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Quantum nilpotent subalgebras of classical quantum groups and affine crystals

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Abstract. We give a new interpretation of RSK correspondence of type D in terms of affine crystals. We show that the crystal of quantum nilpotent subalgebra of $U_q(D_n)$ associated to a maximal Levi subalgebra of type A_{n-1} has an affine $D_n^{(1)}$-crystal structure, and it is isomorphic to a direct limit of perfect Kirillov-Reshetikhin crystal $B^{n,s}$ for $s \geq 1$. An analogue of RSK correspondence for type D due to Burge is naturally defined on this crystal and shown to be an isomorphism of affine crystals. We further obtain a generalization of Greene's formula for type D and as a byproduct a new polytope realization of $B^{n,s}$.

Keywords: quantum groups, quantum nilpotent subalgebra, crystal graphs

1 Introduction

Let $\mathfrak{g}$ be a classical Lie algebra and let $\mathfrak{l}$ be its proper maximal Levi subalgebra of type A (or a sum of type A). Let $\mathfrak{u}^-$ be the negative nilradical of the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. The enveloping algebra $U(\mathfrak{u}^-)$ is an integrable $\mathfrak{l}$-module, which has a multiplicity-free decomposition [8], and the expansion of its character

$$\mathrm{ch}\, U(\mathfrak{u}^-) = \prod_{\alpha \in \Phi(\mathfrak{u}^-)} (1 - e^{\alpha})^{-1} \tag{1.1}$$

into irreducible $\mathfrak{l}$-characters (that is, Schur polynomials or a product of Schur polynomials) gives the celebrated Cauchy identity when $\mathfrak{g}$ is of type A, and Littlewood identities when $\mathfrak{g}$ is of type B, C, D, where $\Phi(\mathfrak{u}^-)$ is the set of roots of $\mathfrak{u}^-$.

The decomposition of $U(\mathfrak{u}^-)$ into $\mathfrak{l}$-modules has a purely combinatorial interpretation by RSK correspondence and its variations, say κ (cf. [2, 5]). In [16], Lascoux showed that

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κ is an isomorphism of $\mathfrak{l}$-crystals, which immediately implies the same result for type B and C [14] by using similarity of crystals . Furthermore, it is shown in [15] that the RSK correspondence κ can be extended to an isomorphism of affine crystals of type $A_n^{(1)}$ when $\mathfrak{g}$ is of type A_n, and of type $D_{n+1}^{(2)}$ and $C_n^{(1)}$ when $\mathfrak{g}$ is of type B_n and C_n, respectively.

In this paper, we establish an analogue of the above result when $\mathfrak{g}$ is of type D. First, we consider the crystal $\mathbf{B}_{\mathbf{i}_0}$ of $\mathbf{i}_0$-Lusztig data, where $\mathbf{i}_0$ is a reduced expression associated to a specific convex order on the set of positive roots of $\mathfrak{g}$. The subcrystal $B(U_q(\mathfrak{u}^-))$ of $\mathbf{B}_{\mathbf{i}_0}$ consisting of Lusztig data on $\Phi(\mathfrak{u}^-)$ has a nice combinatorial realization, and naturally admits an affine crystal structure of type $D_n^{(1)}$ isomorphic to a direct limit of KR crystals $B^{n,s}$ for $s \geq 1$. We give an explicit description of $B^{n,s} \subset B(U_q(\mathfrak{u}^-))$ in terms of double paths on $\Phi(\mathfrak{u}^-)$, which yields a polytope realization of $B^{n,s}$ (Theorem 3.10). Next, we consider an analogue of RSK correspondence for type D due to Burge [2]. We prove that it is an isomorphism of affine crystals of type $D_n^{(1)}$, where a suitable affine crystal structure is defined on the side of tableaux (Theorem 4.1). Furthermore, we present an interesting formula for the shape of a semistandard tableau corresponding to a Lusztig datum on $\Phi(\mathfrak{u}^-)$ in terms of non-intersecting double paths on $\Phi(\mathfrak{u}^-)$ (Theorem 4.4). A full version of this paper including detailed proofs has appeared in [9].

2 PBW crystals

2.1 PBW basis and crystals

We refer the reader to [11, 10] for definitions of crystal base and crystal. Suppose that $\mathfrak{g}$ is of finite type and $U_q(\mathfrak{g})$ is the associated quantized enveloping algebra (see [7, 18]). Let us briefly recall the notion of PBW basis and the crystal of Lusztig data which is isomorphic to $B(\infty)$ (see [17, 19]). Let W be the Weyl group of $\mathfrak{g}$ generated by the simple reflections s_i for $i \in I$. Let w_0 be the longest element in W of length N, and let $R(w_0) = \{\, \mathbf{i} = (i_1, \ldots, i_N) \,|\, w_0 = s_{i_1} \ldots s_{i_N} \,\}$ be the set of reduced expressions of w_0. For $i \in I$, let T_i be the $\mathbb{Q}(q)$-algebra automorphism of $U_q(\mathfrak{g})$, which is given as $T''_{i,1}$ in [18].

For $\mathbf{i} \in R(w_0)$,

$$\Phi^+ = \{\beta_1 := \alpha_{i_1}, \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})\} \tag{2.1}$$

is the set of positive roots of $\mathfrak{g}$. For $\mathbf{c} = (c_{\beta_1}, \ldots, c_{\beta_N}) \in \mathbb{Z}_+^N$. For $1 \leq k \leq N$, put $f_{\beta_k} := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(f_{i_k})$ and let $b_{\mathbf{i}}(\mathbf{c}) = f_{\beta_1}^{(c_{\beta_1})} f_{\beta_2}^{(c_{\beta_2})} \cdots f_{\beta_N}^{(c_{\beta_N})}$. Then the set $B_{\mathbf{i}} := \{\, b_{\mathbf{i}}(\mathbf{c}) \,|\, \mathbf{c} \in \mathbb{Z}_+^N \,\}$ is a $\mathbb{Q}(q)$-basis of $U_q^-(\mathfrak{g})$ called a *PBW basis*.

Let A_0 be the subring of $\mathbb{Q}(q)$ consisting of rational functions regular at $q = 0$. The A_0-lattice $L(\infty)$ of $U_q^-(\mathfrak{g})$ generated by $B_{\mathbf{i}}$ is independent of the choice of $\mathbf{i}$ and invariant under $\widetilde{e}_i$, $\widetilde{f}_i$, and the induced crystal $\pi(B_{\mathbf{i}})$ under a canonical projection $\pi : L(\infty) \to$

$L(\infty)/qL(\infty)$ is isomorphic to $B(\infty)$. We identify $\mathbf{B_i} := \mathbb{Z}_+^N$ with a crystal $\pi(B_{\mathbf{i}})$ under the map $\mathbf{c} \mapsto b_{\mathbf{i}}(\mathbf{c})$, and call $\mathbf{c} \in \mathbf{B_i}$ an $\mathbf{i}$-*Lusztig datum*.

Let $w \in W$ be given with length r. There exists $i = (i_1, \cdots, i_N) \in R(w_0)$ such that $w = s_{i_1} \cdots s_{i_r}$ by the properties of the Bruhat order. The $\mathbb{Q}(q)$-subspace of $U_q^-(\mathfrak{g})$ spanned by $b_{\mathbf{i}}(\mathbf{c})$ for $\mathbf{c} \in \mathbf{B_i}$ with $c_k = 0$ for $r+1 \le k \le N$ is a subalgebra called the *quantum nilpotent subalgebra associated to* $w \in W$ and denoted by $U_q^-(w)$ (see for example, [13] and references therein).

2.2 Description of $\tilde{f}_i$

Let $\mathbf{i} \in R(w_0)$ be given. For $\beta \in \Phi^+$, we denote by $\mathbf{1}_\beta$ the element in $\mathbf{B_i}$ where $c_\beta = 1$ and $c_\gamma = 0$ for $\gamma \in \Phi^+ \setminus \{\beta\}$. The Kashiwara operators $\widetilde{f}_i$ or $\widetilde{f}_i^*$ on $\mathbf{B_i}$ for $i \in I$ is not easy to describe in general except

$$\begin{aligned} \widetilde{f}_i\mathbf{c} &= (c_{\beta_1}+1, c_{\beta_2}, \ldots, c_{\beta_N}) = \mathbf{c} + \mathbf{1}_{\alpha_i}, \quad \text{when } \beta_1 = \alpha_i, \\ \widetilde{f}_i^*\mathbf{c} &= (c_{\beta_1}, \ldots, c_{\beta_{N-1}}, c_{\beta_N}+1) = \mathbf{c} + \mathbf{1}_{\alpha_i}, \quad \text{when } \beta_N = \alpha_i, \end{aligned} \tag{2.2}$$

for $\mathbf{c} \in \mathbf{B_i}$ [18].

Let us review the results in [21], which plays an important role in our paper. For simplicity, let us assume that $\mathfrak{g}$ is simply laced. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ be a sequence with $\sigma_u \in \{+, -, \cdot\}$. We replace a pair $(\sigma_u, \sigma_{u'}) = (+, -)$, where $u < u'$ and $\sigma_{u''} = \cdot$ for $u < u'' < u'$, with $(\cdot, \cdot)$, and repeat this process as far as possible until we get a sequence with no $-$ placed to the right of $+$. We denote the resulting sequence by σ^{red}. For another sequence $\tau = (\tau_1, \ldots, \tau_t)$, we denote by $\sigma \cdot \tau$ the concatenation of σ and τ.

Recall that a total order $\prec$ on Φ^+ is called *convex* if either $\gamma \prec \gamma' \prec \gamma''$ or $\gamma'' \prec \gamma' \prec \gamma$ whenever $\gamma' = \gamma + \gamma''$ for $\gamma, \gamma', \gamma'' \in \Phi^+$. It is well-known that there exists a one-to-one correspondence between $R(w_0)$ and the set of convex orders on Φ^+, where the convex order $\prec$ associated to $\mathbf{i} = (i_1, \ldots, i_N) \in R(w_0)$ is given by

$$\beta_1 \prec \beta_2 \prec \ \ldots \prec \beta_N, \tag{2.3}$$

where β_k is as in (2.1). Recall that there exists a reduced expression $\mathbf{i}'$ obtained from $\mathbf{i}$ by a 3-term braid move $(i_k, i_{k+1}, i_{k+2}) \to (i_{k+1}, i_k, i_{k+1})$ with $i_k = i_{k+2}$ if and only if $\{\beta_k, \beta_{k+1}, \beta_{k+2}\}$ forms the positive roots of type A_2, where the corresponding convex order $\prec'$ is given by replacing $\beta_k \prec \beta_{k+1} \prec \beta_{k+2}$ with $\beta_{k+2} \prec' \beta_{k+1} \prec' \beta_k$. Also there exists a reduced expression $\mathbf{i}'$ obtained from $\mathbf{i}$ by a 2-term braid move $(i_k, i_{k+1}) \to (i_{k+1}, i_k)$ if and only if β_k and β_{k+1} are orthogonal, where the associated convex ordering $\prec'$ is given by replacing $\beta_k \prec \beta_{k+1}$ with $\beta_{k+1} \prec' \beta_k$.

Given $i \in I$, suppose that $\mathbf{i}$ is *simply braided for* $i \in I$, that is, if one can obtain $\mathbf{i}' = (i'_1, \ldots, i'_N) \in R(w_0)$ with $i'_1 = i$ by applying a sequence of braid moves consisting of either a 2-term move or 3-term braid move $(\gamma,\gamma',\gamma'') \to (\gamma'',\gamma',\gamma)$ with $\gamma'' = \alpha_i$. Suppose that

$$\Pi_s = \{\gamma_s, \gamma'_s, \gamma''_s\} \tag{2.4}$$

is the triple of positive roots of type A_2 with $\gamma'_s = \gamma_s + \gamma''_s$ and $\gamma''_s = \alpha_i$ corresponding to the s-th 3-term braid move for $1 \le s \le t$.

For $\mathbf{c} \in \mathbf{B_i}$, let $\sigma_i(\mathbf{c}) = (\underbrace{-\cdots-}_{c_{\gamma'_1}}\ \underbrace{+\cdots+}_{c_{\gamma_1}}\ \cdots\ \underbrace{-\cdots-}_{c_{\gamma'_t}}\ \underbrace{+\cdots+}_{c_{\gamma_t}})$.

Theorem 2.1. *[21, Theorem 4.6] Let $\mathbf{i} \in R(w_0)$ and $i \in I$. Suppose that $\mathbf{i}$ is simply braided for i. Let $\mathbf{c} \in \mathbf{B_i}$ be given.*

(1) If there exists $+$ in $\sigma_i(\mathbf{c})^{\mathrm{red}}$ and the leftmost $+$ appears in c_{γ_s}, then $\widetilde{f}_i\mathbf{c} = \mathbf{c} - \mathbf{1}_{\gamma_s} + \mathbf{1}_{\gamma'_s}$.

(2) If there exists no $+$ in $\sigma_i(\mathbf{c})^{\mathrm{red}}$, then $\widetilde{f}_i\mathbf{c} = \mathbf{c} + \mathbf{1}_{\alpha_i}$.

3 Crystal of quantum nilpotent subalgebra

3.1 Crystal $\mathbf{B}_{\mathbf{i}_0}$

From now on, we assume that $\mathfrak{g}$ is of type D_n ($n \ge 4$). We assume that the weight lattice is $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$, where $\{\,\epsilon_i \,|\, 1 \le i \le n\,\}$ is an orthonormal basis with respect to a symmetric bilinear form $(\ ,\ )$. The set of positive roots is $\Phi^+ = \{\,\epsilon_i \pm \epsilon_j \,|\, 1 \le i < j \le n\,\}$ and let α_i be the i-th simple root given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \ne n$, and $\alpha_n = \epsilon_{n-1} + \epsilon_n$. Recall that W acts faithfully on P by $s_i(\epsilon_i) = \epsilon_{i+1}$, $s_i(\epsilon_k) = \epsilon_k$ for $1 \le i \le n-1$ and $k \ne i, i+1$, and $s_n(\epsilon_{n-1}) = -\epsilon_n$ and $s_n(\epsilon_k) = \epsilon_k$ for $k \ne n-1, n$. The fundamental weights are $\varpi_i = \sum_{k=1}^i \epsilon_k$ for $i = 1, \ldots, n-2$, $\varpi_{n-1} = (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2$ and $\varpi_n = (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2$. Put $J = I \setminus \{n\}$. Let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{g}$ associated to $\{\,\alpha_i \,|\, i \in J\,\}$ of type A_{n-1}. Then

$$\Phi^+ = \Phi^+(J) \cup \Phi^+_J,$$

where $\Phi^+_J = \{\,\epsilon_i - \epsilon_j \,|\, 1 \le i < j \le n\,\}$ is the set of positive roots of $\mathfrak{l}$ and $\Phi^+(J) = \{\,\epsilon_i + \epsilon_j \,|\, 1 \le i < j \le n\,\}$ is the set of roots of the nilradical $\mathfrak{u}$ of the parabolic subalgebra associated to $\mathfrak{l}$.

Throughout this paper, we consider a specific $\mathbf{i}_0 \in R(w_0)$, whose associated convex order on Φ^+ is given by

$$\begin{aligned} &\epsilon_i + \epsilon_j \prec \epsilon_k - \epsilon_l, \\ &\epsilon_i + \epsilon_j \prec \epsilon_k + \epsilon_l \iff (j > l) \text{ or } (j = l,\, i > k), \\ &\epsilon_i - \epsilon_j \prec \epsilon_k - \epsilon_l \iff (i < k) \text{ or } (i = k,\, j < l), \end{aligned} \tag{3.1}$$

for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. An explicit form of $\mathbf{i}_0$ is as follows. For $1 \leq k \leq n-1$, put

$$\mathbf{i}_k = \begin{cases} (n, n-2, \ldots, k+1, k), & \text{if } k \text{ is odd,} \\ (n-1, n-2, \ldots, k+1, k), & \text{if } k \text{ is even,} \\ (n), & \text{if } n \text{ is even and } k = n-1, \end{cases}$$

$$\mathbf{i}'_k = \begin{cases} (n-1, n-2, \ldots, k+1, k), & \text{if } n \text{ is even and } 1 \leq k \leq n-1, \\ (n, n-2, \ldots, k+1, k), & \text{if } n \text{ is odd and } 1 \leq k \leq n-2, \\ (n), & \text{if } n \text{ is odd and } k = n-1. \end{cases}$$

Let $\mathbf{i}^J = \mathbf{i}_1 \cdot \cdots \cdot \mathbf{i}_{n-1}$ and $\mathbf{i}_J = \mathbf{i}'_1 \cdot \cdots \cdot \mathbf{i}'_{n-1}$. Then $\mathbf{i}_0$ as the concatenation $\mathbf{i}^J \cdot \mathbf{i}_J$. We write $\mathbf{i}_0 = (i_1, \ldots, i_N)$, where $i_1 = n$, and put $\mathbf{i}^J = (i_1, \ldots, i_M)$, $\mathbf{i}_J = (i_{M+1}, \ldots, i_N)$ with $N = n^2 - n$ and $M = N/2$. Throughout the paper, we set $\mathbf{B} := \mathbf{B}_{\mathbf{i}_0}$. For $\mathbf{c} = (c_\beta) \in \mathbf{B}$, we also write

$$c_{\beta_k} = \begin{cases} c_{\overline{j}\,\overline{i}}, & \text{if } \beta_k = \epsilon_i + \epsilon_j \text{ for } 1 \leq i < j \leq n, \\ c_{j\,\overline{i}}, & \text{if } \beta_k = \epsilon_i - \epsilon_j \text{ for } 1 \leq i < j \leq n. \end{cases}$$

Proposition 3.1.

(1) *The reduced word $\mathbf{i}_0$ is simply braided for any $i \in I$.*

(2) *For $i \in I \setminus \{n\}$ and $\mathbf{c} \in \mathbf{B}$, we have $\sigma_i(\mathbf{c}) = \sigma_{i,1}(\mathbf{c}) \cdot \sigma_{i,2}(\mathbf{c}) \cdot \sigma_{i,3}(\mathbf{c})$, where*

$$\begin{aligned} \sigma_{i,1}(\mathbf{c}) =& (\underbrace{-\cdots-}_{c_{\overline{n}\,\overline{i}}}\ \underbrace{+\cdots+}_{c_{\overline{n}\,\overline{i+1}}}\underbrace{-\cdots-}_{c_{\overline{n-1}\,\overline{i}}}\ \underbrace{+\cdots+}_{c_{\overline{n-1}\,\overline{i+1}}}\ \cdots\ \underbrace{-\cdots-}_{c_{\overline{i+2}\,\overline{i}}}\ \underbrace{+\cdots+}_{c_{\overline{i+2}\,\overline{i}}}), \\ \sigma_{i,2}(\mathbf{c}) =& (\underbrace{-\cdots-}_{c_{\overline{i}\,\overline{i-1}}}\ \underbrace{+\cdots+}_{c_{\overline{i+1}\,\overline{i-1}}}\underbrace{-\cdots-}_{c_{\overline{i}\,\overline{i-2}}}\ \underbrace{+\cdots+}_{c_{\overline{i+1}\,\overline{i-2}}}\ \cdots\ \underbrace{-\cdots-}_{c_{\overline{i}\,\overline{1}}}\ \underbrace{+\cdots+}_{c_{\overline{i+1}\,\overline{1}}}), \\ \sigma_{i,3}(\mathbf{c}) =& (\underbrace{-\cdots-}_{c_{i+1\,\overline{1}}}\ \underbrace{+\cdots+}_{c_{i\,\overline{1}}}\underbrace{-\cdots-}_{c_{i+1\,\overline{2}}}\ \underbrace{+\cdots+}_{c_{i\,\overline{2}}}\ \cdots\ \underbrace{-\cdots-}_{c_{i+1\,\overline{i-1}}}\ \underbrace{+\cdots+}_{c_{i\,\overline{i-1}}}\underbrace{-\cdots-}_{c_{i+1\,\overline{i}}}). \end{aligned} \tag{3.2}$$

Here c_{ab} is assumed to be zero when it is not defined.

Remark 3.2. *Although we need some 2-term moves on $\mathbf{i}^J$ on showing (1) in Proposition 3.1 (see [9, Example 3.3]), we can apply [21, Theorem 4.6] for $\mathbf{i}_0$ because the bracketing rule is local.*

Set

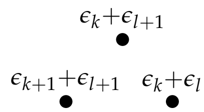
$$\begin{aligned} \mathbf{B}^J &= \{\, \mathbf{c} = (c_\beta) \in \mathbf{B} \,|\, c_\beta = 0 \text{ unless } \beta \in \Phi^+(J) \,\}, \\ \mathbf{B}_J &= \{\, \mathbf{c} = (c_\beta) \in \mathbf{B} \,|\, c_\beta = 0 \text{ unless } \beta \in \Phi_J \,\}. \end{aligned} \tag{3.3}$$

which we regard them as subcrystals of $\mathbf{B}$, where we assume that $\widetilde{e}_n\mathbf{c} = \widetilde{f}_n\mathbf{c} = \mathbf{0}$ with $\varepsilon_n(\mathbf{c}) = \varphi_n(\mathbf{c}) = -\infty$ for $\mathbf{c} \in \mathbf{B}_J$. The subcrystal $\mathbf{B}^J$ is the crystal of the quantum nilpotent subalgebra $U_q^-(w^J)$, where $w^J = s_{i_1} \cdots s_{i_M}$ with $\mathbf{i}^J = (i_1, \ldots, i_M)$, which can be viewed as a q-deformation of $U(\mathfrak{u}^-)$ by definition.

Corollary 3.3. *The map* $\mathbf{B} \longrightarrow \mathbf{B}^J \otimes \mathbf{B}_J$ *sending* $\mathbf{c}$ *to* $\mathbf{c}^J \otimes \mathbf{c}_J$ *is an isomorphism of* $\mathfrak{g}$*-crystals.*

3.2 Subcrystal $\mathbf{B}^J$

Let us consider the subcrystal $\mathbf{B}^J$ in more details. Let Δ_n be the arrangements of dots in the plane to represent the $(n-1)$-th triangular number. We identify Δ_n with $\Phi^+(J)$ in such a way that $\epsilon_{k+1}+\epsilon_{l+1}$, $\epsilon_{k+1}+\epsilon_l$ and $\epsilon_k+\epsilon_l$ for $1 \leq k, l \leq n-1$ are the vertices of a triangle of minimal shape in Δ_n as follows:



(3.4)

We also identify $\mathbf{c} \in \mathbf{B}^J$ with an array of c_β's in $\mathbf{c}$ with c_β at the corresponding dot in Δ_n.

Lemma 3.4. *We have* $\mathbf{B}^J = \{\, \mathbf{c} \,|\, \varepsilon_i^*(\mathbf{c}) = 0 \ (i \in J) \,\}$.

For $s \geq 1$, let

$$\mathbf{B}^{J,s} := \{\, \mathbf{c} \in \mathbf{B}^J \,|\, \varepsilon_n^*(\mathbf{c}) \leq s \,\}, \tag{3.5}$$

which is a subcrystal of $\mathbf{B}^J$. By Lemma 3.4 and [11, Proposition 8.2] , we have

$$B(s\varpi_n) \cong \mathbf{B}^{J,s} \otimes T_{s\varpi_n}, \qquad \bigcup_{s\geq 1} \mathbf{B}^{J,s} = \mathbf{B}^J, \tag{3.6}$$

as $\mathfrak{g}$-crystals. By (3.6), $\mathbf{B}^J$ is a regular $\mathfrak{l}$-crystal, that is, any connected component with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in J$ is isomorphic to the crystal of an integrable highest weight $U_q(\mathfrak{l})$-module, say $B_J(\lambda)$ for some $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i \in P$ with $(\lambda, \alpha_i) \in \mathbb{Z}_+$ for $i \in J$. Then Proposition 3.1 enables us to decompose $\mathbf{B}^J$ into $\mathfrak{l}$-crystals directly as follows, and hence the decomposition of $U_q(w^J)$ into irreducible $U_q(\mathfrak{l})$-modules.

Proposition 3.5. *As an* $\mathfrak{l}$*-crystal, we have* $\mathbf{B}^J \cong \bigsqcup_\lambda B_J(\lambda)$*, where the union is over* $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i \in P$ *such that* $0 \geq \lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq \cdots$.

3.3 Combinatorial description of ε_n^*

Let us give an explicit combinatorial description of ε_n^* on $\mathbf{B}^J$, whose proof is obtained by using the formula of Berenstein-Zelevinsky [1].

Definition 3.6. A *path in* Δ_n is a sequence $p = (\gamma_1, \ldots, \gamma_s)$ in $\Phi^+(J)$ for some $s \geq 1$ such that

(1) $\gamma_1, \ldots, \gamma_s \in \Phi^+(J)$,

(2) if $\gamma_i = \epsilon_k + \epsilon_{l+1}$ for some $k < l$, then $\gamma_{i+1} = \epsilon_{k+1} + \epsilon_{l+1}$ or $\epsilon_k + \epsilon_l$ (see (3.4)),

(3) $\gamma_s = \epsilon_k + \epsilon_{k+1}$ for some k.

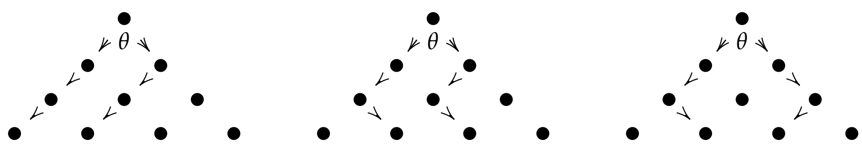
For $\beta \in \Phi^+(J)$, a *double path at β in Δ_n* is a pair of paths $\mathbf{p} = (p_1, p_2)$ in Δ_n of the same length with $p_1 = (\gamma_1, \ldots, \gamma_s)$ and $p_2 = (\delta_1, \ldots, \delta_s)$ such that

(1) $\gamma_1 = \delta_1 = \beta$,

(2) γ_i is located to the strictly left of δ_i for $2 \le i \le s$,

(3) $\gamma_s = \epsilon_{k+1} + \epsilon_{k+2}$, $\delta_s = \epsilon_k + \epsilon_{k+1}$ for some $k \ge 1$.

Example 3.7. The followings are some examples of double paths $\mathbf{p}$ at $\theta = \epsilon_1 + \epsilon_5$ in Δ_5.

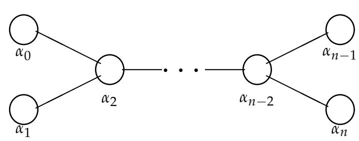


For $\mathbf{c} \in \mathbf{B}^J$ and a double path $\mathbf{p}$, let $||\mathbf{c}||_{\mathbf{p}} = \sum_{\beta \text{ lying on } \mathbf{p}} c_\beta$.

Theorem 3.8. *For $\mathbf{c} \in \mathbf{B}^J$, $\varepsilon_n^*(\mathbf{c}) = \max\{\, ||\mathbf{c}||_{\mathbf{p}} \,|\, \mathbf{p}$ is a double path at θ in $\Delta_n \,\}$, where $\theta = \epsilon_1 + \epsilon_n$.*

3.4 Realization of KR crystals $B^{n,s}$

Let $\hat{\mathfrak{g}}$ be an affine Kac-Moody algebra of type $D_n^{(1)}$ with $\hat{I} = \{\, 0, 1, \ldots, n \,\}$ the index set for the simple roots.



For $r \in \{0, n\}$, let $\hat{\mathfrak{g}}_r$ be the subalgebra of $\hat{\mathfrak{g}}$ corresponding to $\{\, \alpha_i \,|\, i \in \hat{I} \setminus \{r\} \,\}$. Then $\hat{\mathfrak{g}}_0 = \mathfrak{g}$, and $\hat{\mathfrak{g}}_0 \cap \hat{\mathfrak{g}}_n = \mathfrak{l}$. Let $\hat{P} = \bigoplus_{i \in \hat{I}} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ be the weight lattice of $\hat{\mathfrak{g}}$, where δ is the positive imaginary null root and Λ_i is the i-th fundamental weight . We regard $P = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ as a sublattice of $\hat{P}/\mathbb{Z}\delta$ by putting $\epsilon_1 = \Lambda_1 - \Lambda_0$, $\epsilon_2 = \Lambda_2 - \Lambda_1 - \Lambda_0$, $\epsilon_k = \Lambda_k - \Lambda_{k-1}$ for $k = 3, \ldots, n-2$, $\epsilon_{n-1} = \Lambda_{n-1} + \Lambda_n - \Lambda_{n-2}$ and $\epsilon_n = \Lambda_n - \Lambda_{n-1}$. In particular, we have $\alpha_0 = -\epsilon_1 - \epsilon_2$ in P. If ϖ_i' are the fundamental weights for $\hat{\mathfrak{g}}_n$ for $i \in \hat{I} \setminus \{n\}$, then $\varpi_i' = \varpi_i$ for $i \in \hat{I} \setminus \{0, n\}$ and $\varpi_0' = -\varpi_n$.

For $\mathbf{c} \in \mathbf{B}^J$, define

$$\widetilde{e}_0\mathbf{c} = \mathbf{c} + \mathbf{1}_{\epsilon_1+\epsilon_2}, \quad \widetilde{f}_0\mathbf{c} = \begin{cases} \mathbf{c} - \mathbf{1}_{\epsilon_1+\epsilon_2}, & \text{if } c_{\epsilon_1+\epsilon_2} > 0, \\ \mathbf{0}, & \text{otherwise,} \end{cases} \tag{3.7}$$
$$\varphi_0(\mathbf{c}) = \max\{\, k \,|\, \widetilde{f}_0^k\mathbf{c} \neq \mathbf{0} \,\}, \quad \varepsilon_0(\mathbf{c}) = \varphi_0(\mathbf{c}) - (\mathrm{wt}(\mathbf{c}), \alpha_0).$$

Lemma 3.9. *The set* $\mathbf{B}^J$ *is a* $\hat{\mathfrak{g}}$*-crystal with respect to* wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ *for* $i \in \hat{I}$, *where* wt *is the restriction of* $\mathrm{wt} : \mathbf{B} \longrightarrow P$ *to* $\mathbf{B}^J$.

Theorem 3.10. *For* $s \geq 1$, $\mathbf{B}^{J,s} \otimes T_{s\varpi_n}$ *is a regular* $\hat{\mathfrak{g}}$*-crystal and* $\mathbf{B}^{J,s} \otimes T_{s\varpi_n} \cong B^{n,s}$, *where* $B^{n,s}$ *is the Kirillov-Reshetikhin crystal of type* $D_n^{(1)}$ *associated to* $s\varpi_n$ *(cf.[3]).*

Remark 3.11. *By Theorem 3.8, we have* $\mathbf{B}^{J,s} = \bigcap_{\mathbf{p}} \left\{ \mathbf{c} \in \mathbf{B}^J \,|\, ||\mathbf{c}||_{\mathbf{p}} \leq s \right\}$, *where* $\mathbf{p}$ *runs over the double paths in* Δ_n. *This gives a polytope realization of the KR crystal* $B^{n,s}$. *Also, we note that by construction, the crystal* $\mathbf{B}^J$ *is the direct limit of* $\{\mathbf{B}^{J,s} : s \in \mathbb{Z}_+\}$. *By [4],* $\{B^{n,s}\}$ *is a family of perfect KR crystals. It is conjectured that* $\{B^{n,s}\}$ *has the limit in the sense of [12], that is,* $\{B^{n,s}\}$ *is a coherent family.*

4 RSK correspondence for type D and affine crystals

4.1 Burge correspondence

Let $\mathscr{P}$ be the set of partitions $\lambda = (\lambda_i)_{i\geq 1}$, which are often identified with Young diagrams. Let $\lambda' = (\lambda'_i)_{i\geq 1}$ be the conjugate of λ, and let λ^π be the skew Young diagram obtained by 180°-rotation of λ. Let $\ell(\lambda)$ denote the length of λ, and let $\mathscr{P}_n = \{\, \lambda \,|\, \ell(\lambda) \leq n \,\}$. Let $[\overline{n}] := \{\, \overline{n} < \cdots < \overline{1} \,\}$ be a linearly ordered set. Let $\mathcal{W}$ be the set of finite words in $[\overline{n}]$. For a skew Young diagram λ^π, let $SST_{[\overline{n}]}(\lambda^\pi)$ or simply $SST(\lambda^\pi)$ denote the set of semistandard tableaux of shape λ^π with entries in $[\overline{n}]$. For $T \in SST(\lambda^\pi)$, let $w(T)$ be a word in $\mathcal{W}$ obtained by reading the entries of T row by row from top to bottom, and from right to left in each row, and let $\mathrm{sh}(T)$ denote the shape of T. Note that we use English convention for partitions and tableaux. Let $T^{\nwarrow}$ be the unique semistandard tableau such that $\mathrm{sh}(T^{\nwarrow}) \in \mathscr{P}$ and $w(T^{\nwarrow})$ is Knuth equivalent to $w(T)$. We define $T^{\searrow}$ in a similar way such that $\mathrm{sh}(T^{\searrow}) \in \mathscr{P}^\pi$. Note that if $\mathrm{sh}(T^{\nwarrow}) = \nu$, then $\mathrm{sh}(T^{\searrow}) = \nu^\pi$. For $a \in [\overline{n}]$ and $U \in SST(\lambda)$ with $\lambda \in \mathscr{P}_n$, let $a \to U$ be the tableau obtained by applying the Schensted's column insertion of a into U. Similarly, for $V \in SST(\lambda^\pi)$ and $b \in [\overline{n}]$, let $V \leftarrow b$ be the tableau obtained by applying the Schensted's column insertion of b into V in a reverse way starting from the rightmost column. For $w = w_1 \ldots w_r \in \mathcal{W}$, we define $P(w) = (w_r \to (\cdots (w_2 \to w_1) \cdots))$. Note that $P(w)^{\searrow} = ((w_r \leftarrow w_{r-1}) \leftarrow \cdots \leftarrow w_1)$.

Let us recall a variation of RSK correspondence for type D [2]. Set

$$\mathcal{T}^{\searrow} := \bigsqcup_{\substack{\lambda \in \mathscr{P}_n \\ \lambda' : \text{even}}} SST(\lambda^\pi), \qquad \mathcal{T}^{\nwarrow} := \bigsqcup_{\substack{\lambda \in \mathscr{P}_n \\ \lambda' : \text{even}}} SST(\lambda), \tag{4.1}$$

where we say that λ' is even if each part of λ' is even Let Ω be the set of biwords $(\mathbf{a}, \mathbf{b}) \in \mathcal{W} \times \mathcal{W}$ such that

(1) $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$ for some $r \geq 0$,

(2) $a_i < b_i$ for $1 \leq i \leq r$,

(3) $(a_1, b_1) \leq \cdots \leq (a_r, b_r)$,

where $(a,b) < (c,d)$ if and only if ($a < c$) or ($a = c$ and $b > d$) for $(a,b), (c,d) \in \mathcal{W} \times \mathcal{W}$. We denote by $\mathbf{c}(\mathbf{a}, \mathbf{b})$ the unique element in $\mathbf{B}^J$ corresponding to $(\mathbf{a}, \mathbf{b}) \in \Omega$ such that $c_{ab} = |\{ k \,|\, (a_k, b_k) = (a,b) \}|$.

For $(\mathbf{a}, \mathbf{b}) \in \Omega$ with $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$, we define a sequence of tableaux $P_r, P_{r-1}, \ldots, P_1$ inductively as follows:

(1) let P_1 be a vertical domino $\boxed{\begin{smallmatrix} a_r \\ b_r \end{smallmatrix}}$,

(2) if P_{k+1} is given for $1 \leq k \leq r-1$, then define P_k to be the tableau obtained by first applying the column insertion to get $P_{k+1} \leftarrow b_k$, and then adding $\boxed{a_k}$ at the conner of $P_{k+1} \leftarrow b_k$ located above the box $\mathrm{sh}(P_{k+1} \leftarrow b_k)/\mathrm{sh}(P_{k+1})$.

We put $P^{\searrow}(\mathbf{a}, \mathbf{b}) := P_1$. It is not difficult to see from the definition that $P^{\searrow}(\mathbf{a}, \mathbf{b}) \in SST(\lambda^{\pi})$ for some $\lambda \in \mathscr{P}$ such that λ' is even.

For $\mathbf{c} \in \mathbf{B}^J$, let $P^{\searrow}(\mathbf{c}) = P^{\searrow}(\mathbf{a}, \mathbf{b})$ where $\mathbf{c} = \mathbf{c}(\mathbf{a}, \mathbf{b})$. Since the map $(\mathbf{a}, \mathbf{b}) \mapsto P^{\searrow}(\mathbf{a}, \mathbf{b})$ is a bijection from Ω to $\mathcal{T}^{\searrow}$ [2], we have a bijection

$$\kappa^{\searrow} : \mathbf{B}^J \longrightarrow \mathcal{T}^{\searrow}, \tag{4.2}$$

where $\kappa^{\searrow}(\mathbf{c}) = P^{\searrow}(\mathbf{c})$. Similarly, let Ω' be the set of biwords $(\mathbf{a}, \mathbf{b}) \in \mathcal{W} \times \mathcal{W}$ satisfying the same conditions as in Ω except that $<$ is replaced by $<'$, where $(a,b) <' (c,d)$ if and only if ($b < d$) or ($b = d$ and $a < c$) for (a,b) and $(c,d) \in \mathcal{W} \times \mathcal{W}$. We define $\mathbf{c}'(\mathbf{a}, \mathbf{b})$ in the same way as in $\mathbf{c}(\mathbf{a}, \mathbf{b})$. Given $(\mathbf{a}, \mathbf{b}) \in \Omega'$ with $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$, define a sequence of tableaux $P_1, P_2, \ldots, P_r$ inductively as follows:

(1) let P_1 be a vertical domino $\boxed{\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}}$,

(2) if P_{k-1} is given for $2 \leq k \leq r$, then define P_k to be the tableau obtained by first applying the column insertion to get $a_k \rightarrow P_{k-1}$, and then adding $\boxed{b_k}$ at the conner of $a_k \rightarrow P_{k-1}$ located below the box $\mathrm{sh}(a_k \rightarrow P_{k-1})/\mathrm{sh}(P_{k-1})$,

and put $P^{\nwarrow}(\mathbf{a}, \mathbf{b}) := P_r$. For $\mathbf{c} \in \mathbf{B}^J$, let $P^{\nwarrow}(\mathbf{c}) = P^{\nwarrow}(\mathbf{a}, \mathbf{b})$ where $\mathbf{c} = \mathbf{c}'(\mathbf{a}, \mathbf{b})$. Then we also have a bijection

$$\kappa^{\nwarrow} : \mathbf{B}^J \longrightarrow \mathcal{T}^{\nwarrow}, \tag{4.3}$$

where $\kappa^{\nwarrow}(\mathbf{c}) = P^{\nwarrow}(\mathbf{c})$

4.2 Isomorphism of affine crystals

We regard $[\overline{n}] = \{\, \overline{n} < \cdots < \overline{1} \,\}$ as the crystal of dual natural representation of $\mathfrak{l}$ with $\mathrm{wt}(\overline{k}) = -\epsilon_k$. Then $\mathcal{W}$ is a regular $\mathfrak{l}$-crystal, where $w = w_1 \ldots w_r$ is identified with $w_1 \otimes \cdots \otimes w_r$. For $\lambda \in \mathscr{P}_n$, $SST(\lambda)$ is a regular $\mathfrak{l}$-crystal with lowest weight $-\sum_{i=1}^{n} \lambda_i \epsilon_i$, where T is identified with $w(T)$. In particular $\mathcal{T}^{\searrow}$ and $\mathcal{T}^{\nwarrow}$ are regular $\mathfrak{l}$-crystals.

Let us recall the $\hat{\mathfrak{g}}_0$-crystal structure on $\mathcal{T}^{\searrow}$ [15, Section 5.2]. Let $T \in \mathcal{T}^{\searrow}$ be given. For $k \geq 1$, let t_k be the entry in the top of the k-th column of T (enumerated from the right). Consider $\sigma = (\sigma_1, \sigma_2, \ldots)$, where

$$\sigma_k = \begin{cases} + \,, & \text{if } t_k > \overline{n-1} \text{ or the } k\text{-th column is empty,} \\ - \,, & \text{if the } k\text{-th column has both } \overline{n-1} \text{ and } \overline{n} \text{ as its entries,} \\ \cdot \,, & \text{otherwise.} \end{cases}$$

Then $\widetilde{e}_n T$ is obtained from T by removing $\boxed{\overline{n}}\atop\boxed{\overline{n-1}}$ in the column corresponding to the right-most $-$ in σ^{red} (see Section 2.2 for σ^{red}). If there is no such $-$ sign, then we define $\widetilde{e}_n T = \mathbf{0}$, and $\widetilde{f}_n T$ is obtained from T by adding $\boxed{\overline{n}}\atop\boxed{\overline{n-1}}$ column corresponding to the left-most $+$ in σ^{red}. Hence $\mathcal{T}^{\searrow}$ is a $\hat{\mathfrak{g}}_0$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in \hat{I} \setminus \{0\})$, where $\varepsilon_n(T) = \max\{\, k \,|\, \widetilde{e}_n^{\,k} T \neq \mathbf{0} \,\}$ and $\varphi_n(T) = \varepsilon_n(T) + (\mathrm{wt}(T), \alpha_n)$.

Similarly, we have a $\hat{\mathfrak{g}}_n$-crystal structure on $\mathcal{T}^{\nwarrow}$ [15, Section 5.2]. Let $T \in \mathcal{T}^{\nwarrow}$ be given. For $k \geq 1$, let t_k be the entry in the bottom of the k-th column of T (enumerated from the left). Consider $\sigma = (\ldots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} - \,, & \text{if } t_k < \overline{2} \text{ or the } k\text{-th column is empty,} \\ + \,, & \text{if the } k\text{-th column has both } \overline{1} \text{ and } \overline{2} \text{ as its entries,} \\ \cdot \,, & \text{otherwise.} \end{cases}$$

Then $\widetilde{e}_0 T$ is given by adding $\boxed{\overline{2}}\atop\boxed{\overline{1}}$ to the bottom of the column corresponding to the right-most $-$ in σ^{red}, and $\widetilde{f}_0 T$ is obtained from T by removing $\boxed{\overline{2}}\atop\boxed{\overline{1}}$ in the column corresponding to the left-most $+$ in σ^{red}. If there is no such $+$ sign, then we define $\widetilde{f}_0 T = \mathbf{0}$. Hence $\mathcal{T}^{\nwarrow}$ is a $\hat{\mathfrak{g}}_n$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in \hat{I} \setminus \{n\})$, where $\varphi_0(T) = \max\{\, k \mid \widetilde{f}_0^{\,k} T \neq \mathbf{0} \,\}$ and $\varepsilon_0(T) = \varphi_0(T) - (\mathrm{wt}(T), \alpha_0)$.

Theorem 4.1. *The bijection $\kappa^{\searrow}$ in (4.2) is an isomorphism of $\hat{\mathfrak{g}}_0$-crystals, and the bijection $\kappa^{\nwarrow}$ in (4.3) is an isomorphism of $\hat{\mathfrak{g}}_n$-crystals.*

Remark 4.2. *It would be interesting to compare Theorem 4.1 with the result in [20].*

For a semistandard tableau T of skew shape, let $[T]$ denote the equivalence class of T with respect to Knuth equivalence. Let $\mathcal{T} = \{\, [T] \,|\, T \in \mathcal{T}^{\searrow} \,\} = \{\, [T] \,|\, T \in \mathcal{T}^{\nwarrow} \,\}$.

If we define $\widetilde{x}_i[T] = [\widetilde{x}_0 T^{\nwarrow}]$ when $i = 0$, $\widetilde{x}_i[T] = [\widetilde{x}_n T^{\searrow}]$ when $i = n$, and $\widetilde{x}_i[T] = [\widetilde{x}_i T]$ otherwise for $i \in \hat{I}$ and $x = e, f$ (we assume that $[\mathbf{0}] = \mathbf{0}$), then the set $\mathcal{T}$ is a $\hat{\mathfrak{g}}$-crystal with respect to $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in I)$, where wt, ε_i, and φ_i are induced from either $\mathcal{T}^{\searrow}$ or $\mathcal{T}^{\nwarrow}$ [15].

Corollary 4.3. *The map* $\kappa : \mathbf{B}^J \longrightarrow \mathcal{T}$ *sending* $\mathbf{c}$ *to* $[P^{\nwarrow}(\mathbf{c})] = [P^{\searrow}(\mathbf{c})]$ *is an isomorphism of* $\hat{\mathfrak{g}}$*-crystals.*

4.3 Shape formula

For $\mathbf{c} \in \mathbf{B}^J$, let $\lambda(\mathbf{c}) = (\lambda_1(\mathbf{c}) \geq \ldots \geq \lambda_\ell(\mathbf{c}))$ be the partition corresponding to the regular $\mathfrak{l}$-subcrystal of $\mathbf{B}^J$ including $\mathbf{c}$, that is, $\lambda(\mathbf{c}) = \mathrm{sh}(\kappa^{\nwarrow}(\mathbf{c}))$ by Theorem 4.1. Note that $\ell = 2[\frac{n}{2}]$ and $\lambda_{2i-1}(\mathbf{c}) = \lambda_{2i}(\mathbf{c})$ for $1 \leq i \leq [\frac{n}{2}]$.

Theorem 4.4. *For* $\mathbf{c} \in \mathbf{B}^J$ *and* $1 \leq l \leq [\frac{n}{2}]$*, we have*

$$\lambda_1(\mathbf{c}) + \lambda_3(\mathbf{c}) + \cdots + \lambda_{2l-1}(\mathbf{c}) = \max_{\mathbf{p}_1,\ldots,\mathbf{p}_l} \{ ||\mathbf{c}||_{\mathbf{p}_1} + \cdots + ||\mathbf{c}||_{\mathbf{p}_l} \},$$

where $\mathbf{p}_1, \ldots, \mathbf{p}_l$ *are mutually non-intersecting double paths in* Δ_n *and each* $\mathbf{p}_i$ *starts at the* $(2i-1)$*-th row of* Δ_n *for* $1 \leq i \leq l$*.*

This formula can be viewed as an analogue of Greene's formula for the shape of a tableau corresponding to a biword under usual RSK given in terms of disjoint weakly decreasing subwords [6].

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