (t-1)^j \left(\sum_{l \ge 0} \binom{n-j}{l(k-ri)}_{k-ri} t^l \right) \right)}{(1-t)^n}. \tag{2.1}$$

Now we simplify (2.1) to get a simple description for the h^*-vector of $I^n_{r,k}$.

Lemma 2.1. *For positive integers n, m, and a, $\binom{n}{m}_a - \binom{n}{m-1}_a = \binom{n-1}{m}_a - \binom{n-1}{m-a}_a$.*

Proposition 2.2. *For positive integers s and a,*

$$\sum_{j \ge 0} \binom{s}{j} (t-1)^j \left(\sum_{l \ge 0} \binom{n-j}{la}_a t^l \right) = \sum_{l \ge 0} \binom{n}{la - s}_a t^l.$$

Using Proposition 2.2, (2.1) becomes

$$\frac{\sum_{i \ge 0} (-1)^i \binom{n}{i} \sum_{l \ge 0} \binom{n}{l(k-ri)-i}_{k-ri} t^l}{(1-t)^n}.$$

Thus we have

$$h^*_d(I^n_{r,k}) = \sum_{i\geq 0}(-1)^i \binom{n}{i}\binom{n}{(k-ri)d-i}_{k-ri}. \tag{2.2}$$

In Section 2.2, we will prove Conjecture 1.5 which contains Conjecture 1.2 as a special case when $r = 1$. Since we have an explicit formula for $h^*_d(I^n_{r,k})$, our strategy is to count the number of r-hypersimplicial decorated ordered set partitions of type (k, n) with winding number d and compare the formulas.

2.2 Enumeration of r-hypersimplicial decorated ordered set partitions with a fixed winding number

We start with an elementary lemma, skipping the proof.

Lemma 2.3. *The $\mathbb{Z}/n\mathbb{Z}$ action on $\{1, 2, \cdots, n\}$ by cyclic shift does not change the winding number of decorated ordered set partitions.*

For example, decorated ordered set partitions $(\{1, 2, 7\}_2, \{3, 5\}_3, \{4, 6\}_1)$ and $(\{2, 3, 1\}_2, \{4, 6\}_3, \{5, 7\}_1)$ have same the winding number.

Next we will show that a winding vector determines a decorated ordered set partition. We observed that when the winding number is d, then $w_1 + \cdots + w_n = kd$. And $0 \leq w_i \leq k-1$ since the total length of the circle is k ($w_i = k$ would mean that i and $(i+1)$ are in a same block but in that case $w_i = 0$). It turns out that these are the only restrictions for winding vectors.

Proposition 2.4. *Decorated ordered set partitions of type (k, n) with winding number d are in bijection with elements of $\{(w_1, \cdots, w_n) \in \mathbb{Z}^n \mid 0 \leq w_i \leq k-1,\ w_1 + \cdots + w_n = kd\}$.*

Proof. It is enough to construct a decorated ordered set partition of type (k, n) with winding number d from a winding vector satisfying the above conditions. First, draw k spots on the circle in clockwise order and put 1 in one spot. Having put i in some spot, move clockwise w_i spots and put $i+1$ in that spot. After placing all elements, nonempty spots become blocks and the clockwise distance from L_i and L_{i+1} is l_i. □

From Proposition 2.4, we know that the number of decorated ordered set partitions of type (k, n) with winding number d is $|\{(w_1, \cdots, w_n) \in \mathbb{Z}^n \mid 0 \leq w_i \leq k-1, w_1 + \cdots + w_n = kd\}|$. A simple combinatorial argument shows this number is the same as the coefficient of t^{kd} in $(1 + \cdots + t^{k-1})^n$, which is $\binom{n}{kd}_k$. So the number of decorated ordered set partitions of type (k, n) with winding number d is $\binom{n}{kd}_k$.

Recall that we are interested in the number of r-hypersimplicial decorated ordered set partitions of type (k, n) with winding number d. Throughout this section, when we say decorated ordered set partition, **we always assume it is of type (k, n) with winding number d.**

Definition 2.5. For a decorated ordered set partition $P = \{(L_1)_{l_1}, (L_2)_{l_2}, ..., (L_m)_{l_m}\}$, a block L_i is *r-bad* if $l_i \geq r|L_i|$. Let $I_r(P) = \{L_i \mid L_i \text{ is } r\text{-bad}\}$.

For example, $I_1((\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)) = \{\{3,5\}\}$. Recall that r-hypersimplicial decorated ordered set partitions satisfy $1 \leq l_i \leq r|L_i| - 1$ for all blocks. So a decorated ordered set partition is r-hypersimplicial if and only if $I_r(P)$ is empty.

Definition 2.6. For a set T, define $UP(T)$ to be a set of all (unordered) partitions of T. For example, $\{\{1,2,4\}, \{3\}, \{5\}\} \in UP(\{1,2,3,4,5\})$.

Definition 2.7. For $T \subseteq \{1,2,\cdots,n\}$ and $S \in UP(T)$, define
$K_r(S) = \{P\text{: decorated ordered set partition such that } S \subseteq I_r(P)\}$.

In other words $K_r(S)$ is the set of all decorated ordered set partitions having elements of S as r-bad blocks. For example, when $S = \phi$, $K_r(\phi)$ is a set of all decorated ordered set partitions.

Definition 2.8. For $T \subseteq \{1,2,\cdots,n\}$, let $H_r(T) = \sum\limits_{S \in UP(T)} (-1)^{|S|} |K_r(S)|$.

For example, when $T = \{1,2,3\}$,

$$\begin{aligned} H_r(T) = &-|K_r(\{\{1,2,3\}\})| + |K_r(\{\{1,2\},\{3\}\})| + |K_r(\{\{2,3\},\{1\}\})| \\ &+|K_r(\{\{1,3\},\{2\}\})| - |K_r(\{\{1\},\{2\},\{3\}\})|. \end{aligned}$$

Proposition 2.9. *The number of r-hypersimplicial decorated ordered set partitions is*

$$\sum_{T \subseteq \{1,2,\ldots,n\}} H_r(T).$$

Proof. It is enough to compute $\sum\limits_{T \subseteq \{1,2,\cdots,n\}} \left(\sum\limits_{S \in UP(T)} (-1)^{|S|} |K_r(S)| \right)$, by the definition of $H_r(T)$. If decorated ordered set partition P has empty $I_r(P)$ then it will be counted once when $S = \phi$. If $I_r(P)$ is non empty, say $|I_r(P)| = m$. Then P will be counted $\binom{m}{i}$ times with weight $(-1)^i$ as S ranges over all i-element subsets of $I_r(P)$. So the total weight is $\sum\limits_{i=0}^{m} (-1)^i \binom{m}{i} = 0$. So the above sum counts P such that $I_r(P)$ is empty, which means r-hypersimplicial. □

When $S \in UP(\{1,2,\cdots,n\})$, elements of $K_r(S)$ are decorated ordered set partitions $P = ((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ whose blocks are all r-bad, which means $l_i \geq r|L_i|$ for all i. Summing inequalities for all i gives $\sum l_i \geq r \sum |L_i|$ that implies $k \geq rn$ which is impossible as $k < n$. Thus $K_r(S)$ is an empty set so $H_r(\{1,2,\cdots,n\}) = 0$. So we will only consider when T is a proper subset of $\{1,2,\cdots,n\}$. By Lemma 2.3, $H_r(T)$ is invariant under cyclic shifts of $\{1,2,\cdots,n\}$. We may assume that $n \notin T$.

Definition 2.10. For $T \subset \{1, 2, \cdots, n\}$, a *T-singlet block* is a block with only one element t and $t \in T$. A sequence of consecutive T-singlet blocks $(L_i, \cdots, L_{i+j})$ in a decorated ordered set partition P (indices are considered modulo number of blocks in P) is *r-packed* if $l_i = \cdots = l_{i+j-1} = r$ and $l_{i+j} \geq r$. An r-packed sequence is *increasing r-packed* if elements in $(L_i, \cdots, L_{i+j})$ are in increasing order. Such a sequence is *maximal* if it is not a subsequence of another increasing r-packed sequence.

The increasing r-packed condition highly depends on T since it only applies to consecutive T-singlet blocks. Note that T-singlet blocks in r-packed sequence are all r-bad. It is the most concentrated arrangement that makes these blocks all r-bad. We allow increasing r-packed sequence of length 1 by convention.

Lemma 2.11. *Let $S = \{M_1, M_2, \cdots, M_j\} \in UP(T)$, where $T = \{t_1 < t_2 < \cdots < t_m\}$ and $n \notin T$. Enumerate the elements of M_i in increasing order, so $M_i = \{t_{i_1} < t_{i_2} < \cdots < t_{i_w}\}$. Then elements of $K_r(S)$ are in bijection with elements of $K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})$ having increasing r-packed sequence $(\{t_{i_1}\}, \{t_2\}, \cdots, \{t_{i_w}\})$ for all i.*

Proof. Given a decorated ordered set partition $P \in K_r(S)$, P has a block $(M_i)_l$ which is r-bad. So $l \geq r|M_i| = rw$. Change $(M_i)_l$ to $\{t_{i_1}\}_r$,$\{t_{i_2}\}_r$,...,$\{t_{i_w}\}_{l-r(w-1)}$. Since $l - r(w - 1) \geq r$, the sequence $(\{t_{i_1}\}, \{t_{i_2}\}, \cdots, \{t_{i_w}\})$ will be increasing r-packed. This process does not change the winding number and new T-singlet blocks are all r-bad. Repeating this process for all i we get the desired correspondence. □

Example 2.12. See Figure 2. The figure on the left is a decorated ordered partition $(\{1, 2, 4\}_6, \{5, 8, 9, 10, 13\}_1, \{6, 7\}_4, \{11, 12\}_1)$. When $T = \{1, 2, 4, 6, 7\}$ and $r = 2$, the figure on the left has r-bad blocks $\{1, 2, 4\}$ and $\{6, 7\}$, so belongs to $K_r(\{\{1, 2, 4\}, \{6, 7, \}\})$. Under the correspondence stated in Lemma 2.11, the decorated ordered set partition above goes to $(\{1\}_2, \{2\}_2, \{4\}_2, \{5, 8, 9, 10, 13\}_1, \{6, \}_2, \{7\}_2\{11, 12\}_1)$, a decorated ordered set partition for the figure on the right. The winding number does not change.

Remark 2.13. The condition $n \notin T$ is essential for Lemma 2.11. Without this condition, the correspondence changes the winding number as shown in Figure 3. The winding number on the left figure is 1 but the winding number on the right is 2. We spread elements in blocks in increasing order but since there is a cyclic symmetry, "increasing" might not be meaningful if $n \in T$.

Now fix $T = \{t_1 < t_2 < \cdots < t_m\} \subseteq \{1, 2, \cdots, n\}$ such that $n \notin T$. For $S \in UP(T)$, the correspondence in Lemma 2.11 gives an embedding

$$i_S : K_r(S) \to K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\}).$$

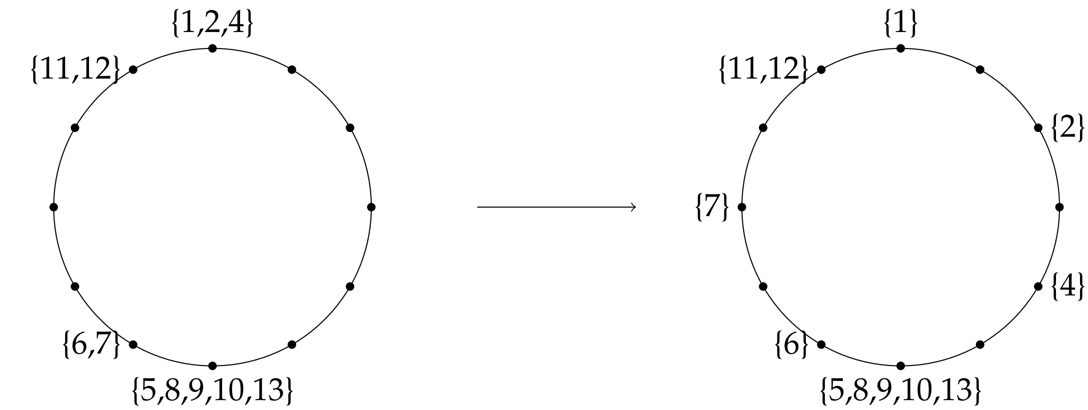

Figure 2: Correspondence in Lemma 2.11 for $T = \{1, 2, 4, 6, 7\}$ and $r = 2$.

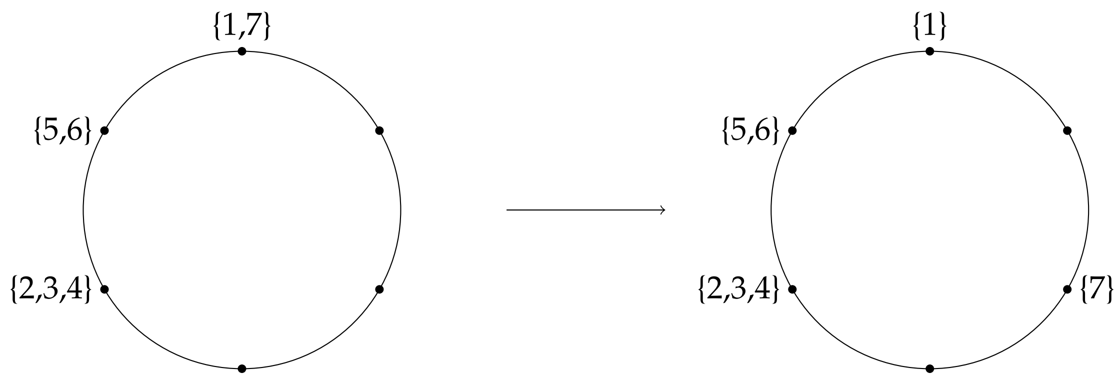

Figure 3: Correspondence in Lemma 2.11 for $T = \{1, 7\}$ and $r = 2$.

Let $\chi_S : K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\}) \to \{0, 1\}$ to be the characteristic function of $i_S(K_r(S))$. In other words, $\chi_S(P) = 0$ if $P \notin i_S(K_r(S))$ and $\chi_S(P) = 1$ if $P \in i_S(K_r(S))$. Then

$$\begin{aligned} H_r(T) &= \sum_{S\in UP(T)} (-1)^{|S|}|K_r(S)| = \sum_{S\in UP(T)} (-1)^{|S|}|i_S(K_r(S))| \qquad (2.3) \\ &= \sum_{S\in UP(T)} (-1)^{|S|} \left(\sum_{P\in K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})} \chi_S(P) \right) \\ &= \sum_{P\in K_r(\{\{t_1\},\{t_2\},\cdots,\{t_m\}\})} \left(\sum_{S\in UP(T)} (-1)^{|S|}\chi_S(P) \right). \end{aligned}$$

Proposition 2.14. *If P does not have increasing r-packed sequence of length greater than 1, then* $\sum_{S\in UP(T)} -(1)^{|S|}\chi_S(P) = (-1)^{|T|}$. *Otherwise it is zero.*

Proof. For $P \in K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})$, define $\hat{S}(P)$ to be unordered partition of T by putting t_i and t_j in same part if they belong to same increasing r-packed sequence (this will partition T by maximal increasing r-packed sequences of P). If $\chi_S(P) = 1$, then S should be finer partition than $\hat{S}(P)$. When P has no increasing r-packed sequence of length greater than 1, $\hat{S}(P) = \{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\}$, the finest unordered partition of T. So $\chi_S(P) = 1$ only when $S = \hat{S}(P)$ thus $\sum_{S \in UP(T)} -(1)^{|S|}\chi_S(P) = (-1)^{|T|}$. Now assume there is $M \in \hat{S}(P)$ such that $|M| = a \geq 2$. To split M into b parts such that resulting finer partition S still satisfies $\chi_S(P) = 1$, we should choose $(b-1)$ spots among $(a-1)$ spaces between adjacent elements of M and put bars in those spots to split M. So there are total $\binom{a-1}{b-1}$ ways to do that. Then we have

$$\sum_{S \in UP(T)} -(1)^{|S|}\chi_S(P) = \prod_{M \in \hat{S}(P), |M| \geq 2} \left(\sum_{b=1}^{|M|} (-1)^b \binom{|M|-1}{b-1} \right) \prod_{M \in \hat{S}(P), |M|=1} (-1).$$

Since $\sum_{b=1}^{|M|} (-1)^b \binom{|M|-1}{b-1} = 0$, $\sum_{S \in UP(T)} -(1)^{|S|}\chi_S(P) = 0$ whenever P has increasing r-packed sequence of length greater than 1, that is, $\hat{S}(P)$ has part with more than one element. $\square$

Let $\hat{K}_r(T)$ be a subset of $K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})$ consisting of decorated ordered set partition without increasing r-packed sequence of length greater than 1. By Proposition 2.14 and (2.3), we have $H_r(T) = (-1)^{|T|}|\hat{K}_r(T)|$. We will count the number of elements in $\hat{K}_r(T)$ by defining the second winding vector for each element. The second winding vector is a modified version of winding vector that we previously defined.

Assume we are given $P \in K_r(\{\{t_1\}, \{t_2\}, \cdots, \{t_m\}\})$. There are k spots total on the circle including empty spots that are recording distances and T-singlet blocks $\{t_1\}, \{t_2\}, \cdots, \{t_m\}$ are r-bad blocks so for each $\{t_i\}$, there will be at least $(r-1)$ empty spots after $\{t_i\}$ as the distance to the next block is at least r. Color these r spots, that is, the spot occupied by $\{t_i\}$ with $(r-1)$ empty spots after that **red**. Doing this for all i, $r|T| = rm$ spots will be colored red. And color the remaining $(k-rm)$ spots **blue**.

Define *second winding vector* $v = (v_1, v_2, \cdots, v_n)$, by setting v_i to be the number of **blue spots** passed while moving from i to $(i+1)$ in clockwise fashion. Do not include the starting point but include the arriving point (if it's blue) and when the starting point and the arriving point are in same block (spot), set $v_i = 0$. Since the winding number is d, the whole path winds around the circle d times. So we have $v_1 + \cdots + v_n = (k-rm)d$.

If $i \notin T$, we are starting from the blue spot so v_i can range from 0 to $(k-rm-1)$. However when $i \in T$, we claim v_i cannot be zero. If $v_i = 0$, then path from i to $i+1$ should not include any blue spots. So the path will be of the form $\{i\}, \phi, \cdots, \phi, \{a_1\}, \phi, \cdots, \phi, \cdots, \{a_q\}, \phi, \cdots, \phi, \{i+1\}$ where ϕ means an empty spot. The sequence $(\{i\}, \{a_1\}, \cdots, \{a_q\}, \{i+1\})$ is r-packed, but P does not have an increasing r-packed sequence of length greater than 1, which is a contradiction. It is possible to have $v_i = k-rm$ since i is not in the blue spot. We conclude $1 \leq v_i \leq k-rm$.

Example 2.15. Figure 4 shows how to read off the second winding vector. We are given $T = \{1, 2, 9\}$, and $r = 2$. The upper left figure is a picture for the decorated ordered set partition $(\{2\}_2, \{1\}_2, \{5, 6\}_1, \{7, 8\}_1, \{9\}_3, \{11, 12, 13\}_1, \{10, 14\}_1, \{3, 4\}_1)$. Note that the sequence $(\{2\}, \{1\})$ is r-pacekd but not increasing r-packed. So P has no increasing r-packed sequence of length greater than 1. After coloring spots with the rule above we get the upper right figure. There will be $r|T| = 6$ red spots and $k - r|T| = 6$ blue spots. To get v_1, wind from 1 to 2 clockwise as shown in the lower figure, and count the number of blue spots passed. Here $v_1 = 6$. Continuing this process we have the second winding vector $v = (6, 6, 0, 1, 0, 1, 0, 0, 3, 5, 0, 0, 1, 1)$.

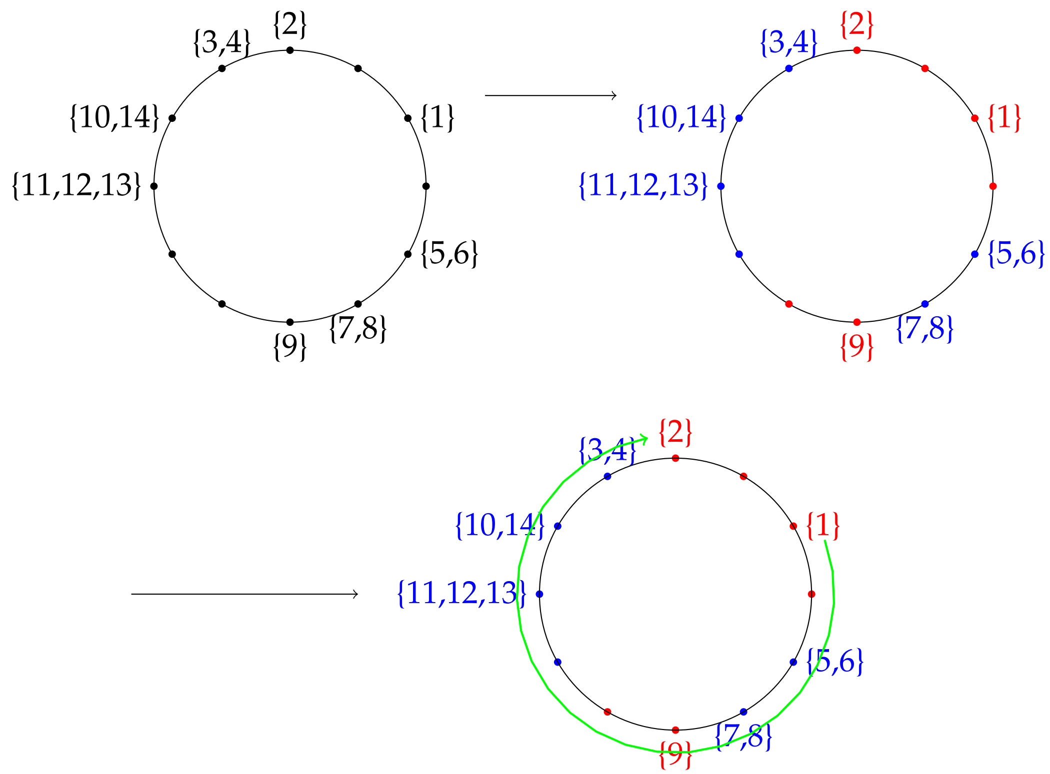

Figure 4: Reading off the second winding vector.

We saw that a second winding vector $v = (v_1, v_2, \cdots, v_n)$ satisfies $v_1 + \cdots + v_n = (k - rm)d$, $0 \leq v_i \leq k - rm - 1$ if $i \notin T$, and $1 \leq v_i \leq k - rm$ if $i \in T$.

It turns out these are the only restrictions for the second winding vectors of the elements of $\hat{K}_r(T)$.

Proposition 2.16. *Elements of* $\hat{K}_r(T)$ *are in bijection with elements of*

$$\{(v_1, v_2, \cdots, v_n) \in \mathbb{Z}^n \mid 0 \leq v_i \leq k - rm - 1 \text{ if } i \notin T, 1 \leq v_i \leq k - rm \text{ if } i \in T, \\ v_1 + \cdots + v_n = (k - rm)d\}.$$

Proof. The forward direction is done by the second winding vector. For the reverse direction, we should recover the decorated ordered set partition (in $\hat{K}_r(T)$) whose second winding vector is the specified vector. First draw $(k - rm)$ spots on the circle (recall $|T| = m$) and put 1 in one spot. Having put i in some spot, move clockwise w_i spots and put $i + 1$ in that spot. After placing every element, let's denote the resulting decorated ordered set partition with P. We construct $\tilde{P} \in \hat{K}_r(T)$ as follows. For each block B of P with $B \cap T \neq \phi$, let $B \cap T = \{i_1 < \cdots < i_s\}$. We replace B with $B \backslash T$ and then add rs spots immediately after $B \backslash T$ as follows: first a T-singlet block $\{i_s\}$ then $(r-1)$ empty spots then T-singlet block $\{i_{s-1}\}$ then $(r-1)$ empty spots $\cdots$ T-singlet block $\{i_1\}$ then $(r-1)$ empty spots. □

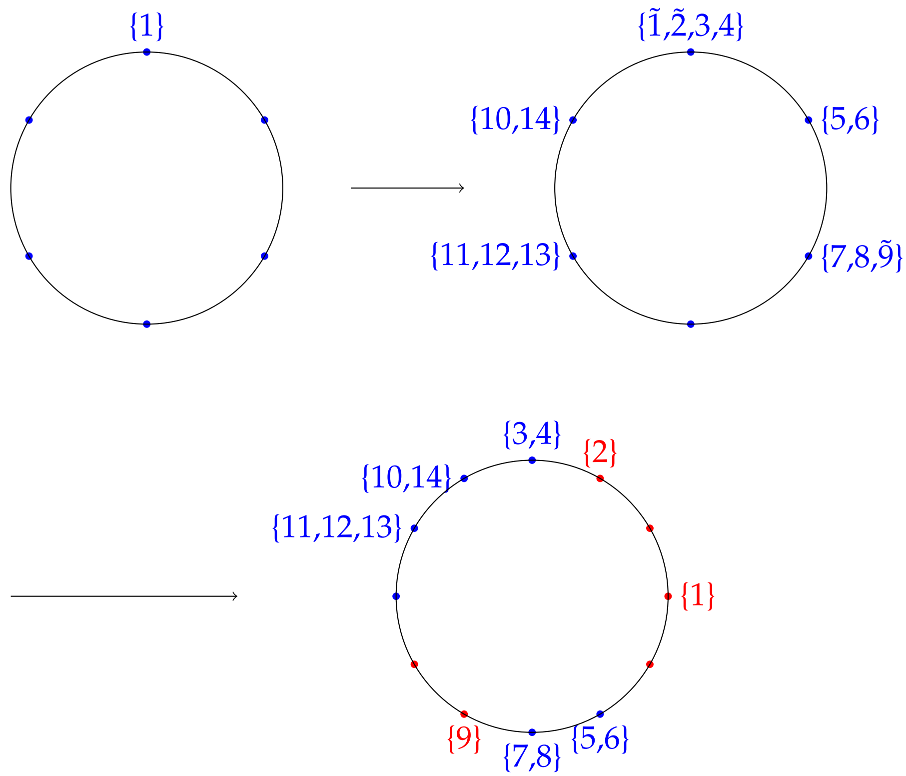

Figure 5: Constructing the decorated ordered set partition associated to the second winding vector $v = (6, 6, 0, 1, 0, 1, 0, 0, 3, 5, 0, 0, 1, 1)$.

Example 2.17. Figure 5 shows how to recover a decorated ordered set partition from a second winding vector as stated in Proposition 2.19. We are given $T = \{1, 2, 9\}$, $r = 2$, and the second winding vector $v = (6, 6, 0, 1, 0, 1, 0, 0, 3, 5, 0, 0, 1, 1)$. In the upper left figure, there are $6 = k - r|T|$ spots ($k = 12$) on the circle and 1 is in one spot. Then put elements according to the second winding vector. The upper right figure shows this. The elements in T are denoted with a tilde. Consider the first block $\{\tilde{1}, \tilde{2}, 3, 4\}$. The numbers 3 and 4 will form a block and 1 and 2 will spread to the right into the space between blocks $\{\tilde{1}, \tilde{2}, 3, 4\}$ and $\{5, 6\}$, making four new red spots. The same thing happens for the block $\{7, 8, \tilde{9}\}$, making two new red spots. The lower figure is the picture for the resulting decorated ordered set partition in $\hat{K}_r(T)$. We recovered Example 2.15.

For a second winding vector $v = (v_1, \cdots, v_n)$, let $v' = (v'_1, \cdots, v'_n)$ be a vector such that $v'_i = v_i$ if $i \notin T$, and $v'_i = v_i - 1$ if $i \in T$. By the property of a second winding vector, we have $0 \leq v'_i \leq k - rm - 1$ and $v'_1 + \cdots + v'_n = (k - rm)d - |T| = (k - rm)d - m$. So the number of such v' is $\binom{n}{(k-rm)d-m}_{k-rm}$ which gives

$$H(T) = (-1)^{|T|} |\hat{K}_r(T)| = (-1)^m \binom{n}{(k-rm)d-m}_{k-rm}. \tag{2.4}$$

Proof of Conjecture 1.5. By Proposition 2.9, and (2.4), the number of r-hypersimplicial decorated ordered set partitions (of type (k, n) with winding number d) is

$$\sum_{T \subseteq \{1,2,\ldots,n\}} H_r(T) = \sum_{m \geq 0} (-1)^m \binom{n}{m} \binom{n}{(k-rm)d-m}_{k-rm}.$$

Now compare with the formula (2.2). □

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