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On the Homogenized Linial Arrangement: Intersection Lattice and Genocchi Numbers

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Abstract. Hetyei recently introduced a hyperplane arrangement (called the homogenized Linial arrangement) and used the finite field method of Athanasiadis to show that its number of regions is a median Genocchi number. These numbers count a class of permutations known as Dumont derangements. Here, we take a different approach, which makes direct use of Zaslavsky's formula relating the intersection lattice of this arrangement to the number of regions. We refine Hetyei's result by obtaining a combinatorial interpretation of the Möbius function of this lattice in terms of variants of the Dumont permutations. This enables us to derive a formula for the generating function of the characteristic polynomial of the arrangement. The Möbius invariant of the lattice turns out to be a (nonmedian) Genocchi number. Our techniques also yield type B, and more generally Dowling arrangement, analogs of these results.

Keywords: hyperplane arrangements, Genocchi numbers, Dowling lattices

1 Introduction

Let $n \geq 1$. The *braid arrangement* is the hyperplane arrangement in $\mathbb{R}^n$ defined by

$$\mathcal{A}_{n-1} := \{x_i - x_j = 0 : 1 \leq i < j \leq n\}.$$

Note that the hyperplanes of $\mathcal{A}_{n-1}$ divide $\mathbb{R}^n$ into $n!$ open cones of the form

$$R_\sigma := \{\mathbf{x} \in \mathbb{R}^n : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\},$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$.

A classical formula of Zaslavsky [19] gives the number of regions of any real hyperplane arrangement $\mathcal{A}$ in terms of the Möbius function of the intersection (semi)lattice $\mathcal{L}(\mathcal{A})$ (which consists of intersections of collections of hyperplanes in $\mathcal{A}$, viewed as affine subspaces of $\mathbb{R}^n$, ordered by reverse containment). Indeed, given any finite, ranked poset P of length r, with a minimum element $\hat{0}$, define the *characteristic polynomial* of P to be

$$\chi_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{r - \mathrm{rk}(x)}, \tag{1.1}$$

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where $\mu_P(x,y)$ is the Möbius function of P and $\mathrm{rk}(x)$ is the rank of x. Zaslavsky's formula is

$$\#\{\text{regions of } \mathcal{A}\} = |\chi_{\mathcal{L}(\mathcal{A})}(-1)|. \tag{1.2}$$

It is well known and easy to see that the lattice of intersections of the braid arrangement $\mathcal{A}_{n-1}$ is isomorphic to the lattice Π_n of partitions of the set $[n] := \{1, 2 \ldots, n\}$. It is also well known that the characteristic polynomial of Π_n is given by

$$\chi_{\Pi_n}(t) = \sum_{k=1}^{n} s(n,k) t^{k-1}, \tag{1.3}$$

where $s(n,k)$ is the Stirling number of the first kind, which is equal to $(-1)^{n-k}$ times the number of permutations in $\mathfrak{S}_n$ with exactly k cycles; see [16, Example 3.10.4]. Hence $\chi_{\Pi_n}(-1) = (-1)^{n-1}|\mathfrak{S}_n|$. Therefore from (1.2), we recover the result observed above that the number of regions of $\mathcal{A}_{n-1}$ is $n!$.

In this extended abstract of [10], we obtain analogous results for a hyperplane arrangement introduced by Hetyei [9]. The *homogenized Linial arrangement* is the hyperplane arrangement in $\mathbb{R}^{2n}$ defined, for $n \geq 2$, by[1]

$$\mathcal{H}_{2n-3} := \{x_i - x_j = y_i \mid 1 \leq i < j \leq n\}.$$

Note that by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_n = 0$ one gets the braid arrangement $\mathcal{A}_{n-1}$. Similarly by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_n = 1$, one gets the Linial arrangement in $\mathbb{R}^n$,

$$\{x_i - x_j = 1 \mid 1 \leq i < j \leq n\}.$$

Postnikov and Stanley [12] show that the number of regions of the Linial arrangement is equal to the number of alternating trees on node set $[n+1]$.

Using the finite field method of Athanasiadis [1], Hetyei [9] obtains a recurrence for $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$ and uses it to show that

$$|\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1)| = h_n, \tag{1.4}$$

where h_n is a median Genocchi number.[2] Barsky and Dumont [2, Theorem 1] obtain the following generating function for the median Genocchi numbers

$$\sum_{n \geq 1} h_n x^n = \sum_{n \geq 1} \frac{n!(n+1)!x^n}{\prod_{k=1}^{n}(1 + k(k+1)x)}. \tag{1.5}$$

The median Genocchi numbers also have numerous combinatorial interpretations. One of these interpretations is given in terms of a class of permutations called Dumont

[1] To justify our our indexing, we note that the length of the intersection lattice is $2n - 3$.

[2] In the literature the median Genocchi number h_n is usually denoted H_{2n+1}.

permutations; see [5] and [6, Corollary 2.4]. Another is given in terms of surjective pistols in [6, Corollary 2.2].

Here, we study the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$. We refine Hetyei's result (1.4) by deriving a combinatorial formula for the Möbius function of $\mathcal{L}(\mathcal{H}_{2n-1})$ in terms of permutations in $\mathfrak{S}_{2n}$ similar to Dumont permutations, which we call D-permutations. A key step in our proof is to show that $\mathcal{L}(\mathcal{H}_{2n-1})$ is isomorphic to the bond lattice of a certain bipartite graph. This bond lattice has a nice description as the induced subposet of the partition lattice Π_{2n} consisting of partitions all of whose nonsingleton blocks have odd smallest element and even largest element.

Our Möbius function result yields a combinatorial formula for the characteristic polynomial of $\mathcal{L}(\mathcal{H}_{2n-1})$ analogous to (1.3) with $\mathfrak{S}_n$ replaced by the D-permutations in $\mathfrak{S}_{2n}$. By constructing a bijection between the D-permutations and surjective pistols, we recover Hetyei's result that $|\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1)|$ is a median Genocchi number. Moreover, we obtain the new result that the (nonmedian) Genocchi number[3] g_n is equal to $|\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0},\hat{1})|$, where $\hat{0}$ and $\hat{1}$ are the minimum and maximum elements of $\mathcal{L}(\mathcal{H}_{2n-1})$, respectively. The bijection also enables us to derive a formula for the generating function $\sum_{n\geq 1}\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)x^n$, which reduces to the Barsky-Dumont formula (1.5) when $t=-1$ and to a similar formula of Barsky and Dumont in [2] for $\sum_{n\geq 1} g_n x^n$ when $t=0$.

Our techniques also yield a type B analog of Hetyei's result and more generally a Dowling arrangement analog. We define the *type B homogenized Linial arrangement* to be the hyperplane arrangement in $\mathbb{R}^{2n}$ defined by

$$\mathcal{H}^B_{2n-1} = \{x_i \pm x_j = y_i : 1 \leq i < j \leq n\} \cup \{x_i = y_i : i = 1\dots,n\}. \tag{1.6}$$

We show that that the intersection lattice of $\mathcal{H}^B_{2n-1}$ is isomorphic to an induced subposet of the signed partition lattice Π^B_{2n-1} and obtain results for the Möbius function and characteristic polynomial analogous to those for $\mathcal{L}(\mathcal{H}_{2n-1})$. We use these results to prove the following generating function formula for the number of regions r^B_n of $\mathcal{H}^B_{2n-1}$,

$$\sum_{n\geq 1} r^B_n x^n = \sum_{n\geq 1} \frac{(2n)!x^n}{\prod_{k=1}^n (1+2k(2k+1)x)}, \tag{1.7}$$

thereby providing a type B analog of (1.5). We also obtain a type B analog of the Barsky-Dumont formula for the generating function of the Genocchi numbers.

Let ω be the primitive mth root of unity $e^{\frac{2\pi i}{m}}$. For $m,n \geq 1$, the *Dowling arrangement* is a hyperplane arrangement in $\mathbb{C}^n$ defined by

$$\{x_i - \omega^l x_j = 0 : 1 \leq i < j \leq n,\, 0 \leq l < m\} \cup \{x_i = 0 : 1 \leq i \leq n\}. \tag{1.8}$$

This is called a Dowling arrangement because its intersection lattice is isomorphic to the classical Dowling lattice $Q_n(\mathbb{Z}_m)$, which is isomorphic to Π_{n+1} when $m=1$, and to Π^B_n

[3]These are the signless Genocchi numbers; g_n is usually denoted $(-1)^{n+1}G_{2n}$ in the literature.

when $m = 2$. By introducing a Dowling analog of the homogenized Linial arrangement, we obtain unifying generalizations of the types A and B results discussed above. These generalizations include a polynomial analog of the formula $g_n = |\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0}, \hat{1})|$ involving a polynomial analog of the Genocchi numbers known as the Gandhi polynomials.

2 Preliminaries

Hyperplane Arrangements. Let k be a field (here k is $\mathbb{R}$ or $\mathbb{C}$). A *hyperplane arrangement* $\mathcal{A} \subseteq k^n$ is a finite collection of affine codimension-1 subspaces of k^n. The *intersection poset* of $\mathcal{A}$ is the poset $\mathcal{L}(\mathcal{A})$ of intersections of hyperplanes in $\mathcal{A}$ (viewed as affine subspaces of k^n), partially-ordered by reverse inclusion. If $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$ then the intersection poset is a geometric lattice, otherwise it's a geometric semilattice.

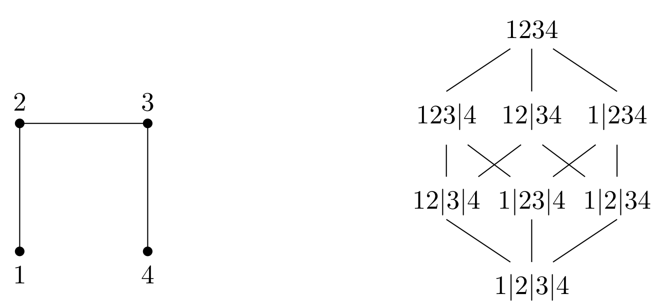
If $\mathcal{A}$ is a real hyperplane arrangement, then $\mathbb{R}^n \setminus \mathcal{A}$ is disconnected. By the number of regions of $\mathcal{A}$ we mean the number of connected components of $\mathbb{R}^n \setminus \mathcal{A}$. This number can be detected solely from $\mathcal{L}(A)$ as Zaslavsky's formula (1.2) shows.

If $\mathcal{A}$ is a complex hyperplane arrangement, its complement $M_{\mathcal{A}} := \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$ is a manifold whose Betti numbers β_i can be detected solely from $\mathcal{L}(\mathcal{A})$. Indeed, this follows from the formula of Orlik and Solomon [11, Theorem 5.2],

$$\sum_{i=0}^{n} \beta_i(M_{\mathcal{A}}) t^i = (-t)^r \chi_{\mathcal{L}(\mathcal{A})}(-t^{-1}), \tag{2.1}$$

where r is the length of $\mathcal{L}(\mathcal{A})$.

The Bond Lattice of a Graph. Let G be a graph on vertex set $[n]$. The *bond lattice of G* is the subposet Π_G of the partition lattice Π_n consisting of partitions $\pi = B_1|\cdots|B_k$ such that $G|_{B_i}$ is connected for all i. Note that Π_n is the bond lattice of the complete graph K_n. Another example is given below.



G $\qquad\qquad$ Π_G

Broken circuits provide a useful means of computing the Möbius function of the bond lattice of a graph (or more generally, of geometric lattices). Let $G = ([n], E)$ be a finite graph. Fix a total ordering of E and let S be a subset of E. Then S is called a *broken circuit* if it consists of a cycle in G with its least edge (with respect to this ordering) removed. If S does not contain a broken circuit, we say that S is a *non-broken circuit* or *NBC* set.

Given any $S \subseteq E$, let π_S be the partition of $[n]$ whose blocks are the vertex sets of the connected components of the graph $([n], S)$. The following formula is due to Whitney [18, Section 7] for graphs and Rota [14, Pg. 359] for general geometric lattices. For $\pi \in \Pi_G$,

$$(-1)^{\mathrm{rk}(\pi)}\mu(\hat{0}, \pi) = \#\{\text{NBC sets } S \text{ of } G : \pi_S = \pi\}. \tag{2.2}$$

Given a rooted tree whose vertex set is a subset of $\mathbb{Z}^+$, we say the tree is *increasing* if each nonroot vertex is larger than its parent. A rooted forest on a subset of $\mathbb{Z}^+$ is said to be *increasing* if it consists of increasing rooted trees. Note that if G is K_n then by ordering the edges lexicographically with the smallest element as the first component, the NBC sets of G are exactly the edge sets of the increasing forests on $[n]$.

Genocchi and median Genocchi Numbers. The Genocchi numbers and median Genocchi numbers are classical sequences of numbers that have been extensively studied in combinatorics. There are many ways to define them. Here we define them in terms of Dumont permutations; see [6, p. 44]. A *Dumont permutation* is a permutation $\sigma \in \mathfrak{S}_{2n}$ such that $2i > \sigma(2i)$ and $2i - 1 \leq \sigma(2i - 1)$ for all $i = 1, \ldots, n$. A *Dumont derangement* is a Dumont permutation without fixed points, i.e., $2i > \sigma(2i)$ and $2i - 1 < \sigma(2i - 1)$ for all $i = 1, \ldots, n$.

Example 2.1. When $n = 2$, the Dumont permutations on $[4]$ (in cycle form) are

$$(1,2)(3,4) \qquad (1,3,4,2) \qquad (1,4,2)(3).$$

When $n = 3$, the Dumont derangements on $[6]$ are:

$$\begin{array}{cccc} (1,3,5,6,4,2) & (1,3,4,2)(5,6) & (1,2)(3,4)(5,6) & (1,2)(3,5,6,4) \\ (1,4,3,5,6,2) & (1,5,6,3,4,2) & (1,5,6,2)(3,4) & (1,4,2)(3,5,6). \end{array}$$

For $n \geq 1$, the (signless) *Genocchi number* g_n is defined to be the number of Dumont permutations on $[2n - 2]$, and for $n \geq 0$, the *median Genocchi number* h_n is defined to be the number of Dumont derangements on $[2n + 2]$. We have,

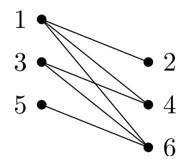
n	0	1	2	3	4	5	6
g_n		1	1	3	17	155	2073
h_n	1	2	8	56	608	9440	198272

3 The (type A) homogenized Linial arrangement

In this section we give a characterization of the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ as an induced subposet of Π_{2n} and compute its Möbius function.

3.1 The intersection lattice is a bond lattice

We begin by showing that $\mathcal{L}(\mathcal{H}_{2n-1})$ is isomorphic to the bond lattice of a nice bipartite graph. Let Γ_{2n} be the bipartite graph[4] on vertex set $\{1, 3, \dots, 2n-1\} \sqcup \{2, 4, \dots, 2n\}$ with an edge between $2i-1$ and $2j$ iff $i \le j$. The graph Γ_6 is given below.



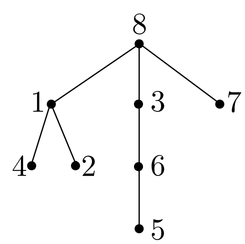
Theorem 3.1. *The intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ is isomorphic to the bond lattice $\Pi_{\Gamma_{2n}}$, which is the induced subposet of Π_{2n} consisting of the partitions $X = B_1 | \cdots | B_k$ in which $\min(B_i)$ is odd and $\max(B_i)$ is even for all nonsingleton B_i.*

In [10], we prove Theorem 3.1 by constructing an invertible $\mathbb{Z}$-linear map that sends $\mathcal{H}_{2n-1}$ to an arrangement whose intersection poset is $\Pi_{\Gamma_{2n}}$.

We use the Rota-Whitney formula (2.2) to compute the Möbius function of $\Pi_{\Gamma_{2n}}$. Our NBC sets have a nice description which we give now. We say that a rooted forest on node set $A \subset \mathbb{Z}^+$ is *increasing-decreasing (ID)* if the trees are rooted at their largest node and for each internal node a,

- if a is odd then a is less than all its descendants and all its children are even,
- if a is even then a is greater than all its descendants and all its children are odd.

$T =$



Theorem 3.2. *For all $\pi \in \Pi_{\Gamma_{2n}}$, we have that $(-1)^{|\pi|}\mu(\hat{0}, \pi)$ equals the number of ID forests on $[2n]$ whose trees have nodes sets equal to the blocks of π. Consequently*

$$\chi_{\mathcal{H}_{2n-1}}(t) = \sum_{k=0}^{2n} (-1)^k f_{2n,k} t^{k-1}.$$

where $f_{2n,k}$ is the number of ID forests on $[2n]$ with exactly k trees.

To prove this, we show that under an appropriate ordering of the edges of Γ_n, the NBC sets of Γ_{2n} are the edge sets of the ID forests on $[2n]$.[5]

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[4] The graph Γ_{2n} belongs to a class of graphs called Ferrers graphs, which were introduced by Ehrenborg and van Willegenburg [7] and further studied in [3], [15]. We have been able to extend some of our results to more general Ferrers graphs and to skew Ferrers graphs.

[5] This connection between NBC sets and ID forests is also implicit in the proof of Theorem 7.3 of [15].

3.2 Dumont-like permutations

Our next step is to introduce a class of permutations similar to the Dumont permutations and then give a bijection between these permutations in $\mathfrak{S}_{2n}$ and the ID forests on $[2n]$.

Let A be a finite subset of $\mathbb{Z}^+$. We say $\sigma \in \mathfrak{S}_A$ is a *D-permutation* on A if $i \leq \sigma(i)$ whenever i is odd and $i \geq \sigma(i)$ whenever i is even. We denote by $\mathcal{D}_A$ the set of D-permutations on A and by $\mathcal{DC}_A$ the set of D-cycles on A. If $A = [n]$, we write $\mathcal{D}_n$ and $\mathcal{DC}_n$.

Note that all Dumont permutations are D-permutations, but not conversely. Indeed, the only difference between the two classes of permutations on $\mathfrak{S}_{2n}$ is that fixed points can be even or odd in a D-permutation, while only odd fixed points are allowed in a Dumont permutation. It follows immediately from the definitions that

$$\mathcal{DC}_{2n} \subseteq \{\text{Dumont derangements in } \mathfrak{S}_{2n}\} \subseteq \{\text{Dumont permutations in } \mathfrak{S}_{2n}\} \subseteq \mathcal{D}_{2n}.$$

Recall that the two sets in the middle of this chain are enumerated by median Genocchi number h_{n-1} and Genocchi number g_{n+1}, respectively. It turns out that the sets on the ends of the chain are also enumerated by Genocchi and median Genocchi numbers. Indeed, there is a bijection between Dumont permutations and D-cycles that yields,

$$|\mathcal{DC}_{2n}| = g_n,$$

for all $n \geq 1$. We use the theory of surjective pistols discussed in [13] and [6] to prove the following result.

Theorem 3.3. *For all $n \geq 0$, $|\mathcal{D}_{2n}| = h_n$.*

Next we define a bijection ψ_A from the set $\mathcal{T}_A$ of ID trees on A to $\mathcal{DC}_A$. For $T \in \mathcal{T}_A$, order the children of each even node in increasing order and the children of each odd node in decreasing order. This turns T into a rooted planar tree, which can be traversed in postorder. Let $\alpha := \alpha_1, \cdots, \alpha_{|A|}$ be the word obtained by traversing T in postorder, that is, α_i is the ith node of T in postorder. Now let $\psi_A(T)$ be the permutation whose cycle form is (α). For the ID tree T given in Section 3.1, we have $\psi_{[8]}(T) = (4,2,1,5,6,3,7,8)$.

Lemma 3.4. *For all $A \subseteq [2n]$, the map $\psi_A : \mathcal{T}_A \to \mathcal{DC}_A$ is a well-defined bijection. Consequently $|\mathcal{T}_A| = |\mathcal{DC}_A|$.*

The *cycle support* of $\sigma \in \mathfrak{S}_n$ is the partition $\operatorname{cyc}(\sigma) \in \Pi_n$ whose blocks are comprised of the elements of the cycles of σ. For example, $\operatorname{cyc}((1,7,2,4)(5)(6,8,9,3)) = 1247|5|3689$. As a consequence of Theorem 3.2 and Lemma 3.4, we have the following result.

Theorem 3.5. *For $\pi \in \Pi_{\Gamma_{2n}}$, where $n \geq 1$,*

$$(-1)^{|\pi|}\mu_{\Pi_{\Gamma_{2n}}}(\hat{0}, \pi) = |\{\sigma \in \mathcal{D}_{2n} \mid \operatorname{cyc}(\sigma) = \pi\}|.$$

Consequently,

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t) = \sum_{k=1}^{2n} s_D(2n,k)t^{k-1}, \tag{3.1}$$

where $(-1)^k s_D(2n,k)$ is equal to the number of D-permutations on $[2n]$ with exactly k cycles.

Next we invoke Theorem 3.3. By setting $t = -1$ in (3.1) we recover Hetyei's result (1.4) that $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1) = -h_n$, and by setting $t = 0$ we obtain the following new result on the Genocchi numbers.[6]

Theorem 3.6. *For all $n \geq 1$,*

$$\mu_{\mathcal{L}(\mathcal{H}_{2n-1})}(\hat{0},\hat{1}) = -|\mathcal{DC}_{2n}| = -g_n.$$

In the full version of the paper [10], we use (3.1) and the theory of surjective pistols in [13] to derive the following result.

Theorem 3.7.

$$\sum_{n\geq 1} \chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)\, x^n = \sum_{n\geq 1} \frac{(t-1)_n(t-1)_{n-1}\, x^n}{\prod_{k=1}^{n}(1-k(t-k)x)}, \tag{3.2}$$

where $(a)_n$ denotes the falling factorial $a(a-1)\cdots(a-n+1)$.

Equation (3.2) reduces to a formula of Barsky and Dumont [2, Lemma 2] for the Genocchi numbers when t is set equal to 0 and to the formula of Barsky and Dumont for the median Genocchi numbers given in (1.5) when t is set equal to -1.

We are also able to obtain the following characterization of the median Genocchi numbers by evaluating $\chi_{\Pi_{\Gamma_{2n}}}(t)$ in another way.

Theorem 3.8. *For all $n \geq 1$, h_n is equal to the number of permutations σ on $[2n]$ whose descents $\sigma(i) > \sigma(i+1)$ occur only when $\sigma(i)$ is even and $\sigma(i+1)$ is odd.*

We now give yet another way in which the median Genocchi numbers arise.

Theorem 3.9. [7] *For all $n \geq 3$,*

$$\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t) = (t-1)^3\chi_{P_n}(t),$$

where P_n is the intersection semilattice of a certain affine hyperplane arrangement in $\mathbb{R}^{2n-4}$. Moreover, $|\chi_{P_n}(1)| = h_{n-3}$; hence the number of bounded regions of this affine arrangement is h_{n-3}.

In the full version of the paper, we compute $\chi_{P_n}(1)$ by applying the theory of shellability to the NBC complex of the geometric semilattice P_n. The consequence follows from Zaslavsky's result on the number of bounded regions of an affine arrangement [19].

[6] For another class of induced subposets of the partition lattice whose Möbius invariant can be expressed in terms of the Genocchi numbers, see [17].

[7] This result was also independently conjectured by Hetyei (personal communication).

4 The homogenized Linial-Dowling arrangement

In this section, we extend the results of the previous section to the Dowling arrangements, which generalize the complexified types A and B braid arrangements.

The *Dowling lattice* $Q_n(\mathbb{Z}_m)$ consists of labeled partitions $B_0|B_1|\ldots|B_k$ of $\{0\} \cup [n]$ such that

- $0 \in B_0$ (B_0 is called the *zero block*),
- the elements of B_i, $i \geq 1$, are labeled with elements of $\{0, 1, \ldots, m-1\}$ and $\min(B_i)$ is labeled with 0.

The cover relation is given by merging blocks as follows. Let $B_0|B_1|\cdots|B_k \in Q_n(\mathbb{Z}_m)$.

- If B_0 and B_i merge, erase all of the labels from B_i and merge the blocks as in Π_n to obtain a new zero block B_0'.
- Suppose $i, j \neq 0$, and $\min(B_i) < \min(B_j)$. There are m ways to merge B_i and B_j. For each $\ell \in \{0, \ldots, m-1\}$, when B_i and B_j merge, the labels of the elements of B_i remain unchanged, while ℓ is added mod m to the labels of the elements of B_j.

Example 4.1. Suppose $m = 3$. Then $05|1^0 3^1|2^0 4^2$ is covered by

$$0135|2^0 4^2 \qquad 0245|1^0 3^1 \qquad 05|1^0 2^0 3^1 4^2 \qquad 05|1^0 2^1 3^1 4^0 \qquad 05|1^0 2^2 3^1 4^1.$$

It is not hard to see that for all $m \geq 1$, the Dowling lattice $Q_n(\mathbb{Z}_m)$ is isomorphic to the intersection lattice of the Dowling arrangement defined in (1.8). See [4] and [8] for further information on Dowling lattices.

Now we introduce a Dowling analog of the homogenized Linial arrangement. Let $\omega = e^{2\pi i/m}$ be a primitive mth root of unity. The *homogenized Linial-Dowling arrangement* is the complex hyperplane arrangement

$$\mathcal{H}_{2n-1}^m = \{x_i - \omega^\ell x_j = y_i \mid 1 \leq i < j \leq n,\ 0 \leq \ell < m\} \cup \{x_i = y_i \mid 1 \leq i \leq n\} \subseteq \mathbb{C}^{2n}.$$

Note that when $m = 2$, the arrangement $\mathcal{H}_{2n-1}^m$ is a complexified version of the type B homogenized Linial arrangement $\mathcal{H}_{2n-1}^B$ defined in the introduction. When $m = 1$, the arrangement $\mathcal{H}_{2n-1}^m$ is the complexified version of the arrangement obtained by intersecting $\mathcal{H}_{2n-1}$ with the coordinate hyperplane $x_{n+1} = 0$. The resulting arrangement on the coordinate hyperplane has the same intersection lattice as $\mathcal{H}_{2n-1}$.

The proof of the following result is similar to that of the type A version.

Theorem 4.2. *For all $n, m \geq 1$, the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1}^m)$ is isomorphic to the induced subposet $\mathcal{L}_{2n-1}^m$ of $Q_{2n-1}(\mathbb{Z}_m)$ consisting of all labeled partitions such that*

- *for nonsingleton B_0, $\min(B_0 \setminus \{0\})$ is odd,*

- *for all nonsingleton B_i, with $i > 0$,* $\min(B_i)$ *is odd and* $\max(B_i)$ *is even.*

To compute the Möbius function of the geometric lattice $\mathcal{L}(\mathcal{H}^m_{2n-1})$, we apply the Rota-Whitney formula (2.2) to $\mathcal{L}^m_{2n-1}$ and then we construct a bijection from the NBC sets of $\mathcal{L}^m_{2n-1}$ to the class of m-labeled D-permutations, which we now define.

An m-labeled D-permutation σ on $[2n]$ is a D-permutation whose entries are decorated with elements of $\{0, 1, \ldots, m-1\}$ such that

- cycle minima are labeled 0,
- if $(a_1, a_2, \ldots, a_r = 2n)$ is the cycle of σ containing $2n$ then all right-to left minima of the word $a_1 a_2 \ldots a_r$ are labeled 0.

For example, let $n = 5$ and $\sigma = (3, 7, 8, 5, 9, 10)(1, 4, 2)(6)$. Since the right-to-left minima of the first cycle are $10, 9, 5, 3$, they must all be labeled 0. Since 1 and 6 are the minima of their respective cycles, they must also be labeled 0. Hence σ with the labeling $(3^0, 7^*, 8^*, 5^0, 9^0, 10^0)(1^0, 4^*, 2^*)(6^0)$, where $*$ denotes any label in $\{0, 1, 2\}$, is a 3-labeled D-permutation.

We write $\mathcal{D}^m_{2n}$ for the set of m-labeled D-permutations on $[2n]$ and $\mathcal{DC}^m_{2n}$ for the set of m-labeled D-cycles on $[2n]$. The *cycle support* of $\sigma \in \mathcal{D}^m_{2n}$ is the m-labeled partition $\mathrm{cyc}(\sigma) = B_0 | \cdots | B_k \in Q_n(\mathbb{Z}_m)$ obtained from σ as follows:

- The set of entries of the cycle of σ that contains $2n$ gives rise to the zero block B_0, with 2n replaced by 0 and all labels removed.
- Each cycle of σ that doesn't contain $2n$ gives rise to a labeled block B for which the labels of the entries of B are the same as the labels of the entries of the cycle.

For example, if $\sigma = (1^0 3^1 4^1 2^2)(5^0)(6^0)(7^0 8^0)$ then $\mathrm{cyc}(\sigma) = 07 | 1^0 2^2 3^1 4^1 | 5^0 | 6^0$.

Theorem 4.3. *Let $n, m \geq 1$. For all $\pi \in \mathcal{L}^m_{2n-1}$,*

$$(-1)^{|\pi|} \mu_{\mathcal{L}^m_n}(\hat{0}, \pi) = |\{\sigma \in \mathcal{D}^m_{2n} \mid \mathrm{cyc}(\sigma) = \pi\}|.$$

Consequently,

$$\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t) = \sum_{k=1}^{2n} s_{D,m}(2n, k) t^{k-1}, \tag{4.1}$$

where $(-1)^k s_{D,m}(2n, k)$ is equal to the number of $\sigma \in \mathcal{D}^m_{2n}$ with exactly k cycles, and

$$\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(-1) = -|\mathcal{D}^m_{2n}| \qquad \text{and} \qquad \mu_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(\hat{0}, \hat{1}) = -|\mathcal{DC}^m_{2n}|.$$

By the Orlik-Solomon formula (2.1), equation (4.1) gives a combinatorial formula for the the Betti numbers of the complement of $\mathcal{H}^m_{2n-1}$ in $\mathbb{C}^{2n}$.

In the full version of the paper [10], we use (4.1) and the theory of surjective pistols in [13] to derive the following m-analog of (3.2). Note that this reduces to (1.7) when we set $m = 2$ and $t = -1$.

Theorem 4.4. *For all $m \geq 1$,*

$$\sum_{n \geq 1} \chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t)\, x^n = \sum_{n \geq 1} \frac{(t-1)_{n,m}(t-m)_{n-1,m}\, x^n}{\prod_{k=1}^n (1 - mk(t-mk)x)}. \tag{4.2}$$

where $(a)_{n,m} = a(a-m)(a-2m)\cdots(a-(n-1)m)$.

There is a well-studied polynomial analog of the Genocchi numbers known as the Gandhi polynomials; see [6, Section 3]. They are defined by $G_1(x) = x^2$ and $G_n(x) = x^2(G_{n-1}(x+1) - G_{n-1}(x))$ for $n \geq 2$. We obtain the following m-analog of Theorem 3.6.

Theorem 4.5. *For all $n \geq 1$,*

$$\mu_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(\hat{0}, \hat{1}) = -m^{2n-1} G_n(m^{-1}).$$

We also obtain an m-analog of Theorem 3.9, in which the intersection semilattice P^m_n of a certain affine arrangement in $\mathbb{C}^{2n-4}$ satisfies

$$\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t) = (t-m)(t-1)^2 \chi_{P^m_n}(t).$$

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