

Restricting Schubert classes to symplectic Grassmannians using self-dual puzzles

Iva Halacheva^{*1}, Allen Knutson^{†2}, and Paul Zinn-Justin^{‡1}

¹*School of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia*

²*The Department of Mathematics, Cornell University, Ithaca, New York*

Abstract. Given a Schubert class on $Gr(k, V)$ where V is a symplectic vector space of dimension $2n$, we consider its restriction to the symplectic Grassmannian $SpGr(k, V)$ of isotropic subspaces. Pragacz gave tableau formulæ for positively computing the expansion of these $H^*(Gr(k, V))$ classes into Schubert classes of the target when $k = n$, which corresponds to expanding Schur polynomials into Q -Schur polynomials. Coşkun described an algorithm for their expansion when $k \leq n$. We give a puzzle-based formula for these expansions, while extending them to equivariant cohomology. We make use of a new observation that usual Grassmannian puzzle pieces are already enough to do some 2-step Schubert calculus, and apply techniques from quantum integrable systems (“scattering diagrams”).

Keywords: Schubert calculus, puzzles, Grassmannian, symplectic Grassmannian

1 Introduction

1.1 Grassmannian duality of puzzles

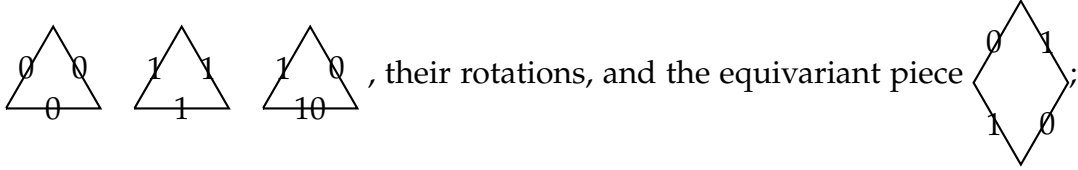
The Littlewood–Richardson coefficients $c_{\lambda\mu}^{\nu}$, where λ, μ, ν are (for now) partitions, satisfy a number of symmetries, one of which is $c_{\lambda\mu}^{\nu} = c_{\mu^T\lambda^T}^{\nu^T}$. One origin of L-R coefficients is as structure constants in the product in $H^*(Gr(k, V))$ of Schubert classes on the Grassmannian of k -planes in V . In that formulation, the **Grassmannian duality** homeomorphism $Gr(k, V) \cong Gr((\dim V) - k, V^*)$, $(U \leq V) \mapsto (U^{\perp} \leq V^*)$ induces an isomorphism of cohomology rings and a correspondence of Schubert bases, giving the symmetry above. This symmetry is not at all manifest in tableau-based computations of the $\{c_{\lambda\mu}^{\nu}\}$, but it is in the “puzzle” rule of [6], which replaces partitions by binary strings and is based on

*iva.halacheva@unimelb.edu.au

†allenk@math.cornell.edu

‡pzinn@unimelb.edu.au. Paul Zinn-Justin was supported by ARC grant FT150100232.

the puzzle pieces



we recall and generalize this rule in [Theorem 2](#) below. Specifically, the **dual** of a puzzle is made by flipping it left-right while exchanging all $0 \leftrightarrow 1$ (in particular, 10-labels again become 10s). The duals of the puzzles counted by $c_{\lambda\mu}^{\nu}$ are exactly those counted by $c_{\mu^T\lambda^T}^{\nu^T}$.

This prompts the question: what do *self-dual* puzzles count? One might expect it is something related to an isomorphism $V \cong V^*$ i.e. a bilinear form, and indeed our main theorems [1A](#), [1B](#), and [1C](#) interpret self-dual puzzles as computing the restrictions of Schubert classes on $Gr(k, 2n)$ to the *symplectic Grassmannian* $SpGr(k, 2n)$. (We will address elsewhere the minimal modifications necessary to handle the orthogonal case.) For $k = n$, there was already a tableau-based formula for these restrictions¹ in [\[8\]](#) which is less simple to state than [Theorem 1A](#); see also [\[4\]](#). This is perhaps another effect of tableaux being less suited to Grassmannian duality than puzzles are.

1.2 Restriction from $Gr(n, 2n)$

Let V be a vector space over \mathbb{C} equipped with a symplectic form, so the Grassmannian $Gr(k, V)$ of k -planes contains the subscheme

$$SpGr(k, V) := \{L \leq V : \dim L = k, L \leq L^\perp\}$$

where \perp means perpendicular with respect to the symplectic form. Then the inclusion $\iota : SpGr(k, V) \hookrightarrow Gr(k, V)$ induces a pullback $\iota^* : H^*(Gr(k, V)) \rightarrow H^*(SpGr(k, V))$ in cohomology. As both cohomology rings possess bases consisting of *Schubert classes* $\{S_\lambda\}$, one can ask about expanding $\iota^*(S_\lambda)$ in the basis of $SpGr(k, V)$'s Schubert classes $\{S_\nu\}$.

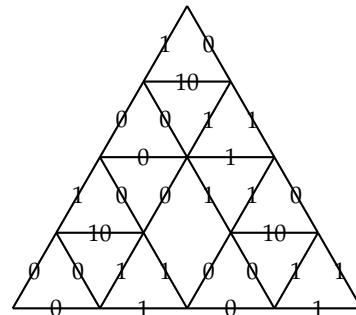
Let $\dim V = 2n$ (necessarily even, since V is symplectic), and for the simplest version of the theorem assume $k = n$. Then the Schubert classes on $Gr(n, V)$ are indexed by the $\binom{2n}{n}$ binary strings with n 0s and n 1s, whereas the Schubert classes on $SpGr(n, V)$ are indexed by the 2^n binary strings of length n (with more detail on this indexing in [Section 2](#)).

Theorem 1A. *Let S_λ be a Schubert class on $Gr(n, 2n)$, indexed by a string λ with content in $0^n 1^n$, and S_ν a Schubert class on $SpGr(n, 2n)$, indexed by a length n binary string. Then the coefficient of S_ν in $\iota^*(S_\lambda)$ is the number of self-dual puzzles with λ on the Northwest side, ν on the left half of the South side (both λ and ν read left to right), and equivariant pieces only allowed along the axis of reflection.*

¹In particular, [\[8\]](#) provides a cohomological interpretation of algebraic results of Stembridge [\[10\]](#) about expanding Schur functions into Schur P - and Q -functions.

Example 1. For $\lambda = 0101$, a self-dual puzzle with λ on the Northwest side has to be of the form $\begin{array}{c} \lambda \\ \triangle \\ \mu \end{array}$ for some μ .

So, it will appear in the usual calculation of $S_{0101}^2 \in H_T^*(Gr(2,4))$, which involves three puzzles. Only one of these puzzles is self-dual, and its only equivariant piece is on the centerline. From this we compute $\iota^*(S_{0101}) = S_{01}$ in $H^*(SpGr(2,4))$.



A surprising aspect of **Theorem 1A** is that equivariant pieces appear in this nonequivariant calculation, albeit only down the centerline.² If we allow them elsewhere (self-dually occurring in pairs), the puzzles compute the generalization of **Theorem 1A** to the map $\iota^* : H_T^*(Gr(n, 2n)) \rightarrow H_T^*(SpGr(n, 2n))$ in (torus-)equivariant cohomology, whose coefficients now live in the polynomial ring $H_T^*(pt) \cong \mathbb{Z}[y_1, \dots, y_n]$. We leave this statement until **Theorem 1C** in **Section 5** because it requires some precision about the locations of the symplectic Schubert varieties.

1.3 Interlude: puzzles with 10s on the South side

To generalize **Theorem 1A** to $SpGr(k, 2n)$, not just $k = n$, we need strings that index its $\binom{n}{k}2^k$ many Schubert classes. We do this using the third edge label, 10: consider strings v of length n with $(n - k)$ 10s, the rest a mix of 1s and 0s.

Before considering self-dual puzzles with Southside 10s, we mention a heretofore unobserved capacity of the puzzle pieces from [6], available once we allow for Southside 10s. It turns out they are already sufficient to compute certain products³ in the T -equivariant cohomology of 2-step flag manifolds! The only necessary new idea is to allow the previously internal label 10 to appear on the South side.

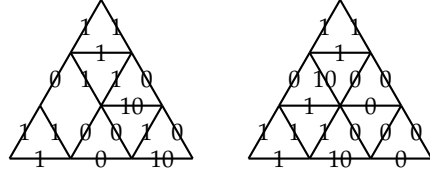
Theorem 2. Let $0 \leq j \leq k \leq n$, and let λ, μ be 0,1-strings with content $0^j1^{n-j}, 0^k1^{n-k}$ respectively, defining equivariant Schubert classes S_λ, S_μ on $Gr(j, \mathbb{C}^n), Gr(k, \mathbb{C}^n)$ respectively. Let π_j, π_k be the respective projections of the 2-step flag manifold $Fl(j, k; \mathbb{C}^n)$ to those Grassmannians. Let v be a string in the ordered alphabet 0, 10, 1 with content $0^j(10)^{k-j}1^{n-k}$, defining a Schubert class S_v in $H_T^*(Fl(j, k; \mathbb{C}^n))$. We emphasize that the alphabet order is 0, 10, 1!

Then as in [6], the coefficient of S_v in the product $\pi_i^*(S_\lambda)\pi_j^*(S_\mu) \in H_T^*(Fl(j, k; \mathbb{C}^n))$ is the sum over puzzles P with boundary labels λ, μ, v , made from the puzzle pieces in **Section 1.1**, of the “fugacities” $fug(P) := \prod_{\text{equivariant pieces } \diamond \text{ in } P} (y_{NE-SW \text{ diagonal of } \diamond} - y_{NW-SE \text{ diagonal of } \diamond})$.

²The number of equivariant pieces down the centerline is in fact fixed and equal to the number of 1s in v , by a weight conservation argument.

³Note that the general 2-step problem has received a puzzle formula [3], but using many more puzzle pieces than we use here. The problem of multiplying classes from different Grassmannians was studied already in [9].

Example 3. If $\lambda = 101$, $\mu = 100$, then their pullbacks give $\pi_1^*(S_{101}) = S_{10,0,1}$, $\pi_2^*(S_{100}) = S_{1,0,10}$, with product $(y_1 - y_2)S_{1,0,10} + S_{1,10,0}$ (note: to compare strings to permutations requires inversion, as in [Section 4](#)).

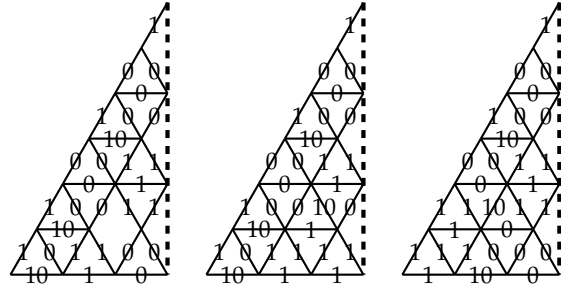


1.4 Restriction from $Gr(k, 2n)$, $k < n$

Theorem 1B. Let λ be a string with content $0^k 1^{2n-k}$, whereas ν is of length n with $(n - k)$ 10s, the rest a mix of 1s and 0s. Consider the puzzles from [Theorem 2](#), where we allow 10 labels to appear on the South side.

Then as before, in $H^*(SpGr(k, 2n))$, the coefficient of S_ν in $\iota^*(S_\lambda)$ is the number of self-dual puzzles with λ on the Northwest side, ν on the left half of the South side (both λ and ν read left to right), and equivariant pieces only allowed along the axis of reflection.

Example 4. In the remainder of the paper we work with the left halves \triangleleft of self-dual puzzles, since the centerline and right half can be inferred. The half-puzzles pictured here (really for equivariant [Theorem 1C](#) to come) show $\iota^*(S_{110101}) = (y_2 - y_3)S_{10,1,0} + S_{10,1,1} + S_{1,10,0}$.



The proof is based on the “quantum integrability” of R -matrices, and closely follows that of [\[7\]](#) (see also [\[11\]](#)); in particular, following the quantum integrable literature, we use graph-dual pictures (scattering diagrams) which are more amenable than puzzles to topological manipulations. The principal new feature is the appearance of K -matrices. The “reflection equation” $RKRK = KRKR$ (more precisely, eq. [\(3.4\)](#) in [Lemma 7](#)) is standard; however since the approach of [\[7\]](#) requires not just R -matrices but the trivalent U -matrix, we need here the possibly novel “ K -fusion equation” [\(3.5\)](#) in [Lemma 7](#).

2 The groups, flag manifolds, and cohomology rings

We take the Gram matrix of our symplectic form to be antidiagonal; this is so that if B_\pm are the upper/lower triangular Borel subgroups of GL_{2n} , then $B_\pm \cap Sp_{2n}$ will be opposed Borel subgroups of Sp_{2n} .

Consider $Gr(k, 2n) = \{0 \leq V \leq \mathbb{C}^{2n} \mid \dim V = k\} \cong GL_{2n}/P$ where $k \leq n$ and P is the parabolic subgroup of block type $(k, 2n - k)$ containing $B = B_+$. Then $P_{Sp_{2n}} = P \cap Sp_{2n}$ is a parabolic for the Lie subgroup Sp_{2n} and the symplectic Grassmannian is $SpGr(k, 2n) = \{0 \leq V < \mathbb{C}^{2n} \mid \dim V = k, V \leq V^\perp\} \cong Sp_{2n}/P_{Sp_{2n}}$. Let $T^{2n} := B_+ \cap B_-$ be the diagonal matrices in GL_{2n} , and $T^n := Sp_{2n} \cap T^{2n}$. Note that

$(GL_{2n}/P)^{T^n} = (GL_{2n}/P)^{T^{2n}}$ since there exist $x \in T^n$ with no repeated eigenvalues. The following diagram of spaces commutes.

$$\begin{array}{ccccc} \{v : \exists a \text{ s.t. } \text{content}(v) = 0^a(10)^{n-k}1^{k-a}\} & \xrightarrow{\sim} & (SpGr(k, 2n))^{T^n} & \hookrightarrow & SpGr(k, 2n) \\ \downarrow \tilde{\iota} & & \downarrow & & \downarrow \iota \\ \{\lambda : \text{content}(\lambda) = 0^k1^{2n-k}\} & \xrightarrow{\text{coord}} & (Gr(k, 2n))^{T^n} & \hookrightarrow & Gr(k, 2n) \end{array}$$

The map $\tilde{\iota}$ takes a sequence v first to its double $v\bar{v}$ where \bar{v} is v reflected and its 0s and 1s are switched; after that, all 10s in $v\bar{v}$ are turned into 1s, e.g.

$$0, 10, 1, 0, 10 \quad \mapsto \quad 0, 10, 1, 0, 10, 10, 1, 0, 10, 1 \quad \mapsto \quad 0, 1, 1, 0, 1, 1, 1, 0, 1, 1$$

The bijective map coord takes a 0, 1-sequence λ to the coordinate k -plane that uses the coordinates in the 0 positions of λ (so, 1, 4, 8 in the above example). Note that $\text{coord} \circ \tilde{\iota}(v) \in SpGr(k, 2n)$ by the anti-diagonality we required of the Gram matrix.

The right-hand square, and the inclusion $T^n \hookrightarrow T^{2n}$, induce the ring homomorphisms

$$\begin{array}{ccccc} H_{T^{2n}}^*(Gr(k, 2n)) & \xrightarrow{f_1} & H_{T^n}^*(Gr(k, 2n)) & \xrightarrow{f_2 = \iota^*} & H_{T^n}^*(SpGr(k, 2n)) \\ g_1 \downarrow & & \downarrow g_2 & & \downarrow g_3 \\ H_{T^{2n}}^*(Gr(k, 2n)^{T^{2n}}) & \xrightarrow{h_1} & H_{T^n}^*(Gr(k, 2n)^{T^n}) & \xrightarrow{h_2} & H_{T^n}^*(SpGr(k, 2n)^{T^n}) \end{array}$$

and since each g_i is injective (see e.g. [5]), we can compute along the bottom row, which is the proof technique used in [7] and Section 5. On each of our flag manifolds, we define our Schubert classes as associated to the closures of orbits of B_- or $B_- \cap Sp_{2n}$.

3 Scattering diagrams and their tensor calculus

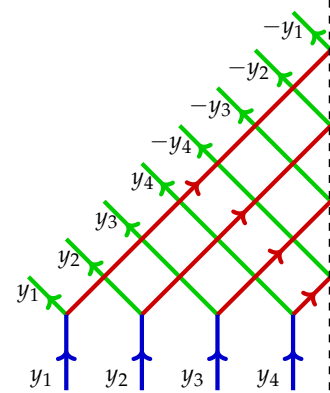
In the statement and proof of Theorem 1B, we work with *half-puzzles*, i.e., labeled half-triangles $2n\triangle$ of size $2n$, tiled with the triangle and rhombus puzzle pieces described in Section 1.1, as well as half-rhombus puzzle pieces obtained by cutting the existing self-dual ones vertically in half. As discussed earlier, a half-puzzle can be considered as half of a self-dual puzzle with all three sides of length $2n$. In our notation, a ‘‘rhombus’’ can also be made of a \triangle and a ∇ triangle glued together.

To linearize the puzzle pictures and relate them back to the restriction of cohomology classes, we consider the puzzle labels $\{0, 10, 1\}$ as indexing bases for three spaces $\mathbb{C}_G^3, \mathbb{C}_R^3, \mathbb{C}_B^3$ (Green, Red, Blue). In our scattering diagrams below, each coloured edge will carry its corresponding vector space.

1. Take an **unlabeled** size $2n$ half-puzzle triangle $2n\mathcal{L}$ tiled by rhombi, half-rhombi (on the East) and triangles (on the South) as before, with assigned “spectral parameters” $y_1, \dots, y_n, -y_n, \dots, -y_1$ on the Northwest side.

2. Consider the dual-graph picture of strands, oriented upwards. Each rhombus corresponds to a crossing of two strands, each half-rhombus to a bounce off the East wall and negates the spectral parameter, and each triangle to a trivalent vertex with all parameters equal.

We also colour the Northwest-pointing strands green, Northeast-pointing red, and North-pointing blue.



3. We let a and b denote two spectral parameters from Step 1. We assign the following linear maps

- to each crossing of two strands with left and right parameters a and b , and colours C and D , a linear map $R_{CD}(a - b) : \mathbb{C}_C^3 \otimes \mathbb{C}_D^3 \longrightarrow \mathbb{C}_D^3 \otimes \mathbb{C}_C^3$;
- to each wall-bounce of a colour C strand with parameter a bouncing to $-a$, a map $K_C(a) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$;
- to each trivalent vertex with incoming blue strand and outgoing green and red strands, all with parameters a , a map $U(a) : \mathbb{C}_B^3 \longrightarrow \mathbb{C}_G^3 \otimes \mathbb{C}_R^3$.

Connecting two strands corresponds to composing the corresponding maps, so the whole $2n\mathcal{L}$ corresponds to a linear map $\Phi : (\mathbb{C}_B^3)^{\otimes n} \longrightarrow (\mathbb{C}_G^3 \otimes \mathbb{C}_R^3)^{\otimes 2n}$.

Definition 5 (The R -, U -, and K -matrices). *In terms of the bases of $\mathbb{C}_G^3, \mathbb{C}_R^3, \mathbb{C}_B^3$ indexed by $\{0, 10, 1\}$, the above sparse matrices can be written compactly as follows (where a labeled diagram corresponds to the coefficient of the map in those basis elements):*

$$R_{CC}(a - b) : \begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ k \quad l \end{array} = \begin{cases} 1 & \text{if } (i, j) = (k, l), \\ b - a & \text{if } (i, j, k, l) \in \{(1, 0, 0, 1), (10, 0, 0, 10), (1, 10, 10, 1)\} \end{cases}$$

where $C \in \{R, G, B\}$ and the two strands are any identical colour

$$R_{RG}(a - b) : \begin{array}{c} i \quad j \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ k \quad l \end{array} = \begin{cases} a - b & \text{if } (i, j, k, l) = (0, 1, 1, 0), \\ 1 & \text{if } (i, j, k, l) \in \left\{ \begin{array}{l} 0^4, 1^4, 0^2 1(10), 0(10)1^2, 0(10)^2 0, \\ 1 \ 0^2 1, 1^2(10)0, (10)1 \ 0^2, (10)1^2(10) \end{array} \right\} \end{cases}$$

$$\begin{aligned}
 K_R(a) : \begin{array}{c} i \\ \diagdown \text{ (green)} \\ \diagup \text{ (red)} \\ j \end{array} &= 1 \text{ if } (i, j) \in \{(1, 0), (0, 1)\} & K_B(a) : \begin{array}{c} i \\ \diagdown \text{ (blue)} \\ \diagup \text{ (blue)} \\ j \end{array} &= \begin{cases} 1 & \text{if } i = j, \\ -2a & \text{if } (i, j) = (1, 0) \end{cases} \\
 U(a) : \begin{array}{c} i \quad j \\ \diagdown \text{ (green)} \quad \diagup \text{ (red)} \\ \diagup \text{ (blue)} \\ k \end{array} &= 1 \text{ if } (i, j, k) \in \{(0, 0, 0), (0, 10, 1), (1, 0, 10), (1, 1, 1), (10, 1, 0)\}
 \end{aligned}$$

The subscripts R, G, B on the maps indicate the colours of the incoming edges (listed counter-clockwise). For each map, the matrix entries which are not listed are zero.

Note that if we take the corresponding bases with lexicographic ordering, with alphabet ordered as $\{0, 10, 1\}$, then the matrices for R_{CC} and K_B are lower-triangular. See [11] for these R -matrices and [7, §3] for their representation-theoretic origins.

Definition 6. With the above notation, let \mathbf{P} be a half-puzzle with boundary labels $\begin{array}{c} \lambda \\ \diagdown \\ v \end{array}$ where $\lambda \in 0^k 1^{2n-k}$ and $v \in (10)^{n-k} \{0, 1\}^k$. The fugacity $\text{fug}(\mathbf{P})$ of \mathbf{P} is the product over all puzzle pieces (dually: vertices) of the entries of the corresponding R -, U -, K -matrices.

In this way, the summation over half-puzzles reproduces the full matrix product, i.e., the (λ, ν) matrix entry of $\Phi = \sum_{\mathbf{P}} \left\{ \text{fug}(\mathbf{P}) \mid \mathbf{P} \text{ is a puzzle with boundary } \begin{array}{c} \lambda \\ \diagdown \\ \nu \end{array} \right\}$.

Lemma 7. The matrices defined in Section 3 satisfy the following identities:

i) The Yang–Baxter equation.

$$\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} \end{array} = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} \quad (3.1)$$

$$\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} \end{array} = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \diagdown & \diagup & \diagup \\ \diagup & \diagdown & \diagdown \\ u_3 & u_2 & u_1 \end{array} \quad (3.2)$$

For example, the linear map form of the Northwest equation is

$$\begin{aligned}
 & (R_{RG}(u_2 - u_1) \otimes \text{Id}) \circ (\text{Id} \otimes R_{RG}(u_3 - u_1)) \circ (R_{RR}(u_3 - u_2) \otimes \text{Id}) \\
 &= (\text{Id} \otimes R_{RR}(u_3 - u_2)) \circ (R_{RG}(u_3 - u_1) \otimes \text{Id}) \circ (\text{Id} \otimes R_{RG}(u_2 - u_1))
 \end{aligned}$$

ii) *Swapping of two trivalent vertices.*

$$(3.3)$$

$$\begin{aligned} & (\text{Id} \otimes R_{RG}(u_1 - u_2) \otimes \text{Id}) \circ (U(u_1) \otimes U(u_2)) \circ R_{BB}(u_2 - u_1) \\ &= (R_{GG}(u_2 - u_1) \otimes R_{RR}(u_2 - u_1)) \circ (\text{Id} \otimes R_{RG}(u_2 - u_1) \otimes \text{Id}) \circ (U(u_2) \otimes U(u_1)) \end{aligned}$$

iii) *The reflection equation.*

$$(3.4)$$

In linear map terms, the left equation says

$$\begin{aligned} & (\text{Id} \otimes K_R(-u_2)) \circ R_{RG}(-u_2 - u_1) \circ (\text{Id} \otimes K_R(-u_1)) \circ R_{RR}(-u_1 + u_2) \\ &= R_{GG}(u_2 - u_1) \circ (\text{Id} \otimes K_R(-u_1)) \circ R_{RG}(-u_1 - u_2) \circ (\text{Id} \otimes K_R(-u_2)) \end{aligned}$$

iv) *K-fusion.*

$$(3.5)$$

$$\begin{aligned} & (\text{Id} \otimes K_R(u_1)) \circ U(u_1) \circ K_B(-u_1) \\ &= \\ & R_{GG}(-2u_1) \circ (\text{Id} \otimes K_R(-u_1)) \circ U(-u_1) \end{aligned}$$

4 AJS/Billey formulæ as scattering diagrams

We first discuss the general AJS/Billey formula for restricting an equivariant Schubert class to a torus-fixed point, and then consider the special cases of types A and C . Let G be an algebraic group and fix a pinning $G \geq B \geq T$, with $W_G = N_G(T)/T$. Let B_- denote the opposite Borel and $P \geq B$ a parabolic, with Weyl group W_P . We recall that Schubert classes are indexed by W_G/W_P , which we identify with strings (or signed strings in type C), on which W_G acts by permuting/negating positions: $\pi W_P \mapsto \omega \circ \pi^{-1}$ (see [Proposition 9](#) for ω). In particular for $P = B$ our indexing is inverse to the usual convention; this inversion is forced on us by the necessary use for general P of strings-with-repeats, e.g. binary strings rather than Grassmannian permutations.

Proposition 8. 1. ([1, 2]) For the Schubert class $S_\pi := [\overline{B-\pi B}/B] \in H_T^*(G/B)$, $\pi, \sigma \in W_G$, and $Q = (q_1, \dots, q_k)$ a reduced word in simple reflections with $\prod Q = \sigma$, the AJS/Billey formula tells us that

$$S_\pi|_\sigma = \sum_{\substack{R \subseteq Q \\ \prod R = \pi}} \prod_{i=1}^k (\widehat{\alpha}_{q_i}^{[q_i \in R]} q_i) \cdot 1 = \sum_{\substack{R \subseteq Q \\ \prod R = \pi}} \prod_{i \in R} \beta_i \in H_T^*(pt) \quad (4.1)$$

where $\beta_i := q_1 q_2 \dots q_{i-1} \cdot \alpha_{q_i}$ and the summation is over reduced subwords R of Q .

2. To compute a point restriction $S_\lambda|_\mu$ on G/P , where $\lambda, \mu \in W_G/W_P$, we use lifts $\tilde{\lambda}, \tilde{\mu} \in W_G$ such that $\tilde{\lambda}$ is the shortest length representative of λ , and observe that $S_\lambda|_\mu = S_{\tilde{\lambda}}|_{\tilde{\mu}}$.

Below we give a diagrammatic description of the formula from [Proposition 8](#) in the cases when (G, W_G) is (GL_{2n}, S_{2n}) or $(Sp_{2n}, S_n \times (\mathbb{Z}/2\mathbb{Z})^n)$ using the tensor calculus setting of [Section 3](#). We first introduce some notation.

Consider σ in W_G (generated by simple reflections $\{s_i^G\}$) and a reduced word for σ , $Q_\sigma = (q_1, \dots, q_k)$ (where $q_i = s_{p_i}$ for some p_i). We can associate to it a wiring diagram $D(Q_\sigma)$ by assigning the diagrams below to the simple reflections $\{s_i^G : 1 \leq i \leq m_G - 1\}$ for $(m_{GL_{2n}}, m_{Sp_{2n}}) = (2n, n)$, and to $s_n^{Sp_{2n}}$ respectively. Then, a word in simple reflections corresponds to a concatenation of such diagrams.

$$s_i^G \mapsto \begin{array}{c} 1 \quad i \quad i+1 \quad m_G \\ \uparrow \dots \uparrow \times \uparrow \dots \uparrow \end{array}, \text{ for } 1 \leq i \leq m_G - 1 \quad \text{and} \quad s_n^{Sp_{2n}} \mapsto \begin{array}{c} 1 \quad n \\ \uparrow \dots \uparrow \times \vdots \end{array}$$

Each wire in a wiring diagram is also assigned a spectral parameter. For $G = GL_{2n}$, they are y_1, \dots, y_{2n} along the top (which we need to later specialize to $y_1, \dots, y_n, -y_n, \dots, -y_1$ as in the maps f_1, h_1 in [Section 2](#)), and for $G = Sp_{2n}$ they are y_1, \dots, y_n .

In the context of the tensor calculus from [Section 3](#), the wiring diagram $D(Q_\sigma)$ can be interpreted as a scattering diagram, i.e., giving a map $(\mathbb{C}_C^3)^{\otimes m_G} \rightarrow (\mathbb{C}_C^3)^{\otimes m_G}$; we replace each crossing with R_{GG} in the GL_{2n} (and $C = G$) case or with R_{BB} in the Sp_{2n} (and $C = B$) case, and replace each bounce with K_B which also negates the spectral parameter. For instance, take $G = Sp_6$ and $\sigma = 31\bar{2}$, $Q_\sigma = (s_2, s_3, s_1)$, then

$$D(Q_\sigma) = \begin{array}{c} y_1 \quad y_2 \quad y_3 \\ \downarrow \times \downarrow \\ y_3 \quad y_1 \quad -y_2 \end{array} \quad (\text{Id} \otimes R_{BB}(y_3 - y_2)) \circ (\text{Id}^{\otimes 2} \otimes K_B(-y_2)) \circ (R_{BB}(y_3 - y_1) \otimes \text{Id}) : (\mathbb{C}_B^3)^{\otimes 3} \rightarrow (\mathbb{C}_B^3)^{\otimes 3}$$

Proposition 9. Let λ, μ be strings in $0, 10, 1$ as in [Section 2](#), which we identify with cosets W_G/W_P where W_G is of type C and P is maximal, or of type A and P is maximal or submaximal.

Let $\omega_{Gr} = 0 \dots 0 1 \dots 1 \in 0^k 1^{2n-k}$ for $G/P = Gr(k, \mathbb{C}^{2n})$, $\omega_{SpGr} = 0 \dots 0 10 \dots 10 \in 0^k (10)^{n-k}$ for $G/P = SpGr(k, \mathbb{C}^{2n})$, or $\omega_{Fl} = 0 \dots 0 10 \dots 10 1 \dots 1 \in 0^j (10)^{k-j} 1^{2n-k}$ for $G/P = Fl(j, k; \mathbb{C}^{2n})$. Make a wiring diagram as just explained, using a reduced word for the shortest lift $\tilde{\mu}$; interpret it as a scattering diagram map, using the $R_{BB}(= R_{GG})$ matrix for crossings and (in type C) K_B for bounces. Then $S_\lambda|_\mu$ is the $(\lambda, \omega_{G/P})$ matrix entry of the resulting product.

The essentially routine rewriting of [Proposition 8](#) to give [Proposition 9](#) will appear elsewhere. The principal thing one checks is that R_{BB} is the correct R -matrix for three labels $\{0, 10, 1\}$. In view of [Proposition 9](#), for $\lambda, \mu, \nu \in W_G/W_P$ as above, we denote

$$\boxed{\begin{array}{c} \lambda \\ \mu \\ \nu \end{array}} := \text{the } (\lambda, \nu) \text{ matrix entry for the scattering diagram map} \\ \text{coming from a reduced word for } \tilde{\mu}.$$

By the proposition, when $\nu = \omega_{G/P}$ this gives $S_\lambda|_\mu$.

5 Proof of [Theorem 1B](#)

The proof of [Theorem 2](#) is very much as in [7, §3] and will appear elsewhere. [Theorem 1A](#) is the $k = n$ special case of [Theorem 1B](#). In fact, we give a more general puzzle rule for equivariant cohomology in [Theorem 1C](#), which in particular implies [Theorem 1B](#).

Theorem 1C. For every $S_\lambda \in H_{T^n}^*(Gr(k, 2n))$, where $\lambda \in 0^k 1^{2n-k}$, and ι^* as in [Section 2](#)

$$\iota^*(S_\lambda) = \sum_{\nu \in (10)^{n-k} \{0,1\}^k} \left(\sum_{\mathbf{P}} \{fug(\mathbf{P}) \mid \mathbf{P} \text{ is a puzzle with boundary } \begin{array}{c} \lambda \\ \omega \end{array}\} \right) S_\nu$$

As explained in [Section 2](#), it suffices to check [Theorem 1C](#)'s equality at each T^n -fixed point $\sigma \in (10)^{n-k} \{0,1\}^k$ of $SpGr(k, 2n)$. To do so, we first prove several preliminary results in the language of [Section 3](#).

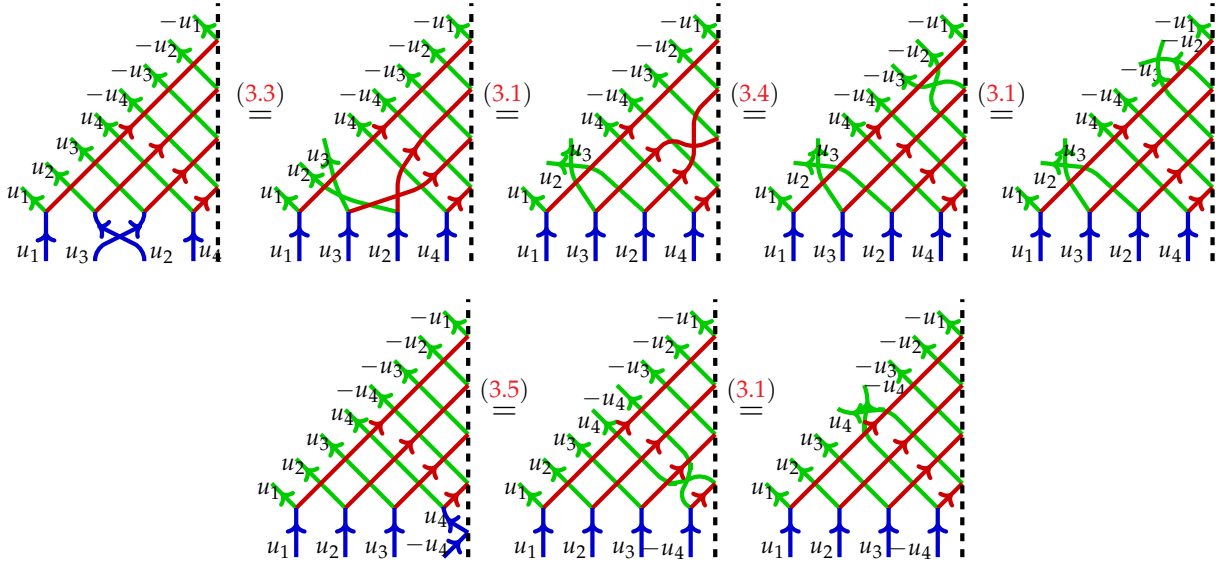
Lemma 10. For $\omega = \omega_{SpGr}$ as in [Proposition 9](#) and $\lambda \in 0^k 1^{2n-k}$, we have $\begin{array}{c} \lambda \\ \omega \end{array} = \delta_{\lambda, \omega \bar{\omega}}$.

Proof. This is a straightforward consequence of [Definition 5](#), when considering the (λ, ω) matrix entry of the product of R -, K -, and U -matrices making up the half-puzzle. Alternatively, note that this is half of a classical triangular self-dual puzzle with NW, NE, S boundaries labelled by $\lambda, \bar{\lambda}, \omega \bar{\omega}$, and so the result follows from [7, Proposition 4]. \square

Proposition 11. Given $\sigma \in (10)^{n-k} \{0,1\}^k$, fixing the Northwest and South boundaries to be strings of length $2n$ and n respectively, one has

$$\begin{array}{c} \diagup \\ \sigma \end{array} = \begin{array}{c} \diagup \\ \sigma \end{array}$$

Proof. It suffices to consider $\tilde{\sigma}$ (from Proposition 8) a simple reflection. For the purposes of illustration, we set $n = 4$ and demonstrate the equality in the case of an s_i where $i < n$, as well as for s_n .



□

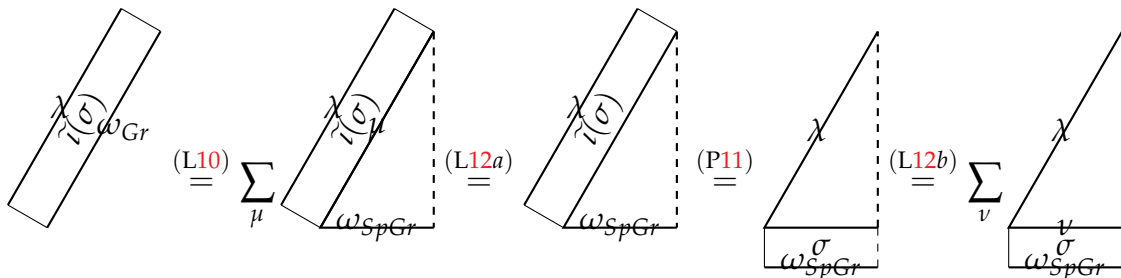
Lemma 12. a) [7, Proposition 4] Type A. Let $\sigma \in 0^k 1^{2n-k}$ and λ be a string of length $2n$:

If $\boxed{\begin{smallmatrix} \lambda \\ \omega_{Gr}^\sigma \end{smallmatrix}} \neq 0$ for ω_{Gr} as in Proposition 9, then λ consists only of 0s and 1s (no 10s).

b) Type C. Let $\sigma \in (10)^{n-k} \{0, 1\}^k$ and λ be a string of length n : If $\boxed{\begin{smallmatrix} \lambda \\ \omega_{SpGr}^\sigma \end{smallmatrix}} \neq 0$ for ω_{SpGr} as in Proposition 9, then λ has the same number of 10s as ω_{SpGr} .

Proof. To prove part b), recall that $\boxed{\begin{smallmatrix} \lambda \\ \omega_{SpGr}^\sigma \end{smallmatrix}}$ is the (λ, ω_{SpGr}) matrix entry for the composition of R_{BB} and K_B maps. From Definition 5, we see that both of these maps preserve the number of 10s in a string, hence so will compositions of these maps. □

Proof of Theorem 1C. In $H_T^*(pt)$, we have the following equality



The left side corresponds to $\iota^*(S_\lambda)|_\sigma$ by [Proposition 9](#). In the second and fourth equality, the strings μ and ν have content $0^k 1^{2n-k}$ and $(10)^{n-k} \{0,1\}^k$ respectively, and all other terms of the sum vanish. \square

Acknowledgements

We thank İzzet Coşkun and Alex Yong for references, and Michael Wheeler for discussions about P - and Q -Schur functions.

References

- [1] H. H. Andersen, J. C. Jantzen, and W. Soergel. *Representations of quantum groups at a p th root of unity and of semisimple groups in characteristic p : independence of p* . Astérisque 220. Soc. Math. France, Paris, 1994.
- [2] S. C. Billey. “Kostant polynomials and the cohomology ring for G/B ”. *Duke Math. J.* **96.1** (1999), pp. 205–224. [Link](#).
- [3] A. S. Buch, A. Kresch, K. Purbhoo, and H. Tamvakis. “The puzzle conjecture for the cohomology of two-step flag manifolds”. *J. Algebraic Combin.* **44.4** (2016), pp. 973–1007. [Link](#).
- [4] İ. Coşkun. “Symplectic restriction varieties and geometric branching rules II”. *J. Combin. Theory Ser. A* **125** (2014), pp. 47–97. [Link](#).
- [5] F. C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry*. Mathematical Notes 31. Princeton University Press, Princeton, NJ, 1984.
- [6] A. Knutson and T. Tao. “Puzzles and (equivariant) cohomology of Grassmannians”. *Duke Math. J.* **119.2** (2003), pp. 221–260. [Link](#).
- [7] A. Knutson and P. Zinn-Justin. “Schubert puzzles and integrability I: invariant trilinear forms”. 2017. [arXiv:1706.10019](#).
- [8] P. Pragacz. “Addendum: A generalization of the Macdonald-You formula”. *J. Algebra* **226.1** (2000), pp. 639–648. [Link](#).
- [9] K. Purbhoo and F. Sottile. “A Littlewood–Richardson rule for Grassmannian permutations”. *Proc. Amer. Math. Soc.* **137.6** (2009), pp. 1875–1882. [Link](#).
- [10] J. R. Stembridge. “Shifted tableaux and the projective representations of symmetric groups”. *Adv. Math.* **74.1** (1989), pp. 87–134. [Link](#).
- [11] P. Zinn-Justin. “Littlewood–Richardson coefficients and integrable tilings”. *Electron. J. Combin.* **16.1** (2009), Research Paper 12, 33 pp. [Link](#).