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Deformations of Coxeter permutahedra and Coxeter submodular functions

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Abstract. We study the cone of deformations of a Coxeter permutahedron. This family contains polyhedral models for the Coxeter-theoretic analogs of compositions, graphs, matroids, posets, and associahedra. Our description extends the known correspondence between generalized permutahedra and submodular functions to any finite reflection group.

Resumen. Estudiamos el cono de deformaciones de un permutaedro de Coxeter. Esta familia contiene modelos poliedrales para las composiciones, grafos, matroides, posets, y asociaedros de tipo Coxeter. Nuestra descripción extiende la correspondencia entre permutaedros generalizados y funciones submodulares a cualquier grupo de reflexiones finito.

Keywords: Permutahedron, generalized permutahedron, polymatroid, Coxeter group, root system, Coxeter complex, polytope deformation, submodular function, nef cone, Mori cone.

1 Introduction

The permutahedron Π_n is the convex hull of the $n!$ permutations of $\{1, \ldots, n\}$ in $\mathbb{R}^n$. This polytopal model for the symmetric group S_n appears in and informs numerous combinatorial, algebraic, and geometric settings. There are two natural generalizations, which we now discuss.

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1. Reflection groups: Instead of the group S_n, we may consider any finite reflection group W with corresponding root system $\Phi \subset V$. This group can similarly be modeled by the Φ-permutahedron, which is the convex hull of the W-orbit of a generic point in V. Most of the geometric and representation theoretic properties of the permutahedron extend to this setting.

2. Deformations: We may deform the polytope by moving its faces while preserving their directions. The resulting family of *generalized permutahedra* or *polymatroids* is special enough to be amenable to combinatorial analysis, and it is flexible enough to include useful geometric models of many combinatorial families of interest, such as partitions, compositions, graphs, matroids, and posets.

The goal of this paper is to initiate a theory of *deformations of* Φ*-permutahedra* or Φ-*polymatroids* generalizing these two directions simultaneously. This theory is motivated by the field of *Coxeter combinatorics*, which recognizes that many classical combinatorial constructions are intimately related to the symmetric group, and have natural generalizations to the setting of reflection groups. There are natural Coxeter-theoretic analogs of compositions, graphs, matroids, and posets, and we observe that they are all part of this geometric framework of *generalized* Φ*-permutahedra*.

A central result is that generalized permutahedra are in bijection with the functions $f : 2^{[n]} \to \mathbb{R}$ that satisfy the *submodular inequalities* $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. This means that the important field of submodular optimization is essentially a study of this family of polytopes. Our main result extends this to all finite reflection groups:

Theorem 1.1. *Let Φ be a root system and $\mathcal{W}$ be its set of weights. Generalized Φ-permutahedra are in bijection with the functions $h : \mathcal{W} \to \mathbb{R}$ that satisfy the* Φ-submodular inequalities*:*

For every element $w \in W$ of the Weyl group and every simple reflection s_i and corresponding fundamental weight λ_i,

$$h(w \cdot \lambda_i) + h(ws_i \cdot \lambda_i) \geq \sum_{j \neq i} -A_{ij}\, h(w \cdot \lambda_j) \tag{1.1}$$

where A is the Cartan matrix. Furthermore this is a minimal set of inequalities; there are

$$\sum_{i=1}^{d} \frac{|W|}{|W_{[d]-N(i)}|}$$

such inequalities, where $N(i)$ is the set of neighbors of i in the Dynkin diagram and $W_{[d]-N(i)}$ is the parabolic subgroup generated by the complement of $N(i)$.

2 Polytopes and their deformations

2.1 Polytopes and their support functions

Let U and V be two real vector space of finite dimension d in duality via a perfect bilinear form $\langle \cdot, \cdot \rangle : U \times V \longrightarrow \mathbb{R}$. A *polyhedron* $P \subset V$ is an intersection of finitely many half-spaces; it is a *polytope* if it is bounded. We will regard each vector $u \in U$ as a linear functional on V, which gives rise to the *u-maximal face* $P_u := \{v \in P : \langle u, v \rangle = \max_{x \in P} \langle u, x \rangle\}$ whenever $\max_{x \in P} \langle u, x \rangle$ is finite.

Let Σ_P be the *(outer) normal fan* in U. For each ℓ-codimensional face F of P, the normal fan Σ_P has a dual ℓ-dimensional face $\Sigma_P(F) = \{u \in U : P_u = F\}$.

A polytope P is *simple* if each vertex is in exactly d facets, or equivalently if every cone in Σ_P is *simplicial* in that its generating rays are linearly independent. Each cone in a fan Σ is called a *face*. Let $\Sigma(\ell)$ be the set of ℓ-dimensional cones of Σ. We call the elements of $\Sigma(d)$ *chambers*, the elements of $\Sigma(d-1)$ *walls*, and the elements of $\Sigma(1)$ *rays*. All fans we consider in this paper will be *projective*, i.e., normal fans Σ_P of polyhedra P.

Given a fan $\Sigma \subset U$, the space of *continuous piecewise linear functions on* Σ is

$$\mathrm{PL}(\Sigma) := \{f : |\Sigma| \to \mathbb{R} \mid f \text{ linear on each cone of } \Sigma \text{ and continuous}\}.$$

The *support function* of a polytope P is an element $h_P \in \mathrm{PL}(\Sigma_P)$ defined by

$$h_P(u) := \max_{v \in P} \langle u, v \rangle. \tag{2.1}$$

Notice that we can recover P from h_P by $P = \{v \in V : \langle u, v \rangle \leq h_P(u) \text{ for all } u \in |\Sigma_P|\}$, so a polyhedron and its support function uniquely determine each other. Also notice that the translation $P + v$ of a polytope P has support function $h_{P+v} = h_P + h_{\{v\}}$, where $h_{\{v\}}$ is the linear functional $\langle \cdot, v \rangle$. Therefore translating a polytope P is equivalent to adding a global linear functional to its support function h_P.

We say two polyhedra P, Q are *normally equivalent* (or *strongly combinatorially equivalent*) if $\Sigma_P = \Sigma_Q$. A fan Σ is a *coarsening* of another fan Σ', or Σ' is a *refinement* of Σ, if each cone of Σ is a union of cones in Σ'; we denote this by $\Sigma \preceq \Sigma'$.

Definition 2.1. *A polytope Q is a* deformation *of P if the normal fan Σ_Q is a coarsening of the normal fan Σ_P.*

2.2 Deformations of zonotopes

Let $\mathcal{A} = \{v_1, \cdots, v_m\} \subset V$ be a set of vectors and let $\mathcal{H} = \{H_1, \cdots, H_m\}$ be the corresponding hyperplane arrangement in U given by the hyperplanes $H_i = \{u \in U : \langle u, v_i \rangle = 0\}$ for $1 \leq i \leq m$.

Definition 2.2. *Let $\mathcal{A} = \{v_1, \cdots, v_m\} \subset V$. The* zonotope *of $\mathcal{A}$ is the Minkowski sum*

$$\mathcal{Z}(\mathcal{A}) := [0, v_1] + \cdots + [0, v_m].$$

Notice that the normal fan of the zonotope $\mathcal{Z}(\mathcal{A})$ is given by the faces of the arrangement $\mathcal{H}$. We can describe the deformations of $\mathcal{Z}(\mathcal{A})$ easily as follows:

Proposition 2.3. *Let $\mathcal{A}$ be a finite set of vectors in V. A polytope P is a deformation of the zonotope $\mathcal{Z}(\mathcal{A})$ if and only if all edges of P are parallel to vectors in $\mathcal{A}$.*

Proof. If P is a deformation of $\mathcal{Z}(\mathcal{A})$ then its normal fan Σ_P coarsens the arrangement $\mathcal{H}$. Every edge e is normal to a codimension 1 wall of Σ_P, which is part of a wall of $\mathcal{H}$, and hence of a hyperplane H_i for some i. Therefore e is parallel to v_i as desired.

Conversely, if every edge of P is parallel to a vector in $\mathcal{A}$, every wall of Σ_P is contained in a hyperplane H_i. We can refine Σ_P by extending each wall to the hyperplane that it spans. The result is a subarrangement of $\mathcal{H}$, which is further refined by $\mathcal{H}$. Thus P is a deformation of $\mathcal{Z}(\mathcal{A})$. □

2.3 Deformation cones

Let P be a polytope in V and $\Sigma = \Sigma_P$ be its normal fan. For each deformation Q of P, the normal fan Σ_Q coarsens Σ, and hence the support function h_Q defined in (2.1) is piecewise-linear on Σ. Thus, by identifying Q with its support function h_Q, we can define the following.

Definition/Theorem 2.4. *[2, Theorems 6.1.5–6.1.7]. Let P be a polytope in V and $\Sigma = \Sigma_P$ be its normal fan. The* deformation cone *of P (or of Σ) is*

$$Def(P) = Def(\Sigma) := \{h_Q \mid Q \text{ is a deformation of } P\} = \{h \in \mathrm{PL}(\Sigma) \mid h \text{ is convex}\}.$$

Remark 2.5. *For each ray $\rho \in \Sigma(1)$ let u_ρ be a vector in the direction of ρ. When Σ is a rational fan, we let u_ρ be the first lattice point on the ray ρ. Let $R = \{u_\rho : \rho \in \Sigma(1)\}$. A piecewise linear function on Σ is determined by its values on the u_ρs, so we may regard it as a function $h : R \to \mathbb{R}$. When the fan Σ is simplicial, those values may be chosen arbitrarily, so $\mathrm{PL}(\Sigma)$ may be identified with $\mathbb{R}^R$.*

2.3.1 The wall crossing criterion

Definition 2.6. (Wall-crossing inequalities) *Let $\tau \in \Sigma(d-1)$ be a wall separating two chambers σ and σ'. Choose any $d-1$ linearly independent rays $\rho_1, \ldots, \rho_{d-1}$ of τ and any two rays ρ, ρ' of σ, σ', respectively, that are not in τ. Up to scaling, there is a unique linear dependence of the form*

$$c \cdot u_\rho + c' \cdot u_{\rho'} = \sum_{i=1}^{d-1} c_i \cdot u_{\rho_i} \tag{2.2}$$

with $c, c' > 0$. To the wall τ we associate the wall-crossing inequality

$$I_\tau^\Sigma(h) := c \cdot h(u_\rho) + c' \cdot h(u_{\rho'}) - \sum_{i=1}^{d-1} c_i \cdot h(u_{\rho_i}) \geq 0, \tag{2.3}$$

which a piecewise linear function $h \in PL(\Sigma)$ must satisfy in order to be convex.

When Σ is complete and simplicial, the element $I_\tau^\Sigma \in \mathrm{PL}(\Sigma)^\vee$ is well-defined up to positive scaling. Notice that $I_\tau^\Sigma(h) = 0$ if and only if h is represented by the same linear functional at both sides on τ, which happens if and only if τ is no longer a wall in the underlying fan of h.

Lemma 2.7. (Wall-Crossing Criterion) *[2, Theorems 6.1.5–6.1.7] Let Σ be a complete fan in U. A continuous piecewise linear function $h \in \mathrm{PL}(\Sigma)$ is a support function of a polytope Q with $\Sigma_Q \preceq \Sigma$ if and only if it satisfies the wall-crossing inequality* (2.3) *for each wall τ of Σ.*

Note that V embeds into $\mathrm{PL}(\Sigma)$ by $v \mapsto \langle v, \cdot \rangle$. The following is a rephrasing of [2, pp. 4.2.12, 6.3.19–22].

Proposition 2.8. *Let Σ be the normal fan of a polytope P. Say $h \sim h'$ for two functions $h, h' \in \mathrm{PL}(\Sigma)$ if $h - h'$ is a globally linear function on U, or equivalently, if $h - h' \in V \subset \mathrm{PL}(\Sigma)$. Then:*

- *Def Cone: Def(Σ) is the polyhedral cone parametrizing deformations of P. It is full dimensional in $\mathrm{PL}(\Sigma)$. Its linearity space is the d-dimensional space $V \subset PL(\Sigma)$ of global linear functions on $|\Sigma| = U$, corresponding to the d-dimensional space of translations of P.*
- *Nef Cone: Nef(Σ) $:= \mathrm{Def}(\Sigma)/V = \mathrm{Def}(\Sigma)/\sim$ is the quotient of Def(Σ) by its linearity space V of globally linear functions. It is a strongly convex cone in $\mathrm{PL}(\Sigma)/V$ parametrizing the deformations of P up to translation.*

Remark 2.9. *When Σ is a rational fan, Nef(Σ) is the* Nef (numerically effective) cone *of the toric variety associated to Σ [2, Chapter 6.3]. The* Mori cone *$\overline{NE}(\Sigma)$ is the cone polar to $Def(\Sigma)$. More precisely,*

$$\overline{NE}(\Sigma) := \mathrm{Cone}\left(I_\tau^\Sigma \mid \tau \in \Sigma(d-1)\right) \subset \mathrm{PL}(\Sigma)^\vee,$$

2.3.2 Batyrev's criterion

When Σ is simplicial, Batyrev's criterion ([2, Lemma 6.4.9]) offers another useful test for convexity, and hence an alternative description of the deformation cone $\mathrm{Def}(\Sigma) = \mathrm{Def}(P)$ when $\Sigma = \Sigma_P$. To state it, we need the following notion.

Definition 2.10. *Let Σ be a simplicial fan. A* primitive collection *F is a set of rays of Σ such that any proper subset $F' \subsetneq F$ forms a cone in Σ but F itself does not. In other words, the primitive collections of a simplicial fan correspond to the minimal non-faces of the associated simplicial complex.*

Lemma 2.11. (Batyrev's Criterion) *[2, Theorem 6.4.9] Let Σ be a complete simplicial fan. A piecewise linear function $h \in \mathrm{PL}(\Sigma)$ is in the deformation cone $\mathrm{Def}(\Sigma)$ (and hence the support function of a polytope) if and only if*

$$\sum_{\rho \in F} h(u_\rho) \geq h\left(\sum_{\rho \in F} u_\rho\right)$$

for any primitive collection F of rays of Σ.

The material in this section can be rephrased in terms of triangulations of point configurations (see [3, Section 5]). Deformation cones are instances of secondary cones for the collection of vectors $\{u_\rho : \rho \in \Sigma(1)\}$. The Wall-Crossing criterion Lemma 2.7 is called the *local folding condition* in [3, Theorem 2.3.20].

3 Reflection groups and Coxeter complexes

In this section we review the combinatorial aspects of finite reflection groups that we will need.

3.1 Root systems and Coxeter complexes

We will identify V with its own dual by means of a positive definite inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. Any vector $v \in V$ defines a linear automorphism s_v on V by reflecting across the hyperplane orthogonal to v; that is,

$$s_v(x) := x - \frac{2\langle x, v\rangle}{\langle v, v\rangle} v. \tag{3.1}$$

Definition 3.1. *A* root system *Φ is a finite set of vectors in an inner product real vector space V satisfying*

(R0) $span(\Phi) = V$,

(R1) for each root $\alpha \in \Phi$, the only scalar multiples of α that are roots are α and $-\alpha$, and

(R2) for each root $\alpha \in \Phi$ we have $s_\alpha(\Phi) = \Phi$.

Each root $\alpha \in \Phi$ gives rise to a hyperplane $H_\alpha = \{x \in V : \langle \alpha, x \rangle = 0\}$. This set of hyperplanes $\mathcal{H}_\Phi = \{H_\alpha : \alpha \in \Phi\}$ is called the *Coxeter arrangement*. The *Coxeter complex* is the associated fan Σ_Φ, which is simplicial. We will often use these two terms interchangeably, and drop the subscript Φ when the context is clear.

The combinatorial structure of the Coxeter complex Σ_Φ is closely related to the algebraic structure of the group W_Φ, as we now explain. Let us fix a chamber (maximal cone) of Σ_Φ to be the *fundamental domain* D; recall that it is simplicial. Then the *simple roots* $\Delta = \{\alpha_1, \cdots, \alpha_d\} \subset \Phi$ are the roots whose positive halfspaces minimally cut out the fundamental domain; that is, $D = \{x \in V : \langle \alpha_i, x \rangle \geq 0 \text{ for } 1 \leq i \leq d\}$. The simple roots form a basis for V, and we call $d = \dim V$ the *rank* of the root system Φ. The *positive roots* are those that are non-negative combinations of simple roots; we denote this set by $\Phi^+ \subset \Phi$. We have that $\Phi = \Phi^+ \sqcup (-\Phi^+)$. The *Cartan matrix* is the $d \times d$ integer matrix A whose entries are $A_{ij} := 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$ for $1 \leq i, j \leq d$.

3.2 Weyl groups, parabolic subgroups, and the Coxeter complex

Proposition 3.2. *Let Φ be a root system spanning V and let $W = W_\Phi$ be the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Phi$. Then W is a finite group, called the* Weyl group *of Φ.*

The action of W on V induces an action on the Coxeter complex Σ_Φ. This action behaves especially well on the top-dimensional faces:

Proposition 3.3. *The Weyl group W acts regularly on the set $\Sigma_\Phi(d)$ of chambers of the Coxeter arrangement; that is, for any two chambers σ and σ' there is a unique element $w \in W$ such that $w \cdot \sigma = \sigma'$. In particular, the chambers of the Coxeter arrangement are in bijection with W.*

The lower dimensional faces of Σ_Φ correspond to certain subgroups of W and their cosets. The *parabolic subgroups* of W are the subgroups $W_I := \langle s_\alpha : \alpha \in I \rangle \subset W$ for each $I \subseteq \Delta$. They are in bijection with the faces of the fundamental domain, where W_I is mapped to the face

$$C_I := \{x \in D : \langle x, \alpha \rangle = 0 \text{ for all } \alpha \in I, \langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta \backslash I\} \qquad \text{for } I \subseteq \Delta.$$

The *parabolic cosets* are the right cosets of parabolic subgroups.

Proposition 3.4. *The faces of the Coxeter complex are in bijection with the parabolic cosets of W, through the labeling $F \mapsto \{w : F \subseteq wD\}$. Under the action of W on the Coxeter complex Σ, the orbit of the face C_I (which is labeled W_I) is the set of faces labeled by the right cosets of W_I. Furthermore, for any v in the interior of C_I, the stabilizer of v under the action of W is the parabolic subgroup W_I.*

Two special cases, stated in the following corollaries, are especially important to us.

Corollary 3.5. *The walls of the Coxeter complex are labeled by the pairs* $\{w, ws_i\} = wW_{\{i\}}$ *for* $w \in W$ *and* $s_i \in S$*. The wall labeled* $\{w, ws_i\}$ *separates the chambers labeled* w *and* ws_i*.*

Definition/Proposition 3.6. *Let the* fundamental weights $(\lambda_1, \cdots, \lambda_d)$ *form the basis of* V *dual to the simple roots* $(\alpha_1, \cdots, \alpha_d)$*; that is,* $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$*. Let the set of* weights *be* $\mathcal{W} = W \cdot \{\lambda_1, \cdots, \lambda_d\}$*. Each weight can be expressed as* $w \cdot \lambda_i$ *for a unique* i *(although the* w *is not unique).*

Corollary 3.7. *The* d *rays of the fundamental domain are spanned by the fundamental weights* $\lambda_1, \ldots, \lambda_d$*, and the rays of the Coxeter complex are spanned by the weights. These correspondences are bijective.*

Theorem 3.8. *If* V' *is any subset of* V *then the subgroup* $W_{V'} \leq W$ *fixing* V' *pointwise is generated by the reflections* s_α *that it contains.*

4 Coxeter permutahedra and some important deformations

Throughout this section, let Φ be a root system and W be its Weyl group.

Definition/Proposition 4.1. *The* standard Coxeter permutahedron of type Φ *or* Φ*-permutahedron is the Minkowski sum of the roots of* Φ*; that is,* $\Pi_\Phi := \sum_{\alpha \in \Phi} [0, \alpha]$*.*

A generalized Φ-permutahedron *or* Φ-polymatroid *is a deformation of the* Φ*-permutahedron* Π_Φ*; that is, a polytope whose normal fan coarsens the Coxeter complex* Σ_Φ*.*

Proposition 4.2. *The following families of polytopes are generalized Coxeter permutahedra:*

1. *the* orbit polytopes $P_\Phi(x) = conv(W \cdot x)$ *for* $x \in V$*.,*
2. *the* Coxeter-graphic zonotopes $\sum_{\alpha \in \Psi} [0, \alpha]$ *for* $\Psi \subseteq \Phi$*,*
3. *the* Coxeter matroids *of Gelfand–Serganova [1],*
4. *the* Coxeter cones *of Reiner [5] and Stembridge [6], and*
5. *the* Coxeter associahedra *of Hohlweg-Lange-Thomas [4].*

These families of polyhedra model the Coxeter-theoretic analogs of compositions, graphs, matroids, posets, and clusters, respectively.

An orbit polytope $P_\Phi(x)$ can always be defined by a point x in the fundamental domain D, and its normal fan depends only on the minimal face of D containing it:

Proposition 4.3. *For* x *in the interior of* C_I*, the chambers of the normal fan of* $P_\Phi(x)$ *are in bijection with* W/W_I*. The chamber of* $\Sigma_{P_\Phi(x)}$ *corresponding to the coset* wW_I *is the union of the* $|W_I|$ *chambers of the Coxeter complex* Σ_Φ *labeled* ww_I *for* $w_I \in W_I$*.*

The following special case will be important below: When x is in the interior of $C_{\{i\}}$, the normal fan of $P_\Phi(x)$ is obtained by merging the chambers $w \cdot D$ and $s_i w \cdot D$ for each $w \in W$.

5 Deformation cone for Coxeter complexes

Our next goal is to describe the deformation cone of Coxeter permutahedra. Recall that a piecewise linear function on a fan is uniquely determined by its restriction to the rays of the fan. Since each ray of the Coxeter complex Σ_Φ contains a weight, and this correspondence is bijective, we may identify the space $\mathrm{PL}(\Sigma_\Phi)$ of piecewise-linear functions on Σ_Φ, with the space $\mathbb{R}^{\mathcal{W}}$ of functions from $\mathcal{W}$ to $\mathbb{R}$.

5.1 Φ-submodular functions

Definition 5.1. *A function $h : \mathcal{W} \to \mathbb{R}$ is* Φ-submodular *if the following equivalent conditions hold:*

- $h \in \mathrm{Def}(\Sigma_\Phi)$.
- *When regarded as a piecewise linear function in $PL(\Sigma_\Phi)$, the function h is convex.*
- *The polytope $\{v \in V : \langle \lambda, v\rangle \leq h(\lambda) \text{ for all } \lambda \in \mathcal{W}\}$ is a generalized Φ-permutahedron.*
- *The polytope $\{v \in V : \langle \lambda, v\rangle \leq h(\lambda) \text{ for all } \lambda \in \mathcal{W}\}$ has edges parallel to roots in Φ.*

We call $\mathrm{Def}(\Sigma_\Phi) \subset \mathbb{R}^{\mathcal{W}}$ *the* Φ-submodular cone.

We now describe the Φ-submodular. The Coxeter complex is simplicial which makes it easier to apply the results in Section 2.

Theorem 5.2. *A function $h : \mathcal{W} \to \mathbb{R}$ is* Φ-submodular *if and only if the following two equivalent sets of inequalities hold:*

1. (Local Φ-submodularity) *For every element $w \in W$ of the Weyl group and every simple reflection s_i and corresponding fundamental weight λ_i,*

$$h(w \cdot \lambda_i) + h(ws_i \cdot \lambda_i) \geq \sum_{j \neq i} -A_{ij}\, h(w \cdot \lambda_j) \tag{5.1}$$

   where A is the Cartan matrix.

2. (Global Φ-submodularity) *For any two weights $\lambda, \lambda' \in \mathcal{W}$*

$$h(\lambda) + h(\lambda') \geq h(\lambda + \lambda') \tag{5.2}$$

   where h is regarded as a piecewise-linear function on Σ_Φ.

Remark 5.3. *To interpret the global Φ-submodular inequalities (5.2) directly in terms of the function $h \in \mathbb{R}^{\mathcal{W}}$, we need to find the minimal cone C of Σ_Φ containing $\lambda + \lambda'$. If $\mathcal{W}_C$ is the set of weights in the cone C, we can write $\lambda + \lambda' = \sum_{w \in \mathcal{W}_C} c_w w$ for a unique choice of positive constants c_w, and (5.2) means that $h(\lambda) + h(\lambda') \geq \sum_{w \in \mathcal{W}_C} c_w h(w)$. In particular, (5.2) holds trivially when λ and λ' span a face of Σ_Φ.*

Proof of Theorem 5.2, Part 1. We know that the deformation cone $\mathrm{Def}(\Sigma_\Phi)$ is given by the wall crossing inequalities of Lemma 2.7. We first compute them for the walls of the fundamental domain D.

Let us apply Definition 2.6 to the wall $H_i = H_{\alpha_i}$ of D orthogonal to the simple root α_i, which separates the chambers D and $s_i \cdot D$. Notice that the only ray of D that is not on the wall H_i is the one spanned by the fundamental weight λ_i. Similarly, the only ray of $s_i D$ that is not on H_i is the one spanned by the weight $s_i \cdot \lambda_i$. Therefore we need to find the coefficients such that

$$c\lambda_i + c' s_i \cdot \lambda_i = \sum_{j \neq i} c_j \lambda_j.$$

Since λ_i and $s_i\lambda_i$ are symmetric with respect to the wall H_i the coefficients c and c' in the equation above are equal, and we may set them both equal to 1. Then, to compute the coefficient c_j for $j \neq i$, let us take the inner product of both sides with α_j. We obtain that

$$\langle s_i \cdot \lambda_i, \alpha_j \rangle = c_j,$$

keeping in mind that the bases $\Delta = \{\alpha_1, \ldots, \alpha_d\}$ and $\{\lambda_1, \ldots, \lambda_d\}$ are dual, so $\langle \alpha_j, \lambda_k \rangle$ equals 1 if $j = k$ and 0 otherwise. Using the formula (3.1) for the reflection s_i we obtain $c_j = -A_{ij}$. It follows that

$$\lambda_i + s_i \cdot \lambda_i = \sum_{i \neq j} -A_{ij} \lambda_j, \tag{5.3}$$

so the wall-crossing inequality is

$$h(\lambda_i) + h(s_i \cdot \lambda_i) \geq \sum_{j \neq i} -A_{ij} h(\lambda_j), \tag{5.4}$$

in agreement with (5.1).

Let us now compute the wall-crossing inequality for a general wall wH_i, which separates chambers $w \cdot D$ and $ws_i \cdot D$. The rays of these chambers that are not on the wall are spanned by $w \cdot \lambda_i$ and $ws_i \cdot \lambda_i$, respectively. Since W acts linearly, (5.3) implies the linear relation

$$w \cdot \lambda_i + ws_i \cdot \lambda_i \geq \sum_{j \neq i} -A_{ij} w \cdot \lambda_j.$$

Therefore the wall-crossing inequalities are indeed the ones given in (5.1). $\square$

Proof of Theorem 5.2, Part 2. Sketch: This follows from Batyrev's criterion (Lemma 2.11) and the fact that the Coxeter complex is flag. $\square$

Example 5.4. *In type A, when $W = S_d$, the weights $\mathcal{W}$ are in bijection with the subsets of $[d]$. The local submodular inequalities produced by Theorem 5.2 say that*

$$h(A \cup b) + h(A \cup c) \geq h(A) + h(A \cup b \cup c) \qquad \text{for } A \subset [d] \text{ and } b, c \notin A,$$

whereas the global submodular inequalities say that

$$h(B) + h(C) \geq h(B \cup C) + h(B \cap C) \qquad \textit{for } B, C \subseteq [d].$$

The second set of inequalities is the usual definition of submodularity, but it is known in the optimization literature that the first subset of conditions (which corresponds to $B = A \cup b$ and $C = A \cup c$) minimally determine the others. The same phenomenon happens in all Coxeter types.

6 Facets of the Φ-submodular cone

In this section we enumerate the facets of the Φ-submodular cone, after proving that they are precisely given by the wall crossing inequalities (5.4). It is not always the case that every wall-crossing inequality for a fan Σ defines a facet of the deformation cone $\mathrm{Def}(\Sigma)$. To prove it in this special case $\Sigma = \Sigma_\Phi$, we will show equivalently that the rays spanned by I_τ are extremal in the Mori cone $\overline{NE}(\Sigma_\Phi) = \mathrm{cone}(I_\tau \,:\, \tau \text{ is a wall of } \Sigma_\Phi)$ in $(\mathrm{PL}(\Sigma_\Phi))^\vee$.

Before proving this, it is useful to remark that the action of W on the Coxeter complex naturally gives rise to actions of W on the vector space $\mathrm{PL}(\Sigma_\Phi)$, the deformation cone $\mathrm{Def}(\Sigma_\Phi) \subset \mathrm{PL}(\Sigma_\Phi)$, and the Mori cone $\overline{NE}(\Sigma_\Phi) \subset (\mathrm{PL}(\Sigma_\Phi))^\vee$.

Theorem 6.1. *Every local Φ-submodular inequality (5.1) is a facet of the W-submodular cone.*

Proof. By Proposition 4.3 and the comment following it, we can produce, for each $1 \leq i \leq d$, a generalized Φ-permutahedron Q_i whose normal fan is obtained from Σ_Φ by removing the walls wH_i separating chambers $w \cdot D$ and $ws_i \cdot D$ for all $w \in W$. The support function of this polytope satisfies

$$I^\Sigma_\tau(h_{Q_i}) \;=\; 0, \quad \text{if } \tau = wH_i \text{ for some } w \in W, \text{ and} \tag{6.1}$$
$$I^\Sigma_\tau(h_{Q_i}) \;>\; 0, \quad \text{otherwise.} \tag{6.2}$$

This means that the set of rays $\{I_{wH_i} \,:\, w \in W\}$ form a face F_i of the Mori cone, so at least one of them must be extremal. But these rays form an orbit of the action of W on the Mori cone, so if one of them is extremal, all are extremal. $\square$

Theorem 6.2. *The number of facets of the Φ-submodular cone is equal to*

$$\sum_{i=1}^{d} \frac{|W|}{|W_{[d]-N(i)}|},$$

where $N(i)$ is the set of neighbors of i in the Dynkin diagram and $W_{[d]-N(i)}$ is the parabolic subgroup generated by the complement of $N(i)$.

Sketch of Proof: We have one inequality for each pair of an element $1 \leq i \leq d$ and a group element $w \in W$, but there are many repetitions. For each i we claim that the set of elements w stabilizing the wall-crossing inequality (5.4) is $W_{[d]-N(i)}$.

If an element w stabilizes (5.4), it must stabilize the support of the right hand side, that is, the set of weights $\{\lambda_j : A_{ij} \neq 0\} = \{\lambda_j : j \in N(i)\}$. Therefore w stabilizes the sum of those weights, which is in the interior of cone $C_{[d]-N(i)}$. By Proposition 3.4, $w \in W_{[d]-N(i)}$.

Conversely, suppose $w \in W_{[d]-N(i)}$. Then for each $j \in N(i)$ we have $w \in W_{[d]-j}$, so w stabilizes λ_j individually. Therefore w does stabilize the right hand side of (5.4). Now, each simple reflection s_k with $k \notin [d] - N(i) - i$ stabilizes λ_i because $k \neq i$, and it stabilizes $s_i \cdot \lambda_i$ since s_i and s_k commute. The remaining reflection s_i interchanges λ_i and $s_i \cdot \lambda_i$. It follows that each generator of $W_{[d]-N(i)}$, and hence the whole parabolic subgroup, stabilizes the left-hand side of (5.4) as well.

We conclude that, for fixed i, each inequality in (5.4) is repeated $|W_{[d]-N(i)}|$, and hence the number of different inequalities is $|W|/|W_{[d]-N(i)}|$. The desired result follows. □

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References

[1] A.-V. Borovik, I. M. Gelfand, and N. White. *Coxeter Matroids*. Progress in Mathematics 216. Birkhäuser, Boston, 2003.

[2] D.-A. Cox, J.-B. Little, and H.-K. Schenck. *Toric Varieties*. Graduate Studies in Mathematics 124. Amer. Math. Soc., Providence, RI, 2011. Link.

[3] J.-A. De Loera, J. Rambau, and F. Santos. *Triangulations: Structures for Algorithms and Applications*. 1st ed. Springer, 2010.

[4] C. Hohlweg, C. Lange, and H. Thomas. "Permutahedra and generalized associahedra". *Adv. Math.* **226**.1 (2011), pp. 608–640. Link.

[5] V. Reiner. *Quotients of Coxeter complexes and P-partitions*. Mem. Amer. Math. Soc. Vol. 95, No. 460. Amer. Math. Soc., Providence, RI, 1992. Link.

[6] J. R. Stembridge. "Coxeter cones and their *h*-vectors". *Adv. Math.* **217**.5 (2008), pp. 1935–1961. Link.