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Cyclic Sieving, Necklaces, and Bracelets

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Abstract. We split the q-Schröder numbers into an "even" and "odd" part. The Schröder numbers are known to enumerate certain necklaces, and the even part turns out to be a q-analogue of the set of bracelets. Both parts are symmetric and unimodal, and we conjecture that there exist posets which explain this phenomenon. Along the way, we find a new cyclic sieving phenomenon on certain double cosets of the symmetric group.

Keywords: Catalan, cyclic sieving, necklace, Schröder, unimodal

1 Introduction

Given a sequence $\alpha = (\alpha_1, \ldots, \alpha_r)$ of positive integers that sums to n, the **multinomial coefficient**

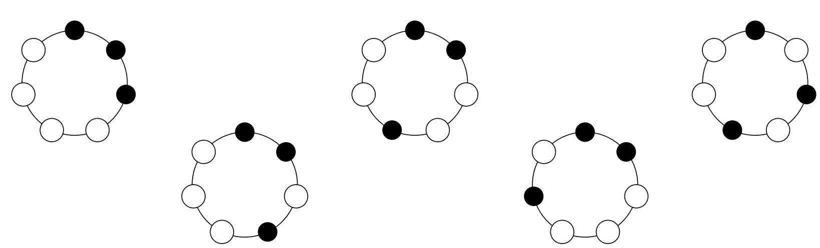
$$\binom{n}{\alpha} = \binom{n}{\alpha_1, \ldots, \alpha_r} = \frac{n!}{\alpha_1! \cdots \alpha_r!}$$

is a positive integer, counting the number of words having exactly α_i occurrences of the letter i for each $i = 1, 2, \ldots, r$. The symmetric group S_n acts on the set of such words permuting positions, and when one restricts this action to the cyclic subgroup $C = \langle c \rangle$ generated by the n-cycle $c = (1, 2, \ldots, n)$, the orbits are called **necklaces** with α_i beads of color i; call these α**-necklaces**. It is easily seen that the C-action on α-necklaces will be free whenever $\gcd(\alpha) = \gcd(\alpha_1, \ldots, \alpha_r) = 1$, and thus the number of α-necklaces in this case is given by $C(\alpha) = \frac{1}{n}\binom{n}{\alpha}$.

When $\alpha = (a, a+1)$, this is the well-known **Catalan number**:

$$C(a, a+1) = \frac{1}{2a+1}\binom{2a+1}{a} = \frac{1}{a+1}\binom{2a}{a}.$$

So, for example, when $\alpha = (3, 4)$, there are $C(3, 4) = \frac{1}{7}\binom{7}{3} = \frac{1}{4}\binom{6}{3} = 5$ such necklaces with 3 black beads and 4 white beads, shown here:



This paper concerns two surprising properties of the q-analogue of $C(\alpha)$ defined by

$$C(\alpha; q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ \alpha \end{bmatrix}_q \tag{1.1}$$

defined in terms of these standard q-analogues:

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \frac{[n]!_q}{[\alpha_1]!_q \cdots [\alpha_r]!_q},$$
$$[n]!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q,$$
$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

1.1 Cyclic Sieving

Recall from [9] that for a set X carrying the action of a cyclic group $\langle \tau \rangle$ of order m, and a polynomial $X(q)$ in $\mathbb{Z}[q]$, we say that $(X, X(q), C)$ exhibits the **cyclic sieving phenomenon** if for every integer d,

$$|\{x \in X : \tau^d(x) = x\}| = X(\zeta_m^d)$$

where $\zeta_m = e^{\frac{2\pi i}{m}}$.

We will be particularly interested in the case where $m = 2$, so that τ is an involution, in which case one says that $(X, X(q), \tau)$ exhibits **Stembridge's** $q = -1$ **phenomenon** [11]. That is,

$$X(1) = \#X,$$
$$X(-1) = \#\{x \in X : \tau(x) = x\}.$$

There is another way to phrase this, in terms of the τ-orbits on X of size one and two, which we will call the **symmetric** and **asymmetric** τ-orbits, respectively. Letting $X(q) = \sum_i a_i q^i$, to say that $(X, X(q), \tau)$ exhibits the $q = -1$ phenomenon is equivalent to saying that the two sums

$$\frac{1}{2}\left(X(1) + X(-1)\right) = a_0 + a_2 + a_4 + \cdots,$$
$$\frac{1}{2}\left(X(1) - X(-1)\right) = a_1 + a_3 + a_5 + \cdots,.$$

count the total number of τ-orbits, and the number of asymmetric τ-orbits on X, respectively.

This lets us phrase our first result, which follows on the observation in [9, §8] that whenever $\gcd(\alpha) = 1$, the q-analogue $C(\alpha; q)$ defined in (1.1) is a polynomial in q with nonnegative coefficients. As noted above, $C(\alpha; 1) = C(\alpha)$ counts the set X of all of α-necklaces. There is a natural involutive action τ on X in which τ reverses a word or **reflects** a necklace over a line; orbits for this τ-action are sometimes called α-**bracelets**. Thus a bracelet is **asymmetric** if it represents a τ-orbit of necklaces that has size two.

Theorem 1.1. *When* $\gcd(\alpha) = 1$*, the set* X *of* α*-necklaces along with* $X(q) := C(\alpha; q) = \sum_i a_i q^i$ *and its* τ*-action by reflection exhibits Stembridge's* $q = -1$ *phenomenon. That is,*

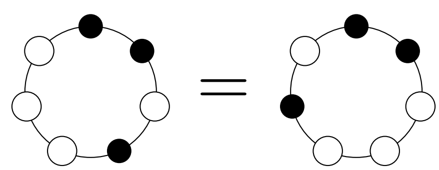
$$\frac{1}{2}\left(C(\alpha;1) + C(\alpha;-1)\right) = a_0 + a_2 + a_4 + \cdots,$$
$$\frac{1}{2}\left(C(\alpha;1) - C(\alpha;-1)\right) = a_1 + a_3 + a_5 + \cdots,$$

count the total number of bracelets, and the number of asymmetric bracelets, respectively.

In the example of $\alpha = (3, 4)$,

$$C(\alpha; q) = \frac{1}{[7]_q} \begin{bmatrix} 7 \\ 3 \end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6,$$

with $\frac{1}{2}\left(C(\alpha;1) + C(\alpha;-1)\right) = 4$ and $\frac{1}{2}\left(C(\alpha;1) + C(\alpha;-1)\right) = 1$. This agrees with the fact that the five necklaces in the previous example give rise to four bracelets, only one of which is asymmetric, namely the one shown here:



Theorem 1.1 will be deduced in Section 2 from a much more general statement. Notice that the reflection τ, thought of as an element of S_n, is contained in the normalizer of C. We show a sufficient condition for other $\tau \in N_{S_n}(C)$ acting on α-necklaces to satisfy a cyclic sieving phenomenon as well.

1.2 Parity Unimodality

We next explain a second surprising property of $C(\alpha; q)$. Say that a polynomial $X(q) = \sum_i a_i q^i$ in q with nonnegative coefficients a_i is **parity-unimodal** if both subsequences $(a_0, a_2, a_4, \ldots)$ and $(a_1, a_3, a_5, \ldots)$ are unimodal.

Conjecture 1.2. *When* $\gcd(\alpha) = 1$*, the polynomial* $C(\alpha; q)$ *is parity-unimodal.*

We have two pieces of evidence for Conjecture 1.2. First, it has been checked for all relevant compositions α of $n \leq 30$. Second, we will explain in Section 3 below why known results in the theory of rational Cherednik algebras imply the conjecture when $\alpha = (k, a-k, b-k)$ with $\gcd(a, b) = 1$ and $0 \leq k \leq a$. Here $C(k, a-k, b-k; q)$ is called a **rational** q**-Schröder polynomial**— the special case $k = 0$ is called the **rational** q**-Catalan polynomial**, whose special case $b = a + 1$ is **MacMahon's** q**-Catalan polynomial**.

Theorem 1.3. *When $\alpha = (k, a, b)$ with $\gcd(a, b) = 1$ and $0 \le k \le a$, the rational q-Schröder polynomial $C(k, a-k, b-k; q)$ is parity-unimodal.*

In this rational Schröder case, something beyond parity unimodality for $C(\alpha; q) = \sum_{i=0}^{s} a_i q^i$ is true. Here the two subsequences $(a_0, a_2, a_4, \dots)$ and $(a_1, a_3, a_5, \dots)$ are not only unimodal, but also symmetric.

This, together with Theorem 1.1, brings to mind Proctor's characterization of Peck posets in [8]. This result suggests there may reasonably be two highly structured ranked posets on α-bracelets whose rank sizes are $(a_0, a_2, a_4, \dots)$ and $(a_1, a_3, a_5, \dots)$, respectively. In particular, the existence of such posets would give a considerably more elementary demonstration of parity-unimodality for the rational q-Schröder polynomials.

In Section 4, we find such a pair of posets for $k = 0$ and $a = 3$, and discuss the challenges with extending to other cases.

2 Cyclic Sieving

Before getting to a more general result that will imply Theorem 1.1, we review a fairly general cyclic sieving phenomenon that specializes a result from [9].

Given any subgroup H of S_n, consider the transitive action of S_n on the coset space $X = S_n/H$, and restrict this action to the cyclic subgroup $C = \langle c \rangle$ of S_n generated by the n-cycle $c = (1, 2, \dots, n)$. Also recall that S_n (and hence H) acts on the graded ring of n-variable polynomials $\mathbb{C}[x]$ by permuting indices, and denote the fixed space of this action by $\mathbb{C}[x]^{S_n}$ (and similarly for $\mathbb{C}[x]^H$). Then [9, Theorem 8.2] implies that the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon, where $\mathcal{H}(V; q) = \sum_{i \ge 0} \dim(V_i) q^i$ is the Hilbert series of a graded vector space $V = \bigoplus_{i \ge 0} V_i$, and

$$X(q) = \frac{\mathcal{H}(\mathbb{C}[x]^H, q)}{\mathcal{H}(\mathbb{C}[x]^{S_n}, q)}. \tag{2.1}$$

We will be particularly interested in the special case where C acts *freely* on $X = S_n/H$. Note that this is equivalent to the condition that no power of the n-cycle c is S_n-conjugate to an element of H, or equivalently, that the permutation group H avoids all cycle types of the form $(d^{\frac{n}{d}})$ for divisors d of n with $d \ge 2$.

In this case, the fact that $(X, X(q), C)$ exhibits a cyclic sieving phenomenon [9, Theorem 8.2] means that $X(\zeta) = 0$ for any nontrivial n^{th} root of unity. Equivalently, we have that $\prod_{i=1}^{n-1}(q - \zeta^i) = [n]_q$ is a factor of of $X(q)$, and so

$$Y(q) = \frac{1}{[n]_q} X(q)$$

is in fact a polynomial.

Remark. Reiner, Stanton, and White use a somewhat different notation: their $X(q)$ is defined as $\mathcal{H}(A(S_n)^H, q)$, where $A(S_n)$ is the coinvariant algebra $\mathbb{C}[x]/\langle f \in \mathbb{C}[x]^{S_n} : f(0) = 0\rangle$. The translation between these two is a standard fact from invariant theory; see, e.g. [2, Corollary 1.2.2].

Notice that elements τ of the normalizer $N_{S_n}(C)$ can act on the set

$$Y = C \backslash S_n / H = \{\text{double cosets } CgH : g \in S_n\}$$

via this rule:

$$\tau \cdot CgH = \tau CgH = C\tau gH. \tag{2.2}$$

Choose a particular involution τ in $N_{S_n}(C)$ having cycle type $(2, 2, \dots, 2, 1)$ for n odd and $(2, 2, \dots, 2, 1, 1)$ for n even, namely let τ be the permutation of the vertices $1, 2, \dots, n$ of a regular n-gon that comes from a reflection fixing vertex 1 (and fixing exactly one other vertex, namely $\frac{n}{2}$, when n is even). We can now state the general result.

Theorem 2.1. *Fix a positive integer $m \geq 2$, and suppose that either $n \equiv 1$ mod m, or n is even and $n \equiv 2$ mod m.*

Assume that H is a subgroup of S_n that avoids the cycle types

- $(d^{\frac{n}{d}})$ *for divisors $d \geq 2$ of n,*
- $(\ell^{\frac{n-2}{\ell}}, 2)$ *for divisors ℓ of m,*
- $(2^{\frac{n-4}{2}}, 4)$.

Further assume that one has some τ in $N_{S_n}(C)$ of order m whose cycle type is

$$\begin{cases} (m^{\frac{n-1}{m}}, 1) & \text{if } n \equiv 1 \bmod m, \\ (m^{\frac{n-2}{m}}, 1, 1) & \text{if } n \equiv 2 \bmod m. \end{cases}$$

Then one has a cyclic sieving phenomenon for the triple $(Y, Y(q), \langle\tau\rangle)$, where $\langle\tau\rangle \cong \mathbb{Z}/m\mathbb{Z}$ acts on Y via the rule (2.2).

Before proving this theorem, we use it to prove Theorem 1.1.

It is a straightforward exercise in commutative algebra to show that if $H = S_{\alpha_1} \times \cdots \times S_{\alpha_r}$ and $X = S_n/H$, then

$$X(q) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q.$$

Notice that as S_n-sets, X is equivalent to the set of words with α_i copies of each letter i, and so the group of rotations $C = \langle c \rangle$ acts freely on X if $\gcd(\alpha_1, \dots, \alpha_r) = 1$. In this case, the associated $Y(q)$ is $C(\alpha; q)$. Moreover, note that H never has an element with cycle type either $(d^{\frac{n}{d}})$ or $(2^{\frac{n-4}{2}}, 4)$, since otherwise each of the α_i would be divisible by d, or 2 respectively. Thus, since τ is the reflection on bracelets, Theorem 2.1 shows that we have a $q=-1$ phenomenon for the triple $(Y, Y(q), \tau)$, as desired.

We now sketch a proof of Theorem 2.1(a), for which $n = 1+km$. The following lemma encapsulates a Molien formula calculation which is used in both cases.

Lemma 2.2. *Let ζ be a primitive m^{th} root of unity, and $k = \left\lfloor \frac{n-1}{m} \right\rfloor$. For any partition $\lambda \vdash n$ of n, write $\operatorname{Fix}_X(\lambda)$ to mean the number of fixpoints in $X \in S_n/H$ for any $g \in S_n$ with cycle type (m^k, λ), and c_i to mean number of parts in λ with size i.*

(a) If $n \not\equiv 0$ mod m, then

$$Y(\zeta) = (1-\zeta)\prod_{j=1}^{n-1-km}(1-\zeta^j)\sum_{\lambda \vdash n-km}\frac{|\operatorname{Fix}_X(\lambda)|}{\prod_{i\geq 1}(i(1-\zeta^i))^{c_i}c_i!}.$$

(b) If $n \equiv 0$ mod m, then

$$Y(\zeta) = (1-\zeta)\sum_{\substack{\lambda \vdash m \\ \lambda_1 \neq m}}\frac{|\operatorname{Fix}_X(\lambda)|}{\prod_{i\geq 1}(i(1-\zeta^i))^{c_i}c_i!} + \frac{(1-\zeta)mk}{4}|\operatorname{Fix}_X(2m)|.$$

The formulas above simplify considerably since there is only one λ that partitions $n-km$, namely $\lambda = (1)$. Therefore, if ζ is a primitive m^{th} root of unity, then

$$Y(\zeta) = (1-\zeta)\frac{|\operatorname{Fix}_X(\tau)|}{1-\zeta} = |\operatorname{Fix}_X(\tau)|.$$

This does not immediately resolve the situation because $\operatorname{Fix}_X(\tau)$ is the set of fixpoints of τ acting on X, rather than the set Y of α-bracelets. Therefore, it remains to show that $|\operatorname{Fix}_X(\tau)| = |\{CgH : \tau CgH = CgH\}|$. The following general lemma reduces this to calculating the size of a centralizer:

Lemma 2.3. *For an arbitrary finite group G, subgroup H, and C acting freely on G/H, let $\tau \in N_G(C)$. Denote the canonical quotient map $G/H \to C\backslash G/H$ by π, and write π_F to be the restriction of π to the set $\operatorname{Fix}_X(\tau) = \{gH : \tau gH = gH\}$.*

Then, for any $gH \in \operatorname{Fix}_X(\tau)$, we have $|\pi_F^{-1}(CgH)| = |Z_C(\tau)|$.

According to this lemma, we have $\pi_F^{-1}(CgH) = \pi_F^{-1}(C\gamma H)$ has size $|Z_C(\tau)|$. We conclude that $|\operatorname{Fix}_X(\tau)| = |Z_C(\tau)| \cdot |\{CgH : \tau CgH = CgH\}|$. Thus, it suffices to show that $Z_C(\tau)$ is the trivial group, which follows from a straightforward calculation.

3 Unimodality in the Rational Schröder Case

We begin by fixing some notation. Any G-representation V gives rise to a **symmetric algebra** SV^*; in coordinates, SV^* is simply a polynomial ring $\mathbb{C}[x_1, \ldots, x_{a-1}]$, where the variables are basis vectors for V^*. The action is given in the natural way: $g \cdot p(x_1, \ldots, x_{a-1}) = p(gx_1, \ldots, gx_{a-1})$. Note that SV^* is a graded vector space, and each graded piece S^iV^* is a G-representation.

It also gives rise to an **exterior representation** $\wedge V$ which as a vector space has a basis element for every subset $S \subseteq \{x_1, \ldots, x_a\}$. The only thing we will use in this paper about its G-representation structure is that as a vector space $\wedge V$ has a natural grading, and each of these pieces $\wedge^i V$ is itself a G-representation.

For the remainder of the section, we let $V = \mathbb{C}^{a-1}$ be the irreducible reflection representation of S_a. The following result is fairly standard; see, e.g. [5, §3].

Theorem 3.1. *Let $U \subseteq SV^*$ be an $(a-1)$-dimensional S_a-subrepresentation contained in degree b and denote by $\langle U \rangle$ the ideal generated by the elements of U. If $SV^*/\langle U \rangle$ is finite-dimensional, then as a graded SV^*-module and $\mathbb{C}[S_a]$-module, it admits a resolution*

$$0 \leftarrow SV^*/\langle U \rangle \leftarrow SV^* \leftarrow \Big(SV^* \otimes \wedge^1 U(-b)\Big) \leftarrow \cdots \leftarrow \Big(SV^* \otimes \wedge^{a-1} U(-b)\Big) \leftarrow 0,$$

where the notation $M(-d)$ denotes the module M with a shifted grading so that $\deg(1) = d$.

Theorem 3.1 is a conditional result, computing a resolution when provided with a "nice" S_a-representation U. In [4], Dunkl proved that if b is coprime to a, then such a U does actually exist, and moreover as an S_a-representation, $U \cong V^*$. The resulting quotient space $SV^*/\langle U \rangle$ has been well-studied. For instance, it is the space of "rational parking functions" in the sense of [1]. In the following section we will introduce the rational Cherednik algebra, and it is true that $SV^*/\langle U \rangle$ is irreducible as a module over this algebra (see, for instance, [3]). In the latter context it is often called $L_{b/a}(1)$; we adopt this notation here.

Using Theorem 3.1 together with a formula of Kirillov-Pak, corrected and simplified by Molchanov in [7], we can determine an alternate model for $C(\alpha; q)$ in the rational Schröder case:

Proposition 3.2. *Let $a < b$ be coprime and $0 \leq k \leq a$. Then the rational q-Schröder polynomial $C(k, a-k, b-k; q)$ coincides with*

$$C^k_{a,b}(q) := \mathcal{H}\Big(\mathrm{Hom}_{\mathbb{C}[S_a]}\big(\wedge^k \mathbb{C}^a, L_{b/a}(1)\big);\, q\Big).$$

3.1 The $\mathfrak{sl}_2$ Action on $L_{b/a}(1)$

Given an algebra A equipped with an action of a group G, the **semidirect product** $A \rtimes G$ is the algebra which as a vector space is $A \otimes \mathbb{C}[G]$, and whose product structure given by $(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh$.

Let $V = \mathbb{C}^{a-1}$ be the irreducible reflection representation of S_a, and $y_1, \ldots, y_{a-1}$ and $x_1, \ldots, x_{a-1}$ be a basis for V and its dual basis (respectively). The **rational Cherednik algebra** is $H_{b/a} = (S(V \oplus V^*) \rtimes S_a)/I$, where I is the ideal generated by the following relations:

$$\begin{aligned} x_i x_j &= x_j x_i \\ y_i y_j &= y_j y_i \\ x_i y_j &= y_j x_i \qquad \text{for all } i \neq j \\ x_i y_i - y_i x_i &= 1 - \frac{b}{a} \sum_{k=1, k\neq i}^{a} (ik). \end{aligned}$$

This algebra can be given a grading via $\deg(w) = 0$ for all $w \in S_n$, and for the variables, $\deg(x_i) = 1$ and $\deg(y_i) = -1$.

We are now ready to prove Theorem 1.3:

Theorem 1.3. *When $\alpha = (k, a, b)$ with $\gcd(a, b) = 1$ and $0 \leq k \leq a$, the rational q-Schröder polynomial $C(k, a-k, b-k; q)$ is parity-unimodal.*

Proof. This argument is loosely based on Haiman [6], which uses simpler tools to obtain the result in the $b = a+1$ case.

There is an action of $\mathfrak{sl}_2(\mathbb{C})$ on $H_{b/a}$ given by left multiplication of certain elements:

$$e = -\frac{1}{2}\sum_{i=1}^{a-1} x_i^2 \qquad\qquad f = \frac{1}{2}\sum_{i=1}^{a-1} y_i^2$$

$$h = \sum_{i=1}^{a-1} x_i y_i + (a-1)\left(\frac{1}{2} - \sum_{i=1}^{a-1}(i, i+1)\right)$$

This extends to an action on SV^* and hence on $L_{b/a}(1)$, and commutes with the action of S_a: e and f are clearly invariant under permuting indices (and thus, so is $h = ef - fe$).

Any $\mathfrak{sl}_2(\mathbb{C})$-module V has a **formal character** $\mathrm{ch}(V) = \sum \dim(V_\ell) q^\ell$. By typical Lie theory arguments (see, e.g. [10]), all $\mathrm{ch}(V)$ are symmetric and parity-unimodal about q^0.

The significance for our situation is that the grading on $H_{b/a}$ descends to a grading on $L_{b/a}(1)$, and since h preserves the grading on $H_{b/a}$ it also does so on $L_{b/a}(1)$. It follows that for any $V \subseteq L_{b/a}(1)$, weight differs from degree only by a constant shift, and therefore $\mathrm{ch}(V)$ is the Hilbert series of V up to a factor of some q^c. Since

$$L_{b/a}(1) = \bigoplus_{\lambda \vdash a}\left(\bigoplus_{\ell \geq 0} m_{\ell,\lambda} V_\ell\right) \otimes S^\lambda,$$

this implies that the space of intertwiners of S^λ with $L_{b/a}(1)$ has (shifted) Hilbert function $P_\lambda(q)$, where the P_λ are each Laurent polynomials, symmetric and parity-unimodal about q^0. In particular, this implies that $C^k_{a,b}(q) = P_{(a-k,1^k)}(q) + P_{(a-k-1,1^{k+1})}(q)$ is symmetric and parity-unimodal about q^0, which is equivalent to the desired statement. $\square$

4 Toward a Poset of Schröder Bracelets

The use of an $\mathfrak{sl}_2(\mathbb{C})$ action to show the symmetry and parity unimodality of $C(k, a-k, b-k)$ brings to mind a classic result of Stanley and Proctor. Let P be a finite ranked poset with maximum rank ρ, and P_i be the set of elements with rank i. The **rank generating function** of P is the polynomial $\sum_{i\geq 0}|P_i|x^i$. We say that P is **rank symmetric** if $|P_i| \leq |P_{\rho-i}|$ for all i, and that P is **rank unimodal** if the sequence $(P_0, P_1, P_2, \dots)$ is unimodal. Finally, P is called **strongly Sperner** if for each $k \geq 1$, there are no k antichains whose union has more elements than the k largest P_i.

Theorem 4.1 (Proctor [8]). *A ranked poset P with maximum rank ρ is rank-symmetric, rank-unimodal, and strongly Sperner if and only if it carries a representation of $\mathfrak{sl}_2(\mathbb{C})$ in the following sense: letting $\mathbb{C}P_i$ be the complex vector space of formal linear combinations of elements with rank i, there exist linear operators E and F acting on $\bigoplus_{i=0}^{\rho}\mathbb{C}P_i$ such that*

$$E(p) = \sum_{p \gtrdot q} c_q q \qquad \text{and} \qquad F(p) = \sum_{\mathrm{rank}(q)=\mathrm{rank}(p)+1} c'_q q$$

for some collections of coefficients $c_q, c'_q \in \mathbb{C}$, and for which each restriction $(EF - FE)|_{\mathbb{C}P_i}$ acts by scalar multiplication $v \mapsto (2i-\rho)v$.

If a poset satisfies either of the equivalent conditions in this theorem, it is said to be **Peck**. Because one of these conditions is the existence of an $\mathfrak{sl}_2(\mathbb{C})$ action, it is reasonable to hope for two Peck posets which explain the parity-unimodality of $C(\alpha; q)$. In particular, assuming the posets and their corresponding representations were reasonably straightforward, this would provide a much more elementary demonstration of unimodality, without delving into the machinery of rational Cherednik algebras.

To help state this question more precisely, we introduce some notation.

Definition. The **even q-Schröder polynomials** are defined to be

$$EC^k_{a,b}(q) = \frac{1}{2}\left[C^k_{a,b}(q^{1/2}) + C^k_{a,b}(-q^{1/2})\right].$$

Similarly the **odd q-Schröder polynomials** are defined to be

$$OC^k_{a,b}(q) = \frac{q^{-\frac{1}{2}}}{2}\left[C^k_{a,b}(q^{1/2}) - C^k_{a,b}(-q^{1/2})\right].$$

These are both polynomials, and up to normalization are simply picking out the terms of $C^k_{a,b}(q)$ with even and odd exponents, respectively.

Additionally, recall that a **symmetric chain decomposition** of a ranked poset P with finite maximum rank ρ is a partition of its ground set $P = \Gamma_1 \coprod \Gamma_2 \coprod \cdots \coprod \Gamma_k$ into saturated chains Γ_i, such that $\mathrm{rank}(\min \Gamma_i) + \mathrm{rank}(\max \Gamma_i) = \rho$ for all i. Having a symmetric chain decomposition is a much stronger condition on P than being Peck, but it is somewhat more elementary, and is satisfied by many combinatorially significant posets.

Question 4.2. *When* $\gcd(a,b)=1$ *and* $0 \le k \le a$, *do there exist "natural" posets* $\beta^k_{a,b}$ *and* $\tilde{\beta}^k_{a,b}$ *with the following properties?*

- *The ground sets of these posets are respectively the* $(k, a-k, b-k)$*-bracelets and asymmetric* $(k, a-k, b-k)$*-bracelets.*
- *The identity map is an order-preserving injection* $\tilde{\beta}^k_{a,b} \to \beta^k_{a,b}$.
- *The rank generating functions of these posets are respectively* $EC^k_{a,b}(q)$ *and* $OC^k_{a,b}(q)$.
- *These posets each admit symmetric chain decompositions.*

We can make this question fully precise, and answer in the affirmative, for $a \le 2$, and for $(a,k) = (3,0)$. Let us say that a poset structure on the set of α-bracelets is **generated by local moves** if for each covering relation $A \lessdot B$, we can obtain B from A by swapping two adjacent beads.

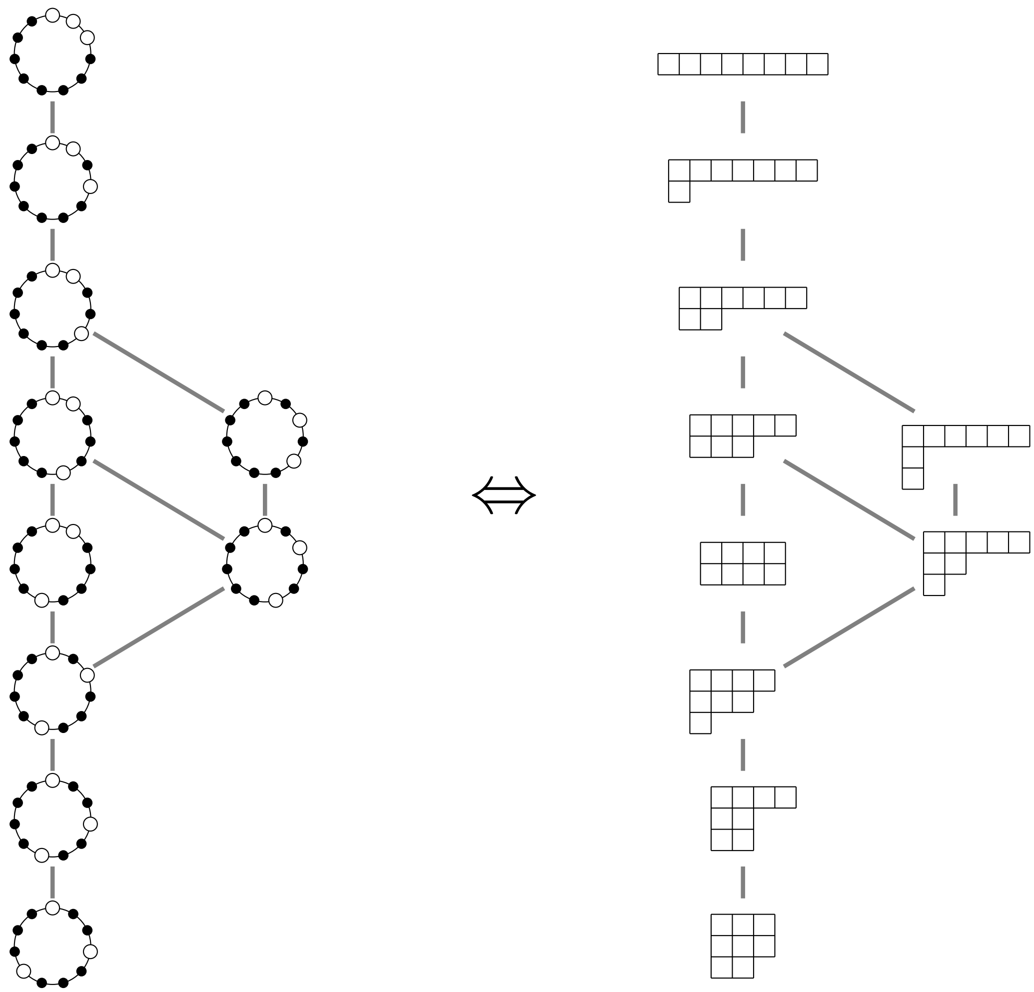
Theorem 4.3. *Suppose that* $\gcd(a,b)=1$ *and either* $0 \le k \le a \le 2$, *or that* $a=3$ *and* $k=0$. *Then there exist posets* $\beta^k_{a,b}$ *and* $\tilde{\beta}^k_{a,b}$ *satisfying the four properties of Question 4.2, which are also generated by local moves.*

For $a \le 2$, all posets involved are chains, so we omit them here. For $a=3$, recall that the **dominance order** is defined in this way: pad all partitions of b with infinitely many zeros, and then $(\lambda_1, \lambda_2, \dots) \le (\mu_1, \mu_2, \dots)$ whenever for all $n \ge 0$ the partitions satisfy the partial sum inequalities $\lambda_1 + \cdots + \lambda_n \le \mu_1 + \cdots + \mu_n$.

We claim that the interval $[\top, \bot]$ in the dominance order is a satisfactory model for $\beta^0_{3,b}$, where the top element is the partition $\top = (b)$ (which is, indeed, the top element of the dominance order itself), and the bottom is the partition $\bot = (\lceil b/3 \rceil, [b/3], \lfloor b/3 \rfloor)$, where $[x]$ is the nearest integer to x.

For any bracelet, each of the three adjacent pairs of white beads has a number of black beads in between them. These numbers can be viewed as the parts of a partition, because the order of these numbers can be freely permuted by rotations and reflections. In particular, $\top$ and $\bot$ correspond respectively to the bracelets where all three white beads are next to each other, and where they are as evenly distributed as possible. Our claim is equivalent to showing that the correspondence described above is in fact a bijection between this interval and the set of $(3,b)$-bracelets.

Example. On the next page is a picture of $\beta_{3,8}$ and the corresponding $[\bot, \top]$. Notice that its rank generating function is $EC_{3,8}(q) = 1 + q + q^2 + 2q^3 + 2q^4 + q^5 + q^6 + q^7$, and the columns form a symmetric chain decomposition of the poset.



4.1 Negative Results

It is possible that the conditions in Theorem 4.3 may be relaxed somewhat, but they do capture some genuine difficulties with the situation. Although we believe that the question may still have an affirmative answer for larger a, we do not currently have an appropriate notion of "naturalness". In particular, the proof above suggests two candidates for making the question into a precise conjecture, both of which are false.

First, the theorem as stated cannot be generalized to just include more values of a and k. Even in the Catalan case when $b = a + 1$ and $k = 0$, being generated by local moves can automatically force the rank counts to be incorrect.

Proposition 4.4. *There is no bounded poset structure on the set of $(4,5)$-bracelets having rank generating function $EC^0_{4,5}(q)$ which is generated by local moves.*

As a second attempt, there is nothing in the definition of dominance which forces us to restrict our attention to partitions. Therefore, even though we may not preserve the bracelet by arbitrarily permuting the gaps between white beads for $a > 3$, we may still try to consider the dominance order on some other representatives of the bracelets. Quick calculations with the $(4,5)$-bracelets already suggest this may be troublesome; for instance, the lexicographically-largest bracelet representatives yields a poset which is not ranked.

On the other hand, there must be some notion of "naturalness" imposed beyond the four properties demanded by Question 4.2. Otherwise, we could make the following trivial construction: arbitrarily assign ranks to bracelets to achieve the correct rank sizes for both $\beta^k_{a,b}$ and $\tilde{\beta}^k_{a,b}$, and then take $A \leq B$ if and only if $\operatorname{rank}(A) \leq \operatorname{rank}(B)$.

References

[1] D. Armstrong, N. A. Loehr, and G. S. Warrington. "Rational Parking Functions and Catalan Numbers". *Ann. Combin.* **20**.1 (2016), pp. 21–58. Link.

[2] A. Broer, V. Reiner, L. Smith, and P. Webb. "Extending the coinvariant theorems of Chevalley, Shephard-Todd, Mitchell, and Springer". *Proc. London Math. Soc (3)* **103**.5 (2011), pp. 747–785. Link.

[3] T. Chmutova and P. Etingof. "On some representations of the rational Cherednik algebra". *Representation Theory* **7** (2003), pp. 641–650. Link.

[4] C. F. Dunkl. "Intertwining operators and polynomials associated with the symmetric group". *Monatshefte für Mathematik* **126**.3 (1998), pp. 181–209. Link.

[5] P. Etingof and X. Ma. "Lecture notes on Cherednik algebras". 2010. arXiv:1001.0432.

[6] M. D. Haiman. "Conjectures on the Quotient Ring by Diagonal Invariants". *J. Algebraic Comb.* **3**.1 (1994), pp. 17–76. Link.

[7] V. F. Molchanov. "On the Poincaré series of representations of finite reflection groups". *Functional Analysis Appl.* **26**.2 (1992), pp. 143–145. Link.

[8] R. A. Proctor. "Representations of $\mathfrak{sl}(2,\mathbb{C})$ on Posets and the Sperner Property". *SIAM J. Algebraic Discrete Methods* **3**.2 (1982), pp. 275–280. Link.

[9] V. Reiner, D. Stanton, and D. White. "The cyclic sieving phenomenon". *J. Comb. Theory Ser. A* **108**.1 (2004), pp. 17–50. Link.

[10] R. P. Stanley. "Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry". *Ann. New York Acad. Sci.* **576**.1 (1989), pp. 500–535. Link.

[11] J. R. Stembridge. "Some hidden relations involving the ten symmetry classes of plane partitions". *J. Comb. Theory, Ser. A* **68**.2 (1994), pp. 372–409. Link.