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# A $q$-deformed type B Cauchy identity and Chow's quasisymmetric functions

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**Abstract.** The Cauchy identity is a fundamental formula in algebraic combinatorics that captures all the nice properties of the RSK correspondence. In particular, expanding both sides of the identity with Gessel's quasisymmetric functions allows to recover the descent preserving property, an essential tool to prove the Schur positivity of sets of permutations. We look at the type B generalisation of these results that involves the domino insertion algorithm. We introduce a $q$-deformation of the modified domino functions of our previous works to extend a type B Cauchy identity by Lam and link it with Chow's quasisymmetric functions. We apply this result to a new framework of type B $q$-Schur positivity and to prove new equidistribution results for some sets of domino tableaux.

**Résumé.** L'identité de Cauchy est un formule fondamentale en combinatoire algébrique qui capture les propriétés intéressantes de la correspondance RSK. Notamment, développer les deux membres de l'identité à l'aide des fonctions quasisymétriques de Gessel permet de retrouver la propriété de conservation de la descente, un outil essentiel pour prouver la Schur positivité d'ensembles de permutations. Nous nous concentrons sur la généralisation de ces résultats au type B utilisant l'algorithme d'insertion de dominos. Nous introduisons une $q$-déformation des fonctions de domino de nos travaux précédents pour étendre une identité de Cauchy de type B due à Lam et la relions aux fonctions quasisymétriques de Chow. Nous appliquons ensuite ces résultats à un nouveau cadre de $q$-Schur positivité de type B et prouvons de nouveaux résultats d'équidistribution de certains ensembles de tableaux dominos.

**Keywords:** Type B Cauchy identity, Domino tableaux, Chow's quasisymmetric functions, Schur-positivity

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# 1 Background

## 1.1 Permutations, Young tableaux and descent sets

For any positive integer $n$ write $[n] = \{1, \cdots, n\}$, $S_n$ the symmetric group on $[n]$. A **partition** $\lambda$ of $n$, denoted $\lambda \vdash n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ of $\ell(\lambda) = p$ parts sorted in decreasing order such that $|\lambda| = \sum_i \lambda_i = n$. A partition $\lambda$ is usually represented as a Young diagram of $n = |\lambda|$ boxes arranged in $\ell(\lambda)$ left justified rows so that the $i$-th row from the top contains $\lambda_i$ boxes. A Young diagram whose boxes are filled with positive integers such that the entries are increasing along the rows and strictly increasing down the columns is called a **semistandard Young tableau**. If the entries of a semistandard Young tableau are restricted to the elements of $[n]$ and strictly increasing along the rows, we call it a **standard Young tableau**. The partition $\lambda$ is the **shape** of the tableau and we denote $SYT(\lambda)$ (resp. $SSYT(\lambda)$) the set of standard (resp. semistandard) Young tableaux of shape $\lambda$.

One important feature of a permutation $\pi$ in $S_n$ is its **descent set**, i.e. the subset of $[n-1]$ defined as $Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$. Similarly, define the **descent set of a standard Young tableau** $T$ as $Des(T) = \{1 \leq i \leq n-1 \mid i$ is in a strictly higher row than $i+1\}$.

## 1.2 Descent preserving Cauchy identity

Let $X = \{x_1, x_2, \cdots\}$ and $Y = \{y_1, y_2, \cdots\}$ be two alphabets of commutative indeterminates. Denote also $XY$ the product alphabet $\{x_i y_j\}_{i,j}$ ordered by the lexicographical order. Let $F_I(X)$ be the **Gessel's fundamental quasisymmetric function** on $X$ indexed by the subset $I \subset [n-1]$. One has

$$F_I(X) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ k \in I \Rightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{1.1}$$

The power series $F_I(X)$ is not symmetric in $X$ but verifies the property that for any strictly increasing sequence of indices $i_1 < i_2 < \cdots < i_p$ the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}$ is equal to the coefficient of $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p}$. Gessel shows in [6] that their coproduct table is given by the structure constants of Solomon's descent algebra. Indeed, for $\pi \in S_n$,

$$F_{Des(\pi)}(XY) = \sum_{\sigma, \rho \in S_n;\, \sigma\rho = \pi} F_{Des(\sigma)}(X) F_{Des(\rho)}(Y). \tag{1.2}$$

Let $s_\lambda(X)$ be the Schur polynomial indexed by partition $\lambda$ on alphabet $X$. Schur polynomials are the generating functions for semistandard Young tableaux. As a result,

$$s_\lambda(X) = \sum_{T \in SSYT(\lambda)} X^T = \sum_{T \in SYT(\lambda)} F_{Des(T)}(X). \tag{1.3}$$

The **Cauchy identity** equates the sum over all integer partitions of the product of pairs of Schur polynomials and the Cauchy kernel. Namely,

$$\sum_{\lambda} s_\lambda(X) s_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \tag{1.4}$$

where the sum runs over all integer partitions. By interpreting the right-hand side of (1.4) as the generating function for *biwords* one recover that they are in weight preserving bijection with pairs of semistandard Young tableaux of the same shape, the main consequence of the *Robinson-Schensted-Knuth* correspondence.
For any integer $n$, the identity can be also expressed as

$$\sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) = s_{(n)}(XY). \tag{1.5}$$

Using (1.3) and (1.2), one can rewrite (1.5) as

$$\sum_{\substack{\lambda \vdash n \\ T,U \in SYT(\lambda)}} F_{Des(T)}(X) F_{Des(U)}(Y) = F_{\emptyset}(XY) = \sum_{\pi \in S_n} F_{Des(\pi)}(X) F_{Des(\pi^{-1})}(Y). \tag{1.6}$$

This gives a direct analytical proof of one important property of the restriction to permutations of the RSK correspondence, i.e. that permutations $\pi$ in $S_n$ are in bijection with pairs of standard Young tableaux $T, U$ of the same shape such that $Des(T) = Des(\pi)$ and $Des(U) = Des(\pi^{-1})$.

## 1.3 Schur positivity

Given any subset $\mathcal{A} \subset S_n$, Gessel also introduces in [6] the quasisymmetric function

$$Q(\mathcal{A})(X) = \sum_{\pi \in \mathcal{A}} F_{Des(\pi)}(X), \tag{1.7}$$

and looks at the problem of characterising the sets $\mathcal{A}$ for which $Q(\mathcal{A})$ is symmetric. Further the question of determining **Schur-positive** sets, i.e. the sets $\mathcal{A}$ for which $Q(\mathcal{A})$ can be expanded with non-negative coefficients in the Schur basis received significant attention. Classical examples of Schur-positive sets include inverse descent classes and Knuth classes. Both of these examples can be solved directly using successively (1.6) and (1.3). As a consequence their Schur positivity property can be seen as a **direct consequence of the Cauchy identity**.

# 2 Type B extension and $q$-deformation

The goal of this paper is to provide a **type B extension** of (1.6) that involves an additional parameter $q$ to keep track of some significant statistics. We then relate it to the question of type B ($q$-) Schur positivity and derive additional results. Such a type B extension naturally involves signed permutations instead of classical ones. However two options are available for the definition of their descent set with a consequence on the choice of appropriate type B quasisymmetric functions and type B Young tableaux. As a first option, in [1], the authors use the notion of *signed descent set*. The analogue of the RS correspondence, in this case is a signed descent preserving bijection relating signed permutations and *bi-tableaux*, i.e. pairs of Young tableaux with specific constraints. They prove an analogue of Equality (1.3) between their generating function and *Poirier's signed quasisymmetric functions*. While they succeed in extending most of the Schur positivity results known in type A, there is no equivalent of the Cauchy identity and this definition of descent is not the standard one for Coxeter groups of type B. We focus on the standard definition of descent. This leads us to *Chow's type B quasisymmetric functions* and an analogue of the RS correspondence relating signed permutations and pairs of *domino tableaux*. We proceed with the introduction of both of these objects.

## 2.1 Signed permutations and domino tableaux

Let $B_n$ be the **hyperoctahedral group** of order $n$. $B_n$ is composed of all signed permutations $\pi$ on $\{-n, \cdots, -2, -1, 0, 1, 2, \cdots, n\}$ such that $\pi(-i) = -\pi(i)$ for all $i$. The **total color** of $\pi \in B_n$ is the number $tc(\pi) = |\{i \geq 1; \pi(i) < 0\}|$. Its **descent set** is the subset of $\{0\} \cup [n-1]$ equal to $Des(\pi) = \{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$. The main difference with respect to the case of the symmetric group is the possible descent in position 0 when $\pi(1)$ is a negative integer.

For $\lambda \vdash 2n$, a **standard domino tableau** $T$ of shape $\lambda$ is a Young diagram of shape $shape(T) = \lambda$ tiled by **dominoes**, i.e. $2 \times 1$ or $1 \times 2$ rectangles filled with the elements of $[n]$ such that the entries are strictly increasing along the rows and down the columns. In the sequel we consider only the set $\mathcal{P}^0(n)$ of *empty 2-core partitions* $\lambda \vdash 2n$ that fit such a tiling. A standard domino tableau $T$ has a descent in position $i > 0$ if $i+1$ lies strictly below $i$ in $T$ and has descent in position 0 if the domino filled with 1 is vertical. We denote $Des(T)$ the set of all its descents. A **semistandard domino tableau** T of shape $\lambda \in \mathcal{P}^0(n)$ and weight $w(T) = \mu = (\mu_0, \mu_1, \mu_2, \cdots)$ with $\mu_i \geq 0$ and $\sum_i \mu_i = n$ is a tiling of the Young diagram of shape $\lambda$ with horizontal and vertical dominoes labelled with integers in $\{0, 1, 2, \cdots\}$ such that labels are non decreasing along the rows, strictly increasing down the columns and exactly $\mu_i$ dominoes are labelled with $i$. If the top leftmost domino is vertical, **it cannot be labelled 0**. Denote $SDT(\lambda)$ (resp. $SSDT(\lambda)$) the set of standard (resp. semistandard) domino tableaux of shape $\lambda$. Finally, we denote

$sp(T)$, the **spin** of (semi-) standard domino tableau $T$, i.e. **half the number** of its vertical dominoes.

**Remark 1.** *The possible labelling of top leftmost horizontal dominoes in semistandard domino tableaux with* 0 *was first introduced by the authors in [9] and is essential to connect their generating functions with Chow's quasisymmetric functions.*

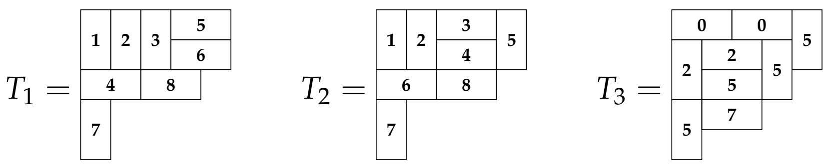


**Figure 1:** Two standard domino tableaux $T_1$ and $T_2$ of shape $(5,5,4,1,1)$, descent set {0,3,5,6}, a semistandard tableau $T_3$ of shape $(5,5,4,3,1)$ and weight $\mu = (2,0,2,0,0,4,0,1)$. All of the tableaux have a spin of 2.

There is a natural analogue of the RSK-correspondence for signed permutations involving **domino tableaux**. Barbash and Vogan ([2]) built a bijection between signed permutations of $B_n$ and pairs of standard domino tableaux of equal shape in $\mathcal{P}^0(n)$. An independent development on the subject is due to Garfinkle. Taşkin ([12, Prop. 26]) showed that the two standard domino tableaux $T$ and $U$ associated to a signed permutation $\pi$ by the algorithm of Barbash and Vogan have respective descent sets $Des(T) = Des(\pi^{-1})$ and $Des(U) = Des(\pi)$ while Shimozono and White showed in [11] the **color-to-spin** property i.e. that

$$tc(\pi) = sp(T) + sp(U) \tag{2.1}$$

## 2.2 Chow's type B quasisymmetric functions

Chow defines in [4] an analogue of Gessel's algebra of quasisymmetric functions that is dual to Solomon's descent algebra of type B.
Let $X = \{\cdots, x_{-i}, \cdots, x_{-1}, x_0, x_1, \cdots, x_i, \cdots\}$ be an alphabet of totally ordered commutative indeterminates with the assumption that $x_{-i} = x_i$ and let $I$ be a subset of $\{0\} \cup [n-1]$, he defines a type B analogue of the fundamental quasisymmetric functions

$$F_I^B(X) = \sum_{\substack{0=i_0\leq i_1\leq i_2\leq\ldots\leq i_n \\ j\in I\Rightarrow i_j<i_{j+1}}} x_{i_1}x_{i_2}\ldots x_{i_n}.$$

Note the particular role of the variable $x_0$.

## 2.3 Type B $q$-Cauchy identity

To build our $q$-deformed framework, we introduce the following constraint of **q-symmetry** on alphabets.

**Definition 1** ($q$-symmetry)**.** *Let $X = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$ be an alphabet of commutative indeterminates. We say that X is q-symmetric if for all $i > 0$,*

$$x_{-i} = qx_i. \tag{2.2}$$

In the sequel we let $X = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$ and $Y = \{\cdots, y_{-1}, y_0, y_1, \cdots\}$ be two $q$-symmetric alphabets. We proceed with the definition of the $q$-deformation of the modified domino functions introduced in [9].

**Definition 2** ($q$-domino functions)**.** *Given a semistandard domino tableau $T$ of weight $\mu$, denote $X^T$ the monomial $x_0^{\mu_0}x_1^{\mu_1}x_2^{\mu_2}\cdots$. For $\lambda \in \mathcal{P}^0(n)$ define the q-domino function indexed by $\lambda$ on alphabet $X$ as*

$$\mathcal{G}_\lambda(X;q) = \sum_{T \in SSDT(\lambda)} q^{sp(T)} X^T. \tag{2.3}$$

We have the following lemma.

**Lemma 1.** *For $\lambda \in \mathcal{P}^0(n)$, the q-domino function indexed by $\lambda$ can be expanded in the basis of Chow's quasisymmetric functions as*

$$\mathcal{G}_\lambda(X;q) = \sum_{T \in SDT(\lambda)} q^{sp(T)} F^B_{Des(T)}(X). \tag{2.4}$$

*Proof.* The proof is similar to the one in [9] in the case $q = 1$. □

**Remark 2.** *Our domino functions resemble the* **LLT-polynomials** *introduced in [8] but* **they are not equal** *because of the possible 0 labels in the semistandard domino tableaux of Remark 1. In particular our domino functions are* **not symmetric** *as the variable $x_0$ has a particular role. More precisely let $\lambda$ and $\mu$ be two integer partitions such that $\lambda/\mu$ is a skew shape, $|\lambda| - |\mu| = 2n$ and such that the Young diagrams of shape $\mu$ and $\lambda/\mu$ (and by extension $\lambda$) may be tiled by horizontal and vertical dominoes. By abuse of notation, we also write $\lambda/\mu \in \mathcal{P}^0(n)$ when all these conditions are fulfilled. For $\lambda/\mu \in \mathcal{P}^0(n)$ denote $SSDT^*(\lambda/\mu)$ the set of semistandard domino tableaux of shape $\lambda/\mu$ that* **do not have** *0 labels. The LLT-polynomial indexed by $\lambda/\mu$ on alphabet $X$ is*

$$LLT_{\lambda/\mu}(X;q) = \sum_{T \in SSDT^*(\lambda/\mu)} q^{sp(T)} X^T. \tag{2.5}$$

*For $\lambda \in \mathcal{P}^0(n)$, the two families of functions are related through*

$$\mathcal{G}_\lambda(X;q) = \sum_{\substack{k \geq 0, \\ \lambda/2k \in \mathcal{P}^0(n-k)}} x_0^k LLT_{\lambda/2k}(X;q). \tag{2.6}$$

Observe that positive indices in the product alphabet $XY = \{x_i y_j\}_{i,j}$ ordered by the lexicographical order (i.e. indices $(i,j) > (0,0)$) may include a negative index for $Y$ e.g. $x_2 y_{-3}$ has a positive index within the alphabet $XY$. Our $q$-deformed type B Cauchy identity can be stated as follows.

**Theorem 1.** *Let $X$ and $Y$ be two $q$-symmetric alphabets, we have:*

$$\mathcal{G}_{(2n)}(XY;q) = \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q). \tag{2.7}$$

*Proof.* Lam shows in [7] the following skew domino Cauchy formula for LLT polynomials

$$\sum_\lambda LLT_{\lambda/\alpha}(X;q)LLT_{\lambda/\beta}(Y;q) = \prod_{i,j>0} \frac{1}{(1-x_iy_j)(1-qx_iy_j)} \sum_\mu LLT_{\beta/\mu}(X;q)LLT_{\alpha/\mu}(Y;q). \tag{2.8}$$

Denote $Z_+^* = (z_i)_{i>0}$, $Z_+ = (z_i)_{i\geq 0}$, $Z_- = (z_i)_{i<0}$ for $Z = X, Y$ and $z = x, y$. As a consequence to (2.8), one has:

$$\begin{aligned}
\sum_\lambda \mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q) &= \sum_{k,l,\lambda} x_0^k y_0^l LLT_{\lambda/2k}(X;q)LLT_{\lambda/2l}(Y;q) \\
&= \prod_{i,j>0} \frac{1}{(1-x_iy_j)(1-qx_iy_j)} \sum_{k,l,m} x_0^k y_0^l s_{l-m}(X_+^*)s_{k-m}(Y_+^*) \\
&= \prod_{i,j>0} \frac{1}{(1-x_iy_j)(1-qx_iy_j)} \sum_m (x_0y_0)^m \sum_{k'} x_0^{k'} s_{k'}(Y_+^*) \sum_{l'} y_0^{l'} s_{l'}(X_+^*) \\
&= \prod_{i,j>0} \frac{1}{(1-x_iy_j)(1-qx_iy_j)} \left(\frac{1}{1-x_0y_0}\right) \prod_{i>0} \frac{1}{1-x_0y_i} \prod_{i>0} \frac{1}{1-y_0x_i} \\
&= \prod_{i,j\geq 0} \frac{1}{(1-x_iy_j)} \prod_{i,j>0} \frac{1}{(1-qx_iy_j)}
\end{aligned}$$

Looking at homogeneous parts of the same degree, we derive

$$\begin{aligned}
\sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X;q)\mathcal{G}_\lambda(Y;q) &= s_n(X_+Y_+ \cup qX_+^*Y_+^*) \\
&= s_n(X_+Y_+ \cup X_+^*Y_-) \\
&= s_n((XY)_+) \\
&= \mathcal{G}_{(2n)}(XY;q)
\end{aligned} \tag{2.9}$$

This is the desired formula. □

**Corollary 1.** *For any integer $n$, the following identity is a consequence to Theorem 1.*

$$\sum_{\pi \in B_n} q^{tc(\pi)} F^B_{Des(\pi)}(X) F^B_{Des(\pi^{-1})}(Y) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T,U \in SDT(\lambda)}} q^{sp(T)+sp(U)} F^B_{Des(T)}(X) F^B_{Des(U)}(Y). \tag{2.10}$$

*Proof.* To prove (2.10), one should develop $F^B_\emptyset(XY)$ in two different ways. Firstly, according to Lemma 1, $F^B_\emptyset(XY) = \mathcal{G}_{(2n)}(XY;q)$. Then, using (2.9) and Lemma 1 yields the right-hand side. But using the definition of Chow's quasisymmetric functions one has:

$$F^B_\emptyset(XY) = \sum_{(0,0)\leq(i_1,j_1)\leq\ldots\leq(i_n,j_n)} x_{i_1}y_{j_1}\ldots x_{i_n}y_{j_n}.$$

The constraints on the indices $(i_k, j_k)$ can be split in two disjoint cases:

(i) $i_k > i_{k+1}$ & $j_k < j_{k-1}$

(ii) $i_k \geq i_{k+1}$ & $j_k \geq j_{k-1}$

Then using the theory of *Type B P-partitions* as in [4, Thm. 2.3.4] and the fact that $Y$ is $q$-symmetric yields the left-hand side of (2.10). □

# 3 Application to type B Schur positivity and statistics on domino tableaux

## 3.1 Type B $q$-Schur positivity

Given any subset $\mathcal{B}$ of $B_n$ and a statistic *stat* defined on the set $\mathcal{B}$ we look at the type B quasisymmetric function

$$Q(\mathcal{B}, stat)(X) = \sum_{\pi\in\mathcal{B}} q^{stat(\pi)} F^B_{Des(\pi)}(X).$$

**Definition 3** (Type B $q$-Schur positivity)**.** *We say that a pair composed of a set $\mathcal{B} \subset B_n$ and a statistic stat defined on the set $\mathcal{B}$ is* **type B $q$-Schur positive** *or* $q$-$\mathcal{G}$ **positive** *if the function $Q(\mathcal{B}, stat)$ can be written as*

$$Q(\mathcal{B}, stat)(X) = \sum_\lambda c_\lambda(q)\mathcal{G}_\lambda(X;q), \tag{3.1}$$

*where the $c_\lambda(q)$ are polynomials in $q^{\frac{1}{2}}$ with non-negative integer coefficients.*

**Remark 3.** *The family $\{\mathcal{G}_\lambda(X;q)\}_{\lambda\in\mathcal{P}^0(n)}$ is not linearly independent. As a result, there may be several ways to expand a type B quasisymmetric function as in (3.1).*

In our previous work dealing with the case $q = 1$ ([10]), we show the type B Schur positivity of some classical sets of permutations. More precisely, let $C^B_T$ be the **type B Knuth class** indexed by standard domino tableau $T$, i.e. the set of signed permutations whose first domino tableau obtained by the Barbash and Vogan algorithm is $T$. Define also the **type B inverse descent class** indexed by $J \subset \{0\} \cup [n-1]$ as the set of signed permutations $\pi$, $D^n_J = \{\pi \in B_n; Des(\pi^{-1}) = J\}$. Finally, a permutation $\pi \in B_n$ is called **a signed arc permutation** if for $1 \leq i \leq n$ the set $\{|\pi(1)|, |\pi(2)|, \cdots |\pi(i-1)|\}$ is an interval in $Z_n$ and $\pi(i) > 0$ if $|\pi(i)| - 1 \in \{|\pi(1)|, |\pi(2)|, \cdots |\pi(i-1)|\}$ and $\pi(i) < 0$ otherwise. We have the following property.

**Proposition 1** ([10])**.** *In the case $q = 1$, type B Knuth classes, type B inverse descent classes and signed arc permutations are type B 1-Schur positive.*

We extend part of this result for general $q$.

**Proposition 2** (Type B $q$-Schur positivity of Knuth and inverse descent classes)**.** *Set $stat = tc$. Type B inverse descent classes and type B Knuth classes are type B $q$-Schur positive.*

*Proof.* Let $J \subset \{0\} \cup [n-1]$. Extracting the coefficient in $F^B_J(Y)$ in (2.10) yields:

$$\sum_{\pi \in D^n_J} q^{tc(\pi)} F^B_{Des(\pi)}(X) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T,U \in SDT(\lambda) \\ Des(U)=J}} q^{sp(T)+sp(U)} F^B_{Des(T)}(X) \tag{3.2}$$

$$= \sum_{Des(U)=J} q^{sp(U)} \sum_{T \in SDT(sh(U))} q^{sp(T)} F^B_{Des(T)}(X)$$

$$= \sum_{Des(U)=J} q^{sp(U)} \mathcal{G}_{sh(U)}(X;q). \tag{3.3}$$

Furthermore, for a standard domino tableau $T$, we have

$$\sum_{\pi \in C^B_T} q^{tc(\pi)} F^B_{Des(\pi)}(X) = q^{sp(T)} \mathcal{G}_{sh(T)}(X;q).$$

This gives the desired result. □

**Remark 4.** *As the proof for signed arc permutations in [10] is not based on the bijection of Barbash and Vogan, the extension of their type B Schur positivity to general q is not obvious and may not even be true for non-trivial choices of stat.*

**Proposition 3.** *For a permutation $\pi \in B_n$, let $l(\pi)$ be its type B Coxeter length. The pair $(B_n, l)$ is type B q-Schur positive. Furthermore, given a standard domino tableau $T$, denote $maj(T) = \sum_{i \in Des(T)} i$ (the same definition applies to signed permutations). We have*

$$\sum_{\pi \in B_n} q^{l(\pi)} F^B_{Des(\pi)}(X) = \sum_{\substack{\lambda \in \mathcal{P}^0(n), \\ Q \in SDT(\lambda)}} q^{2maj(Q)+sp(Q)} \mathcal{G}_{sh(Q)}(X;q). \tag{3.4}$$

*Proof.* For $\pi \in B_n$, let $fmaj(\pi) = 2maj(\pi) + tc(\pi)$. We use a result from Foata in [5] proving a bijection $\varphi$ between signed permutations with a given Coxeter length $k$ and signed permutation with $fmaj$ statistic equal to $k$ such that for $\pi \in B_n$, $Des(\varphi(\pi)) = Des(\pi^{-1})$. As a result,

$$\sum_{\pi \in B_n} q^{l(\pi)} F^B_{Des(\pi)}(X) = \sum_{\pi \in B_n} q^{l(\pi^{-1})} F^B_{Des(\pi)}(X)$$
$$= \sum_{\sigma \in B_n} q^{2maj(\sigma)+tc(\sigma)} F^B_{Des(\sigma)}(X).$$

Then using the fact that the statistic $maj$ depends only on the descent set and applying (3.3) prove the property. □

**Remark 5.** *Propositions 2 and 3 are seen as consequences of Theorem 1 but could be proved independently using the Barbash and Vogan algorithm and its descent preserving and color-to-spin properties. This makes Section 3.1 independent of Section 2.3.*

## 3.2 Two equidistributed statistics on pairs of domino tableaux

There is a well known (not descent preserving) bijection between semistandard domino tableaux of weight $\mu$ and **bi-tableaux**, i.e. pairs of semistandard Young tableaux of respective weights $\mu^-$ and $\mu^+$ such that $\mu_i^+ + \mu_i^- = \mu_i$ for all $i$ (see e.g. [3, Algorithm 6.1]). The respective shapes of the two Young tableaux depend only on the shape of the initial semistandard domino tableau. Denote $(T^-, T^+)$ the bi-tableau associated to a semistandard domino tableau $T$ of shape $\lambda$ and $(\lambda^-, \lambda^+)$ their respective shapes. $(T^-, T^+)$ (resp. $(\lambda^-, \lambda^+)$) is the 2**-quotient** of $T$ (resp. $\lambda$). Semistandard Young tableaux $T^-$ and $T^+$ are built by filling each box of $T$ (a domino is composed of two boxes) by a '–' or a '+' sign such that the top leftmost box is filled with '–' and two adjacent boxes have opposite signs. $T^-$ (resp. $T^+$) is obtained from the sub tableau of $T$ composed of the dominoes with top rightmost box filled with '–' (resp. '+'). As standard domino tableaux are special cases of semistandard domino tableaux, this bijection naturally specialises to the standard case.

**Definition 4** (Number of negative dominos)**.** *For a standard domino tableau $T$ with 2-quotient $(T^-, T^+)$ given by the algorithm of Carré and Leclerc, let $neg(T)$ be the number of boxes in $T^-$. The statistic $neg(T)$ depends only on the shape of $T$.*

**Example 1.** *Figure 2 shows a semistandard domino tableau $T$ of weight $\mu = (2, 0, 2, 0, 0, 4, 0, 1)$ and its 2-quotient. In this example $neg(T) = 3$.*

Equation (2.9) actually implies another more surprising result.

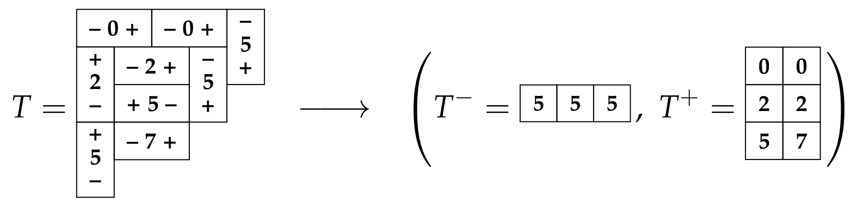

**Figure 2:** A semistandard domino tableau and its 2-quotient.

**Theorem 2.** *Let $n$ be a positive integer. There is a one-to-one correspondence between pairs of standard domino tableaux $T, U$ of the same shape $\lambda \in \mathcal{P}^0(n)$ and pairs of standard domino tableaux $R, S$ of the same shape $\mu \in \mathcal{P}^0(n)$ (possibly $\lambda \neq \mu$) such that $Des(T) = Des(R)$, $Des(U) = Des(S)$ and $sp(T) + sp(U) = neg(R) = neg(S)$.*

*Proof.* By successive application of Theorem 1 and Lemma 1, one may compute

$$\mathcal{G}_{(2n)}(XY;q) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T,U \in SDT(\lambda)}} q^{sp(T)+sp(U)} F^B_{Des(T)}(X) F^B_{Des(U)}(Y). \tag{3.5}$$

Then we use standard properties of Schur polynomials and the fact that (see [9]) for $\lambda \in \mathcal{P}^0(n)$,

$$\mathcal{G}_\lambda(X;1) = s_{\lambda^+}(X_+) s_{\lambda^-}(X_+^*)$$

to expand $\mathcal{G}_{(2n)}(XY;q)$ differently.

$$\begin{aligned}
\mathcal{G}_{(2n)}(XY;q) &= s_{(n)}(X_+Y_+ \cup qX_+^*Y_+^*) \\
&= \sum_k s_{(n-k)}(X_+Y_+) s_{(k)}(qX_+^*Y_+^*) \\
&= \sum_k q^k s_{(n-k)}(X_+Y_+) s_{(k)}(X_+^*Y_+^*) \\
&= \sum_{\substack{\lambda^-,\lambda^+ \\ |\lambda^-|+|\lambda^+|=n}} q^{|\lambda^-|} s_{\lambda^+}(X_+) s_{\lambda^-}(X_+^*) s_{\lambda^+}(Y_+) s_{\lambda^-}(Y_+^*) \\
&= \sum_{\lambda \in \mathcal{P}^0(n)} q^{|\lambda^-|} \mathcal{G}_\lambda(X;1) \mathcal{G}_\lambda(Y;1) \\
&= \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ R,S \in SDT(\lambda)}} q^{neg(R)} F^B_{Des(R)}(X) F^B_{Des(S)}(Y).
\end{aligned} \tag{3.6}$$

Equating (3.5) and (3.6) yields the desired result. □

**Remark 6.** *Combining the two classical extensions of the RS correspondence to signed permutations, there is a natural one-to-one relation between pairs of bi-tableaux and pairs of domino tableaux such that the number of boxes in the first Young tableau of each bi-tableau and the sum of the spins of the domino tableaux are equal. The surprising fact of Theorem 2 is the descent preservation in the sense of domino tableaux (the descent set of a bi-tableau is in general not equal to the one of the domino tableau obtained by reversing the 2-quotient bijection).*

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